Multivariable Calculus

MTH201

Virtual University of Pakistan

Knowledge beyond the boundaries

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LECTURE No. 1

INTRODUCTION

– Calculus is the mathematical tool used to analyze changes in physical quantities.
– Calculus is also Mathematics of Motion and Change.
– Where there is motion or growth, where variable forces are at work producing acceleration, Calculus is right mathematics to apply.

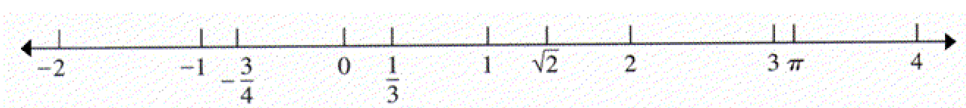
Differential Calculus deals with the Problem of Finding
(1) Rate of change
(2) Slope of curve

Velocities and acceleration of moving bodies. Firing angles that give cannons their maximum range. The times when planets would be closest together or farthest apart.

Integral Calculus deals with the Problem of determining a *function* from information about its *rates of Change.*

Integral Calculus enables us
(1) To calculate lengths of curves.
(2) To find areas of irregular regions in plane.
(3) To find the volumes and masses of arbitrary solids
(4) To calculate the future location of a body from its present position and knowledge of the forces acting on it.

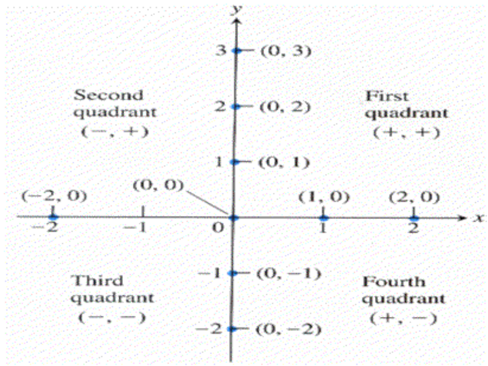
Reference Axis System
Before giving the concept of Reference Axis System, we recall you the concept of real line and locate some points on the real line as shown in the figure below, also remember that the real number system consist of both Rational and Irrational numbers that is we can write set of real numbers as union of rational and irrational numbers.



Here in the above figure, we have located some of the rational as well as irrational numbers and also note that there are infinite real numbers between every two real numbers.

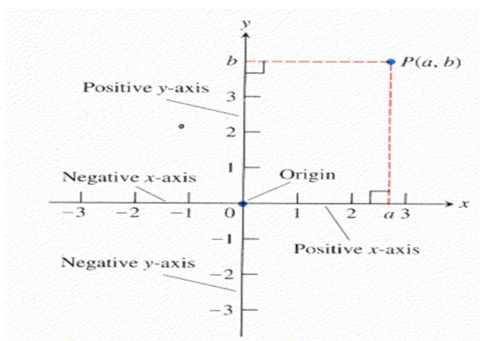
Now if you are working in two dimensions, then you know that we take the two mutually perpendicular lines and call the horizontal line as *x*-axis and vertical line as *y*-axis and where these lines cut we take that point as origin. Now any point on the *x*-axis will be denoted by an order pair whose first element which is also known as abscissa is a real number and other element of the order pair which is also known as ordinate will have 0 values, i.e. $(x,\ 0)$

Similarly any point on the y-axis can be representing by an order pair $(0, y)$. Some points are shown in the figure below. Also note that these lines divide the plane into four regions: First, Second, Third and Fourth quadrants respectively. We take the positive real numbers at the right side of the origin and negative to the left side, in the case of x-axis. Similarly for y-axis and also shown in the figure.



Location of a point

Now we will illustrate how to locate the point in the plane using x- and y- axes. Draw two perpendicular lines from the point whose position is to be determined. These lines will intersect at some point on the x-axis and y-axis and we can find out these points. Now the distance of the point of intersection of x-axis and perpendicular line from the origin is the x-coordinate of the point P and similarly the distance from the origin to the point of intersection of y-axis and perpendicular line is the y-coordinate of the point P as shown in the figure below.

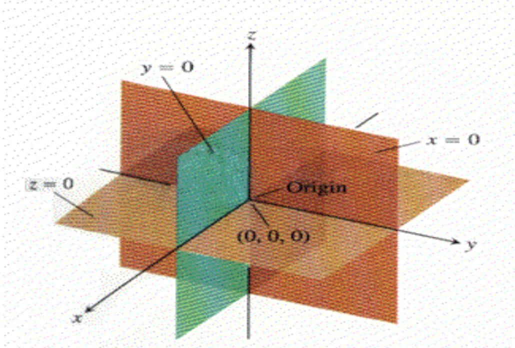


Cartesian coordinates.

In space, we have three mutually perpendicular lines as reference axes, namely x ,y and z axis. Now you can see from the figure below that the planes x = 0 ,y = 0 and z = 0 divide the space into eight octants. Also note that in this case we have (0,0,0) as origin and any point in the space will have three coordinates.

Signs of coordinates in different octants

First of all note that the equation x = 0 represents a plane in the 3d space and in this plane every point has its x-coordinate as 0, also that plane passes through the origin as shown in the figure above. Similarly y = 0 and z = 0 also define a plane in 3d space and have properties similar to that of x = 0 such that these planes also pass through the origin and any point in the plane y = 0 will have y-coordinate as 0 and any point in the plane z = 0 has z-coordinate as 0.



The planes $x = 0$, $y = 0$, and $z = 0$ divide space into eight octants.

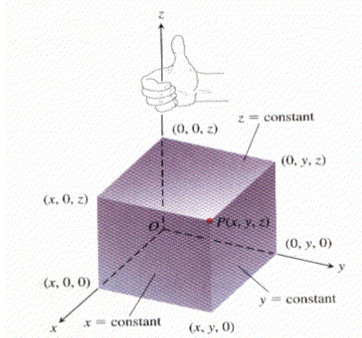
Also remember that when two planes intersect we get the equation of a line and when two lines intersect then we get a plane containing these two lines. Now note that by the intersection of the planes x = 0 and z = 0 we get the line which is our y-axis.
Also by the intersection of x = 0 and y = 0 we get the line which is z-axis, similarly you can easily see that by the intersection of z = 0 and y = 0 we get line which is x-axis.

Now these three planes divide the 3d space into eight octants depending on the positive and negative direction of axis. The octant in which every coordinate of any point has positive sign is known as first octant formed by the positive x, y and z –axes. Similarly in second octant every point has x-coordinate as negative and other two coordinates as positive correspond to negative x-axis and positive y and z axis.

Now one octant is that in which every point has x and y coordinate negative and z-coordinate positive, which is known as the third octant. Similarly we have eight octants depending on the sign of coordinates of a point. These are summarized below.

First octant	$(+,\ +,\ +)$	Formed by positive sides of the three axis.
Second octant	$(-,\ +,\ +)$	Formed by –ve x-axis and positive y and z-axis.
Third octant	$(-,\ -,\ +)$	Formed by –ve x and y axis with positive z-axis.
Fourth octant	$(+,\ -,\ +)$	Formed by +ve x and z axis and –ve y-axis.
Fifth octant	$(+,\ +,\ -)$	Formed by +ve x and y axis with -ve z-axis.
Sixth octant	$(-,\ +,\ -)$	Formed by –ve x and z axis with positive y-axis.
Seventh octant	$(-,-,\ -)$	Formed by –ve sides of three axis.
Eighth octant	$(+,\ -,\ -)$	Formed by -ve y and z-axis with +ve x-axis.

(Remember that we have two sides of any axis one of positive values and the other is of negative values) Now as we told you that in space we have three mutually perpendicular lines as reference axis. So far you are familiar with the reference axis for 2d which consist of two perpendicular lines namely x-axis and y-axis. For the reference axis of 3d space we need another perpendicular axis which can be obtained by the cross product of the two vectors, now the direction of that vector can be obtained by Right Hand rule. This is illustrated below with diagram.



The Cartesian coordinate system is right-handed.

Concept of a Function

Historically, the term, function denotes the dependence of one quantity on other quantity. The quantity x is called the independent variable and the quantity y is called the dependent variable. We write it as $y = f(x)$ and we read y is a function of x.

For example, the equation $y = 2x$ defines y as a function of x because each value assigned to x determines unique value of y.

Examples of function

– The area of a circle depends on its radius r by the equation A= πr^2 so, we say that A is a function of r.

– The volume of a cube depends on the length of its side x by the equation V= x^3 so, we say that V is a function of x.

– The velocity *V* of a ball falling freely in the earth's gravitational field increases with time t until it hits the ground, so we say that V is function of t.

– In a bacteria culture, the number n of present after one day of growth depends on the number N of bacteria present initially, so we say that N is function of n.

Function of Several Variables

Many functions depend on more than one independent variable.

Examples

1) The area of a rectangle depends on its length *l* and width *w* by the equation
$A = l\,w$, so we say that *A* is a function of *l* and *w.*

2) The volume of a rectangular box depends on the length *l*, width *w* and height *h* by the equation $V = l\,w\,h$ so, we say that *V* is a function of *l* , *w* and *h.*

3) The area of a triangle depends on its base length l and height h by the equation

$$A=\frac{1}{2}\, l\times h$$, so we say that A is a function of l and h.

4) The volume V of a right circular cylinder depends on its radius r and height h by the equation $V = \pi r^2 h$ so, we say that V is a function of r and h.

LECTURE No. 2

VALUES OF FUNCTIONS

Example 1: Consider the function $f(x) = 2x^2 - 1$, then $f(1) = 2(1)^2 - 1 = 1$

$f(4) = 2(4)^2 - 1 = 31, \quad f(-2) = 2(-2)^2 - 1 = 7$

$f(t-4) = 2(t-4)^2 - 1 = 2t^2 - 16t + 31$

These are the values of the function at some points.

Example 2 : Now we will consider a function of two variables, so consider the function

$f(x,y) = x^2 y + 1$ then $f(2,1) = (2)^2 1 + 1 = 5, \quad f(1,2) = (1)^2 2 + 1 = 3,$

$f(0,0) = (0)^2 0 + 1 = 1, \quad f(1,-3) = (1)^2(-3) + 1 = -2,$

$f(3a,a) = (3a)^2 a + 1 = 9a^3 + 1, \quad f(ab, a-b) = (ab)^2(a-b) + 1 = a^3b^2 - a^2b^3 + 1$

These are values of the function at some points.

Example 3: Now consider the function $f(x,y) = x + \sqrt[3]{xy}$, then

(a) $f(2,4) = 2 + \sqrt[3]{(2)(4)} = 2 + \sqrt[3]{8} = 2 + 2 = 4$

(b) $f(t,t^2) = t + \sqrt[3]{(t)(t^2)} = t + \sqrt[3]{t^3} = t + t = 2t$

(c) $f(x,x^2) = x + \sqrt[3]{(x)(x^2)} = x + \sqrt[3]{x^3} = x + x = 2x$

(d) $f(2y^2, 4y) = 2y^2 + \sqrt[3]{(2y^2)(4y)} = 2y^2 + \sqrt[3]{8y^3} = 2y^2 + 2y$

Example 4: Now again we take another function of three variables

$f(x,y,z) = \sqrt{1 - x^2 - y^2 - z^2}$ then $f(0, \frac{1}{2}, \frac{1}{2}) = \sqrt{1 - 0 - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}}$

Example 5: Consider the function $f(x,y,z) = xy^2z^3 + 3$, then at certain points we have

$f(2,1,2) = (2)(1)^2(2)^3 + 3 = 19, \quad f(0,0,0) = (0)(0)^2(0)^3 + 3 = 3,$

$f(a,a,a) = (a)(a)^2(a)^3 + 3 = a^6 + 3, \quad f(t,t^2,-t) = (t)(t^2)^2(-t)^3 + 3 = -t^8 + 3,$

$f(-3,1,1) = (-3)(1)^2(1)^3 + 3 = -3 + 3 = 0$

Example 6: Consider the function $f(x,y,z) = x^2y^2z^4$, where $x(t) = t^3$, $y(t) = t^2$ and $z(t) = t$

$(a) \quad f(x(t), y(t), z(t)) = [x(t)]^2 [y(t)]^2 [z(t)]^4 = [t^3]^2 [t^2]^2 [t]^4 = t^{14}$

$(b) \quad f(x(0), y(0), z(0)) = [x(0)]^2 [y(0)]^2 [z(0)]^4 = [0^3]^2 [0^2]^2 [0]^4 = 0$

Example 7: Let us consider the function $f(x,y,z) = xyz + x$, then

$$f\left(xy, \frac{y}{x}, xz\right) = (xy)\left(\frac{y}{x}\right)(xz) + xy = xy^2z + xy$$

Example 8: Let us consider $g(x,y,z) = z\ Sin(xy)$, $u(x,y,z) = x^2z^3$,

$$v(x,y,z) = Pxyz,\ w(x,y,z) = \frac{xy}{z}, \text{ then}$$

$$g\big(u(x,y,z),\ v(x,y,z),\ w(x,y,z)\big) = w(x,y,z)\ Sin\big(u(x,y,z)\ v(x,y,z)\big)$$

Now by putting the values of these functions from the above equations, we get

$$g\big(u(x,y,z),\ v(x,y,z),\ w(x,y,z)\big) = \frac{xy}{z} Sin\left[\left(x^2z^3\right)\left(Pxyz\right)\right] = \frac{xy}{z} Sin\left(Px^3yz^4\right)$$

Example 9: Consider the function $g(x,y) = y\ Sin\left(x^2y\right)$ and $u(x,y) = x^2y^3,\ v(x,y) = \pi\,xy$,

$$\text{then } g\big(u(x,y),\ v(x,y)\big) = v(x,y)\ Sin\left(\left[u(x,y)\right]^2 v(x,y)\right)$$

By putting the values of these functions we get

$$g\big(u(x,y),\ v(x,y)\big) = \pi\,xy\ Sin\left(\left[x^2y^3\right]^2 \pi xy\right) = \pi xy\ Sin\left(\pi x^5y^7\right)$$

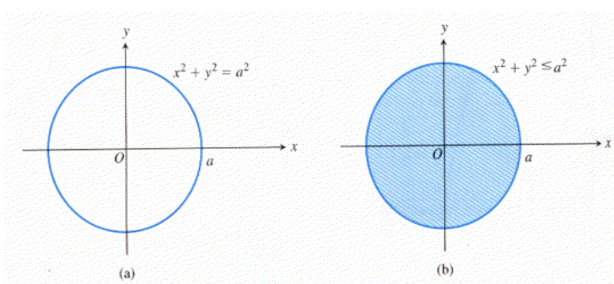
<u>Function of One Variable</u>: A function f of one real variable x is a rule that assigns a unique real number f(x) to each point x in some set D of the real line.

<u>Function of two Variables</u>: A function f in two real variables x and y, is a rule that assigns unique real number $f(x,y)$ to each point (x,y) in some set D of the xy-plane.

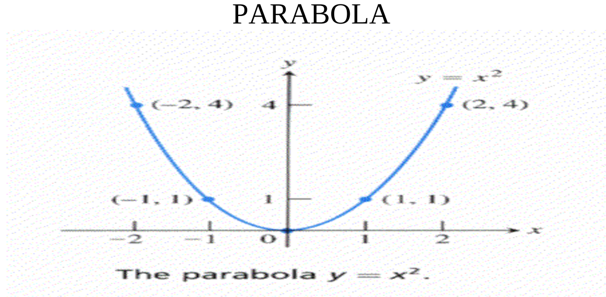
<u>Function of three variables</u>: A function f in three real variables x, y and z, is a rule that assigns a unique real number $f(x,y,z)$ to each point (x,y,z) in some set D of three dimensional space.

<u>Function of n variables</u>: A function f in n variable real variables $x_1, x_2, x_3, \ldots\ldots, x_n$, is a rule that assigns a unique real number $w = f(x_1, x_2, x_3, \ldots\ldots, x_n)$ to each point (x_1, x_2, x_3,……, x_n) I n some set D of n dimensional space.

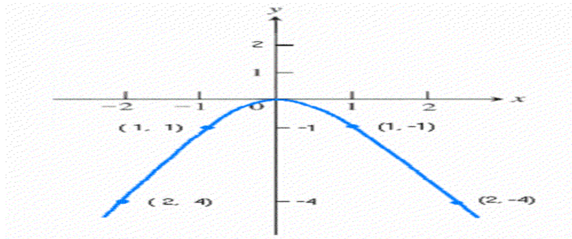
Circles and Disks:



PARABOLA



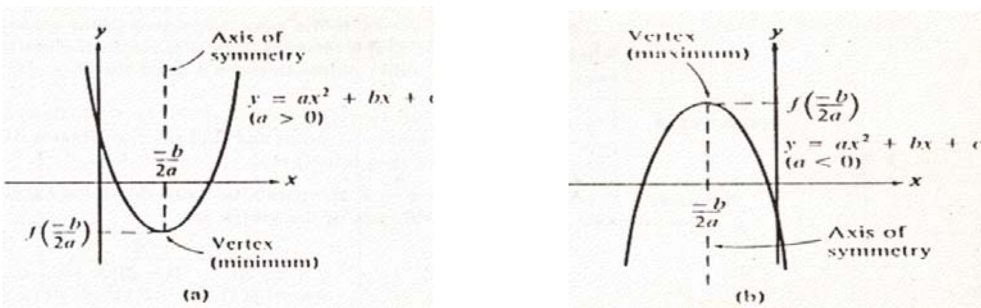
Parabola y = -x^2



General equation of the Parabola opening upward or downward is of the form

$y = f(x) = ax^2 + bx + c.$ Opening upward if $a > 0$, Opening downward if $a < 0$

The x-coordinate of the vertex is given by $x_0 = -\dfrac{b}{2a}$. So the y-coordinate of the vertex is $y_0 = f(x_0)$ The axis of symmetry is $x = x_0$.



Sketching of the graph of parabola y = ax²+bx + c

Finding vertex: x-coordinate of the vertex is given by $x_0 = -\dfrac{b}{2a}$

The y-coordinate of the vertex is $y_0 = a\,x_0^{\,2} + b\,x_0 + c$. Hence vertex is $V(x_0, y_0)$.

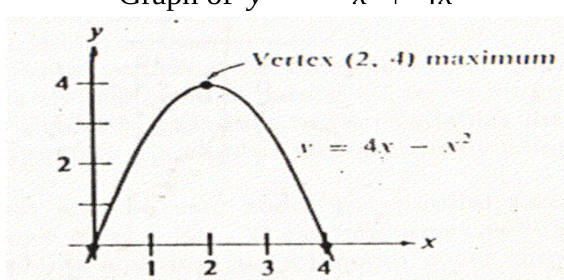
Example 10: Sketch the parabola $y = -x^2 + 4x$

Solution: Since $a = -1 < 0$ because parabola is opening downward. Vertex occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$$ Axis of symmetry is the vertical line $x = 2$.

The y-coordinate of the vertex is $y = -(2)^2 + 4(2) = 4$. Hence vertex is V(2 , 4). The zeros of the parabola (i.e. the point where the parabola meets x-axis) are the solutions to $-x^2 + 4x = 0$, so $x = 0$ and $x = 4$. Therefore, $(0,0)$ and $(4,0)$ lie on the parabola. Also $(1,3)$ and $(3,3)$ lie on the parabola.

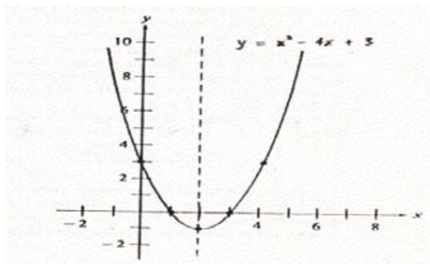
Graph of $y = -x^2 + 4x$



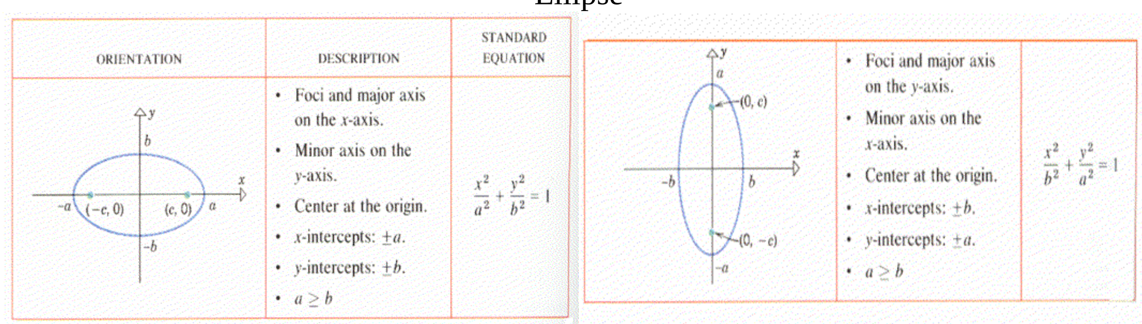
Example 11: Sketch the parabola $y = x^2 - 4x + 3$

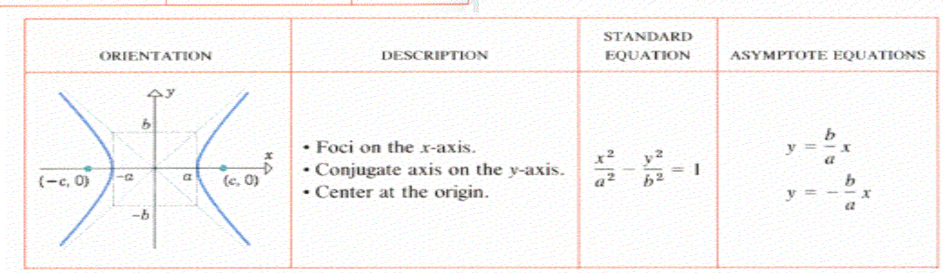
Solution: Since $a = 1 > 0$, parabola is opening upward. Vertex occurs at

$x = -\dfrac{b}{2a} = -\dfrac{(-4)}{2(1)} = 2.$ Axis of symmetry is the vertical line $x = 2.$ The y co-ordinate of the vertex is $y = (2)^2 - 4(2) + 3 = -1$. Hence vertex is $V(2\ ,\ -1)$. The zeros of the parabola (i.e. the point where the parabola meets x-axis) are the solutions to $x^2 - 4x + 3 = 0$, so $x = 1$ and $x = 3$ .Therefore $(1,0)$ and $(3,0)$ lie on the parabola. Also $(0\ ,3)$ and $(4,\ 3)$ lie on the parabola. *Graph of* $y = x^2 - 4x + 3$



Ellipse





Hyperbola

Home Assignments:

In this lecture we recall some basic geometrical concepts which are prerequisite for this course and you can find all these concepts in the chapter # 12 of your book **Calculus** By **Howard Anton**.

LECTURE No.3

ELEMENTS OF THREE DIMENSIONAL GEOMETRY

Distance formula in three dimensions

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points such that PQ is not parallel to one of the coordinate axis Then $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ Which is known as Distance fromula between the points P and Q.

Example of distance formula

Let us consider the points $A(3, 2, 4)$, $B(6, 10, -1)$ and $C(9, 4, 1)$, then

$$|AB| = \sqrt{(6-3)^2 + (10-2)^2 + (-1-4)^2} = \sqrt{98} = 7\sqrt{2}$$

$$|AC| = \sqrt{(9-3)^2 + (4-2)^2 + (1-4)^2} = \sqrt{49} = 7$$

$$|BC| = \sqrt{(9-6)^2 - (4-10)^2 - (1+1)^2} = \sqrt{49} = 7$$

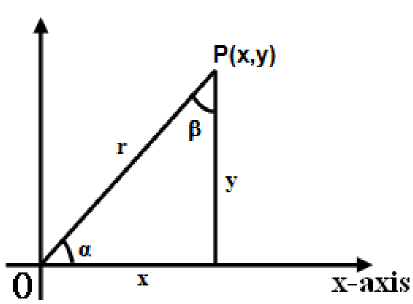
Mid point of two points

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points. If $R(x, y, z)$ is the middle point of the line segment PQ, then the coordinates of the middle point $R(x, y, z)$ are given below:

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}, \qquad z = \frac{z_1 + z_2}{2}$$

Example 2: Let us consider two points $A(3, 2, 4)$ and $B(6, 10, -1)$, then the coordinates of mid point of AB are $\left(\frac{3+6}{2}, \frac{2+10}{2}, \frac{4-1}{2}\right) = \left(\frac{9}{2}, 6, \frac{3}{2}\right)$

Given a point, finding its Direction Cosines



From triangle we can write
cos α = x/r
cos β = y/r

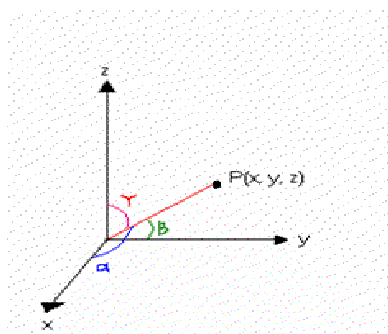
Direction Angles

The direction angles α, β, γ of a line is defined as

$\alpha =$ Angle between line and the positive x-axis

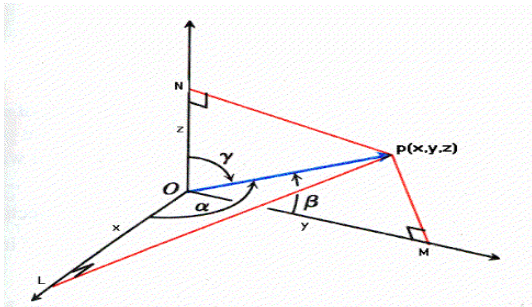
$\beta =$ Angle between line and the positive y-axis

$\gamma =$ Angle between line and the positive z-axis



By definition, each of these angles lie between 0 and π.
Direction Ratios: Cosines of direction angles are called direction cosines. Any multiple of direction cosines are called direction numbers or direction ratios of the line L.

Direction angles of a Line



The angles which a line makes with positive x, y and z-axis are known as Direction Angles. In the above figure, the blue line has direction angles as α, β and γ which are the angles which blue line makes with x, y and z-axes respectively.

Direction Cosines
Now if we take the cosine of the Direction Angles of a line, then we get the Direction cosines of that line. So the Direction Cosines of the above line are given by

$$\cos\alpha=\frac{x}{OP}=\frac{x}{\sqrt{x^2+y^2+z^2}},\quad \cos\beta=\frac{y}{OP}=\frac{y}{\sqrt{x^2+y^2+z^2}},\quad \cos\gamma=\frac{z}{OP}=\frac{z}{\sqrt{x^2+y^2+z^2}}$$

Since, by distance formula, $OP = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = \sqrt{x^2+y^2+z^2}$
Squaring and adding these equations (1), (2) and (3), we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)^2 + \left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)^2$$

$$= \frac{x^2+y^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^2} = \frac{x^2+y^2+z^2}{x^2+y^2+z^2} = 1$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Direction cosines and direction ratios of a line joining two points
For a line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$,
the direction ratios are $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$
and the directions cosines are $\frac{x_2-x_1}{|PQ|}, \frac{y_2-y_1}{|PQ|}$ and $\frac{z_2-z_1}{|PQ|}$.

Example 3: Find direction cosines and direction ratios for a line joining two points $P(1,3,2)$ and $Q(7,-2,3)$.

Solution: For a line joining two points $P(1,3,2)$ and $Q(7,-2,3)$, the direction ratios are

$$x_2 - x_1 = 7-1=6, \qquad y_2 - y_1 = -2-3 = -5, \qquad z_2 - z_1 = 3-2=1$$

and the directions cosines are

$$\frac{6}{\sqrt{6^2+(-5)^2+1^2}}, \frac{-5}{\sqrt{6^2+(-5)^2+1^2}}, \frac{1}{\sqrt{6^2+(-5)^2+1^2}}$$

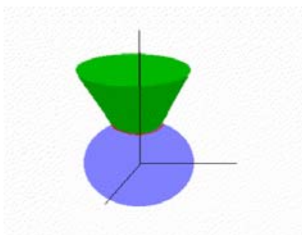
$$\frac{6}{\sqrt{62}}, \frac{-5}{\sqrt{62}}, \frac{1}{\sqrt{62}}$$

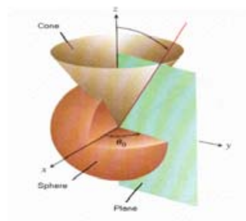
Intersection of two surfaces

•Intersection of two surfaces is a curve in three dimensional space.

•It is the reason that a curve in three dimensional space is represented by two equations representing the intersecting surfaces.

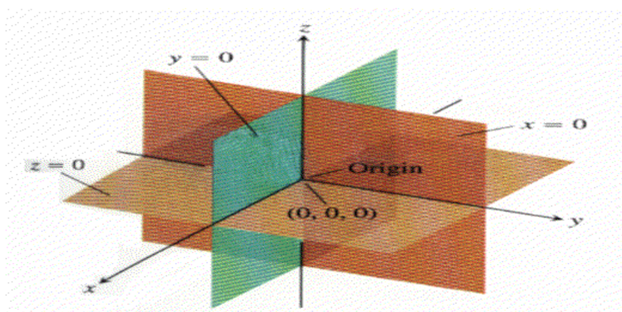
Intersection of Cone and Sphere





Intersection of Two Planes

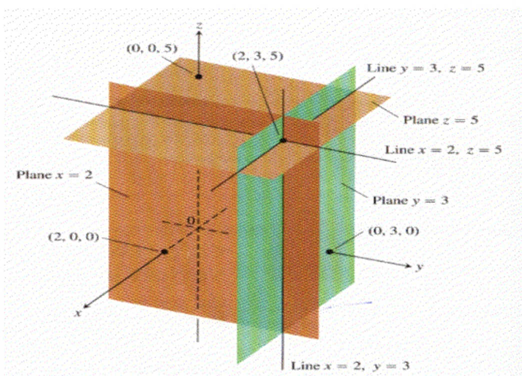
If the two planes are not parallel, then they intersect and their intersection is a straight line. Thus, two non-parallel planes represent a straight line given by two simultaneous linear equations in x, y and z and are known as non-symmetric form of equations of a straight line.



The planes $x = 0$, $y = 0$, and $z = 0$ divide space into eight octants.

REGION	DESCRIPTION	EQUATION
xy-plane	Consists of all points of the form (x, y, 0)	z = 0
xz-plane	Consists of all points of the form (x, 0, z)	y = 0
yz-plane	Consists of all points of the form (0, y, z)	x = 0
x-axis	Consists of all points of the form (x, 0, 0)	y = 0, z = 0
y-axis	Consists of all points of the form (0, y, 0)	z = 0, x = 0
z-axis	Consists of all points of the form (0, 0, z)	x = 0, y = 0

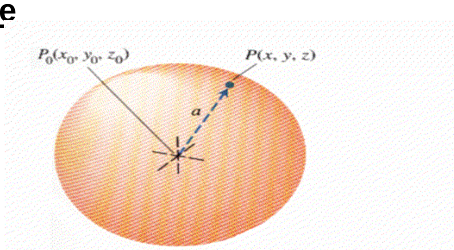
<u>Planes parallel to Coordinate Planes</u>



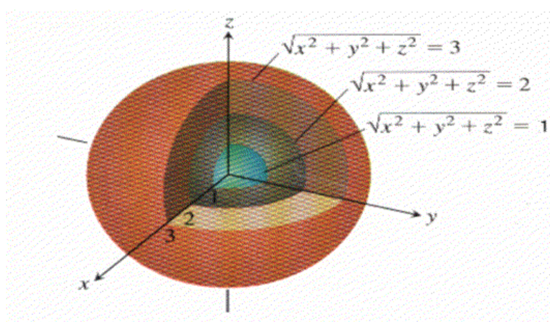
<u>General Equation of Plane</u>

Any equation of the form $ax + by + cz + d = 0$ represents a plane, where a, b, c, d are real numbers.

<u>Sphere</u>

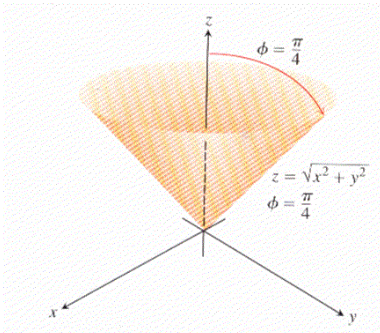


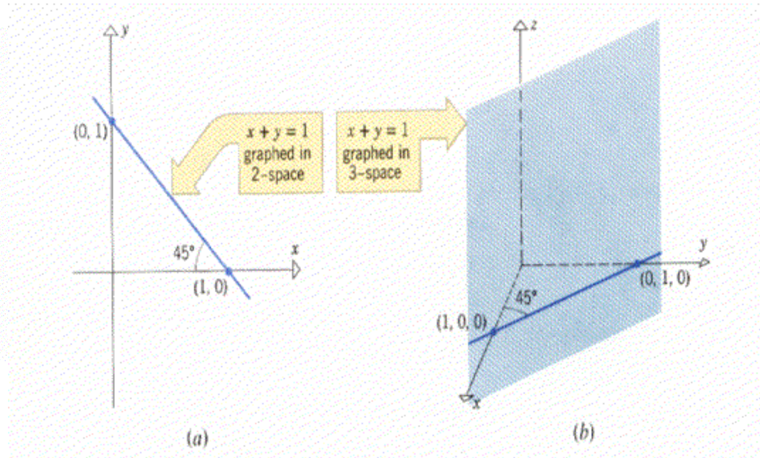
The standard equation of the sphere of radius a centered at (x_0, y_0, z_0) is $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$.

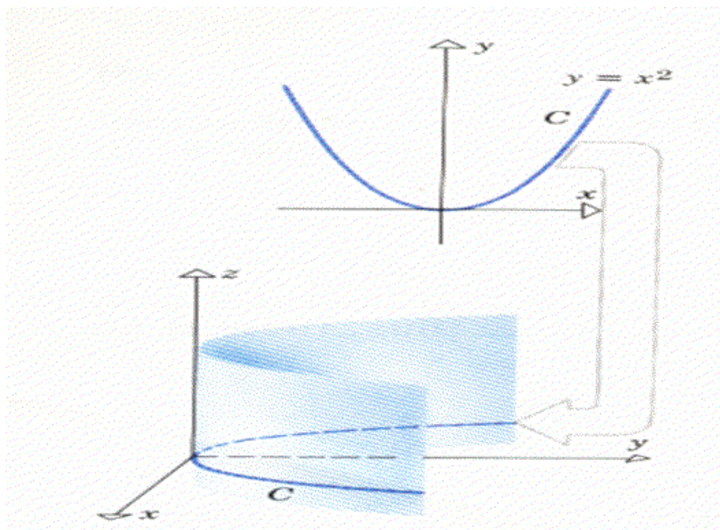


The level surfaces of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ are concentric spheres.

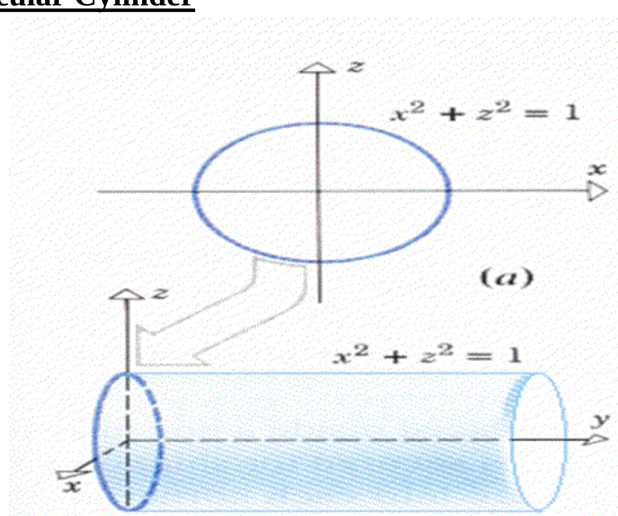
Right Circular Cone



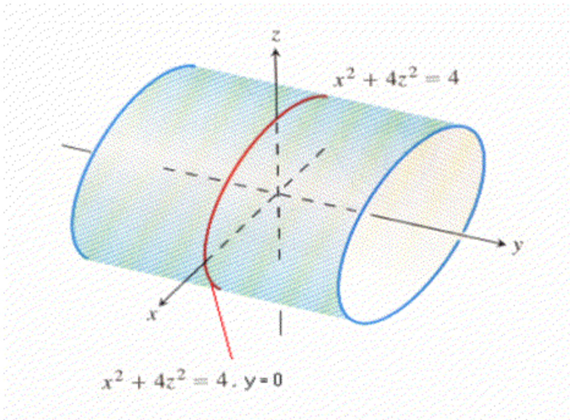


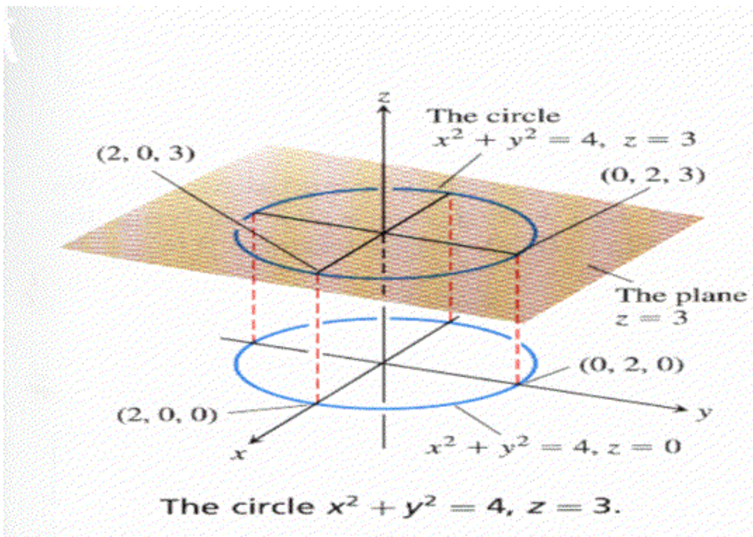


Horizontal Circular Cylinder



Horizontal Elliptic Cylinder





The circle $x^2 + y^2 = 4, z = 3$.

Overview of Lecture # 3

Chapter # 14
Three Dimensional Space
Page # 657

Book **CALCULUS by HOWARD ANTON**

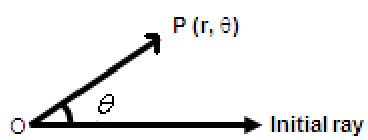
LECTURE No.4

POLAR COORDINATES

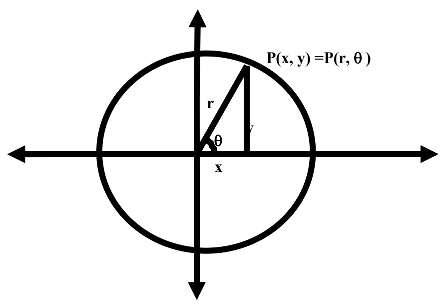
Outlines of the lecture:

- **Spherical Polar Coordinate**
- **Cylindrical Polar Coordinate**

You know that position of any point in the plane can be obtained by the two perpendicular lines known as x and y axes and together we call it as Cartesian coordinates for plane. Beside this coordinate system, we have another coordinate system which can also be used for obtaining the position of any point in the plane. It is called Polar coordinate system. In this coordinate system, we represent position of each particle in the plane by r and θ where r the distance from a fixed point known as pole is O and θ is the measure of the angle.



Conversion formula from polar to Cartesian coordinates and vice versa



Now we convert the polar coordinates $P(r,\theta)$ to Cartesian coordinates $P(x, y)$.

From above diagram and remembering the trigonometric ratios we can write

$$\frac{x}{r} = \cos\theta \quad \Rightarrow \quad x = r\cos\theta \qquad ---(1)$$

$$\frac{x}{r} = \sin\theta \quad \Rightarrow \quad y = r\sin\theta \qquad ---(2)$$

The equations (1) and (2) are used to convert the polar coordinates $P(r,\theta)$ to Cartesian coordinates $P(x, y)$. Now we convert the Cartesian coordinates $P(x, y)$ into polar coordinates $P(r,\theta)$. Squaring equations (1) and (2), and adding them, we get,

$$x^2 + y^2 = \left(r\cos\theta\right)^2 + \left(r\sin\theta\right)^2 = r^2\left(\cos^2\theta + \sin^2\theta\right)$$

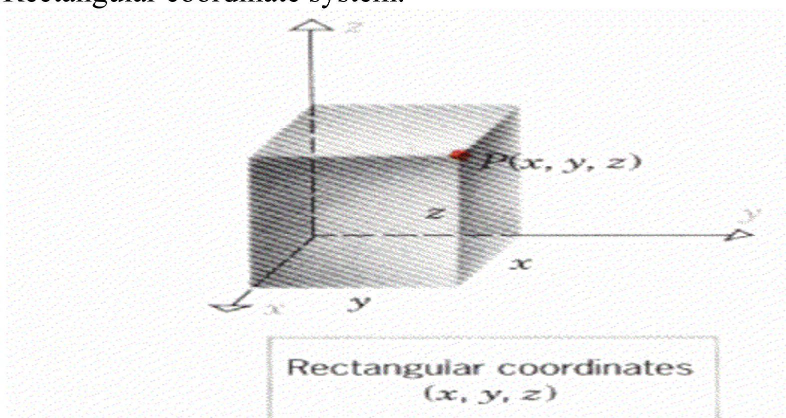
$$x^2 + y^2 = r^2 \qquad \text{or} \qquad r = \sqrt{x^2 + y^2} \qquad ----(3)$$

Dividing equation (2) by equation (1), we get

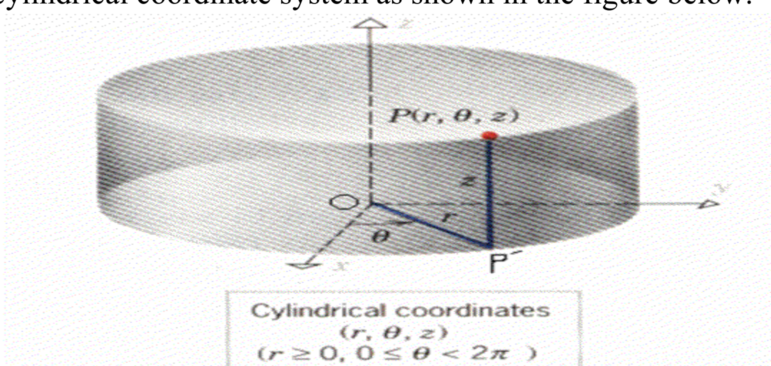
$$\frac{y}{x} = \tan\theta \qquad ----(4)$$

The equations (3) and (4) are used to convert the Cartesian coordinates $P(x, y)$ to polar coordinates $P(r,\theta)$.

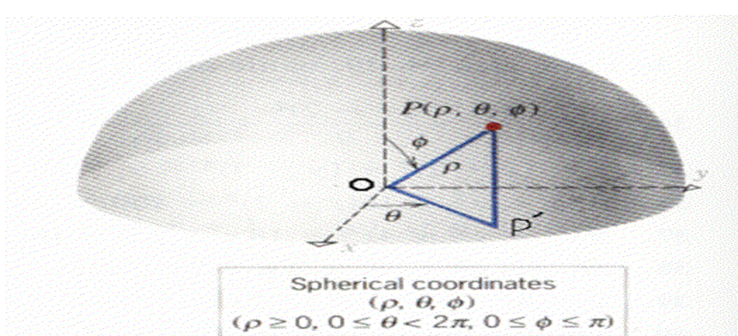
Rectangular coordinates for three dimensions: Since you know that the position of any point in the three dimensions can be obtained by the three mutually perpendicular lines known as x, y and z – axes and also shown in figure below. These coordinate axes are known as Rectangular coordinate system.



Cylindrical Coordinates: Beside the Rectangular coordinate system, we have another coordinate system which is used for getting the position of the any particle in space, known as the cylindrical coordinate system as shown in the figure below.



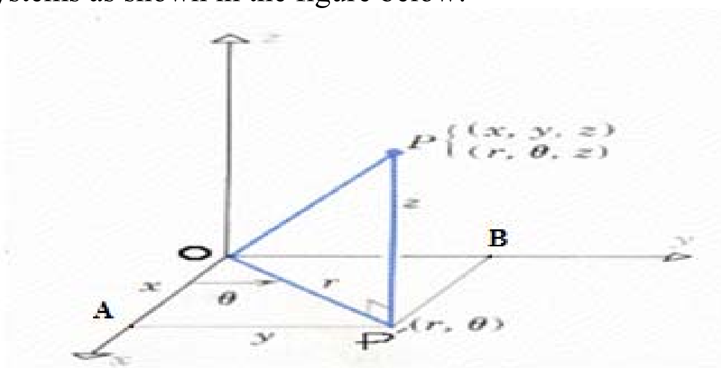
Spherical Coordinates: Beside the Rectangular and Cylindrical coordinate systems, we have another coordinate system which is used for getting the position of the any particle in space, known as the spherical coordinate system as shown in the figure below.



Conversion formulas between Rectangular and Cylindrical coordinates

Now we will find out the relation between the Rectangular coordinate system and Cylindrical coordinate system.

For this, consider any point P in the space and consider the position of this point in both the coordinate systems as shown in the figure below:



In the above figure, we have the projection $P'(r,\theta)$ of the point $P(x,y,z)$ in the xy-plane and write its position in plane polar coordinates and also represent the angle θ.
Now from that projection, we draw perpendiculars $P'A$ and $P'B$ to both of the axes and using the trigonometric ratios, find out the following relations:

$$\frac{x}{r} = \cos\theta, \qquad \frac{x}{r} = \sin\theta, \qquad z = z$$

Therefore, $\quad x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

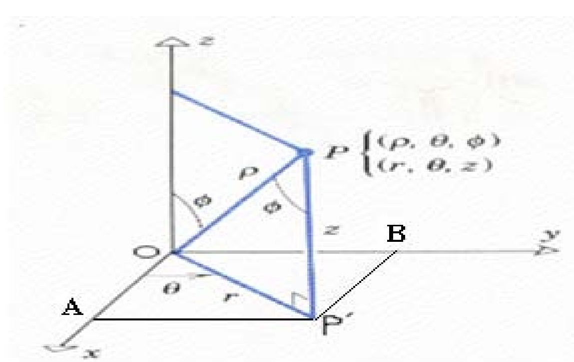
These equations convert the polar coordinates $P(r,\theta,z)$ to Cartesian coordinates $P(x,y,z)$.

$$r = \sqrt{x^2 + y^2}, \qquad \tan\theta = \frac{y}{x}, \qquad z = z$$

These equations convert the Cartesian coordinates $P(x,y,z)$ to polar coordinates $P(r,\theta,z)$.

Conversion formulas between cylindrical and spherical coordinates

Now we will find out the relation between spherical coordinate system and cylindrical coordinate system.



First we will find the relation between Planes polar to spherical. From the above figure, you can easily see that from the two right angled triangles we have the following relations: $(\rho,\theta,\phi)\to(r,\theta,z)$

In the triangle, $Cos\,\theta=\dfrac{x}{r} \Rightarrow x=r\,Cos\,\theta$

$$Sin\,\theta=\frac{y}{r} \Rightarrow y=r\,Sin\,\theta$$

In the triangle, $Sin\,\varphi=\dfrac{r}{\rho} \Rightarrow r=\rho\,Sin\,\varphi \qquad ---(a)$

$$Cos\,\varphi=\frac{z}{\rho} \Rightarrow z=\rho\,Cos\,\varphi \qquad ---(b)$$

Therefore, $r=\rho\,Sin\,\phi,\quad \theta=\theta,\quad z=\rho\,Cos\,\phi$

Now from these equations we will solve the first and second equation for ρ and ϕ. Thus we have $(r,\theta,z)\to(\rho,\theta,\phi)$

Squaring and adding equations (a) and (b), we get

$$\rho=\sqrt{r^2+z^2}$$

Divide equation (a) by equation (b), $\tan\phi=\dfrac{r}{z}$

Therefore, $\rho=\sqrt{r^2+z^2},\quad \theta=\theta,\quad \tan\phi=\dfrac{r}{z}$

Conversion formulae between Rectangular and Spherical coordinates (ρ, θ, Φ) → (x, y, z)

Since we know that the relation between Cartesian coordinates and Polar coordinates are

$$x=r\cos\theta,\quad y=r\sin\theta,\quad z=z \qquad ----(A)$$

We also know that the relation between Spherical and cylindrical coordinates are,

$$r=\rho\,Sin\,\phi,\quad \theta=\theta,\quad z=\rho\,Cos\,\phi \qquad ----(B)$$

Now putting this value of r and z from (B) in (A), we get

$$x=\rho\,Sin\,\phi\;Cos\,\theta,\quad y=\rho\,Sin\,\phi\;Sin\,\theta,\quad z=\rho\,Cos\,\phi \qquad ----(C)$$

It is the relation between spherical coordinate system and Cartesian coordinate system.

Now we will find $(x,y,z)\to(\rho,\theta,\phi)$

Squaring and adding the equations in (C),

$$\begin{aligned} x^2+y^2+z^2 &= (\rho\,Sin\,\phi\;Cos\,\theta)^2+(\rho\,Sin\,\phi\;Sin\,\theta)^2+(\rho\,Cos\,\phi)^2 \\ &=\rho^2\left[Sin^2\phi\left(Cos^2\theta+Sin^2\theta\right)+Cos^2\phi\right] \\ &=\rho^2\left[Sin^2\phi+Cos^2\phi\right] \\ &=\rho^2 \end{aligned}$$

$$\rho=\sqrt{x^2+y^2+z^2}$$

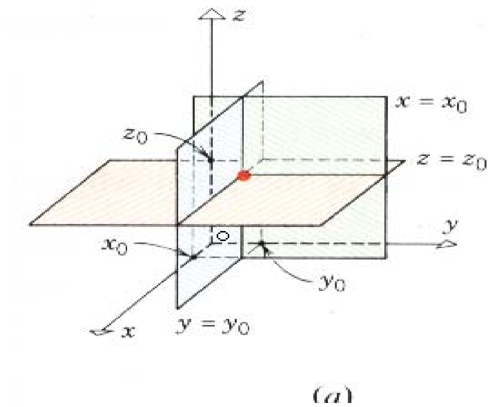
Also, $Tan\,\theta=\dfrac{y}{x}$ And $Cos\,\phi=\dfrac{z}{\rho}=\dfrac{z}{\sqrt{x^2+y^2+z^2}}$

Constant Surfaces in Rectangular Coordinates

The surfaces represented by equations of the form

$$x = x_0, \quad y = y_0, \quad z = z_0$$

where x_0, y_0, z_0 are constants, and are planes parallel to the *yz*-plane, *xz*-plane and *xy*-plane, respectively. Also shown in the figure,

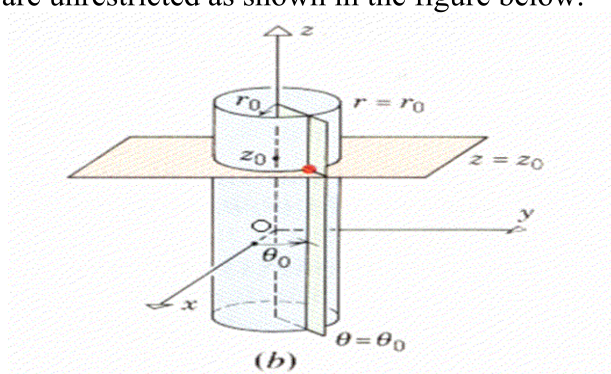


Constant Surfaces in Cylindrical Coordinates

The surface **$r = r_0$ is a right cylinder** of radius r_0 centered on the z-axis. At each point (r, θ, z), this surface on this cylinder, r has the value r_0, z is unrestricted and $0 \le \theta < 2\pi$.

The surface **$\theta = \theta_0$ is a half plane** attached along the z-axis and making angle θ_0 with the positive x-axis. At each point (r, θ, z) on the surface, θ has the value θ_0, z is unrestricted and $r \ge 0$.

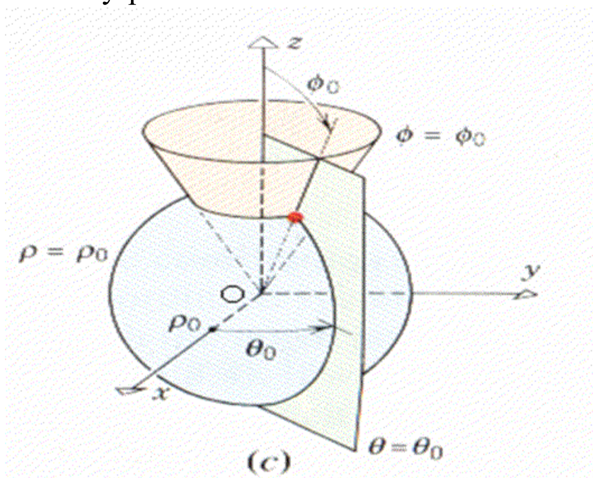
The surfaces **$z = z_0$ is a horizontal plane.** At each point (r, θ, z) this surface z has the value z_0, but r and θ are unrestricted as shown in the figure below.



Constant Surfaces in Spherical Coordinates

The surface **$\rho = \rho_0$ consists of all points whose distance ρ from origin is ρ_0.** Assuming that ρ_0 to be nonnegative, this is a sphere of radius ρ_0 centered at the origin. The surface **$\theta = \theta_0$ is a half plane** attached along the z-axis and making angle θ_0 with the positive x-axis. The surface $\Phi = \Phi_0$ consists of all points from which a line segment to the origin makes an angle of Φ_0 with the positive z-axis. Depending on whether $0 < \Phi_0 < \frac{\pi}{2}$ or

$\frac{\pi}{2} < \Phi_0 < \pi$, this will be a cone opening up or opening down. If $\Phi_0 = \frac{\pi}{2}$, then the cone is flat and the surface is the xy-plane.



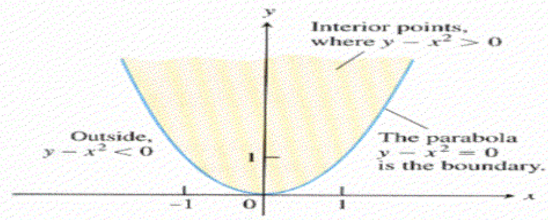
Spherical Coordinates in Navigation
Spherical coordinates are related to longitude and latitude coordinates used in navigation. Let us consider a right handed rectangular coordinate system with origin at earth's center, positive z-axis passing through the North Pole, and x-axis passing through the prime meridian. Considering earth to be a perfect sphere of radius $\rho = 4000$ miles, then each point has spherical coordinates of the form $(4000, \theta, \Phi)$ where Φ and θ determine the latitude and longitude of the point. Longitude is specified in degree east or west of prime meridian and latitudes is specified in degree north or south of the equator.

Domain of the Function
- In the above definitions, the set D is the domain of the function.
- The Set of all values which the function assigns for every element of the domain is called the Range of the function.
- When the range consists of real numbers, the functions are called the real valued function.

Note:
- o If a function is of single variable i.e. $y = f(x)$, then domain is a subset of real line and its graph is a curve.
- o If a function is of two variables i.e. $z = f(x, y)$, then domain will be from xy-plane.
- o If a function is of three variables i.e. $w = f(x, y, z)$, then domain is a subset of space.

NATURAL DOMAIN
Natural domain consists of all those points at which the formula has no divisions by zero and produces only real numbers.
Example

Consider the function $\omega = \sqrt{y - x^2}$. Then the domain of the function is $y \geq x^2$ which can be shown in the plane as parabola opening upwards. It includes the shaded area and its boundary is $y = x^2$ and the range of the function is $[0, \infty)$.



The domain of $f(x, y) = \sqrt{y - x^2}$ consists of the shaded region and its bounding parabola $y = x^2$.

Example

Consider the function $w = \dfrac{1}{xy}$

Domain of function $w = \dfrac{1}{xy}$ is the whole xy-plane, excluding x-axis and y-axis because at x-axis, $y = 0$ and at y-axis, $x = 0$.

Domain: $xy \neq 0 \quad \Rightarrow \quad x \neq 0, \quad y \neq 0$

Domain is entire xy-plane except x-axis and y-axis.

Range is $(-\infty, 0) \cup (0, \infty)$

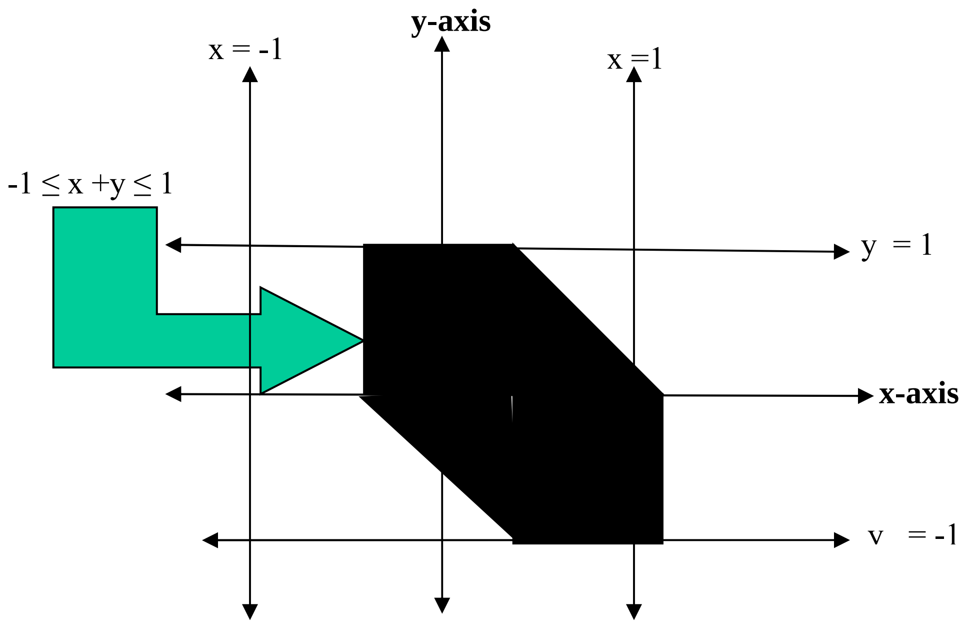
LECTURE No. 5

LIMIT OF MULTIVARIABLE FUNCTION

Example 1:

$f(x, y) = Sin^{-1}(x + y)$

Domain of f is the region in which $-1 \le x + y \le 1$



Domains and Ranges

Functions	*Domain*	*Range*
1) $\omega = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
2) $\omega = \dfrac{1}{x^2 + y^2 + z^2}$	Entire space except origin $(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
3) $\omega = xy \ln z$	Half space, $z > 0$	$(-\infty, \infty)$

Examples of domain of a function

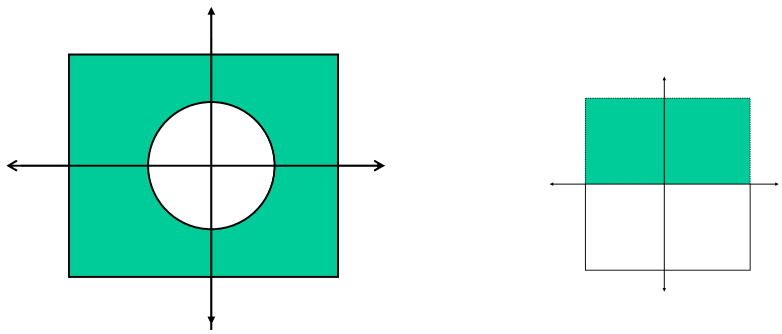
Example 2: $f(x, y) = xy\sqrt{y-1}$

Domain of f consists of the region in xy-plane where $y \ge 1$.

(Here we take $y - 1 \ge 0$ for real values.)

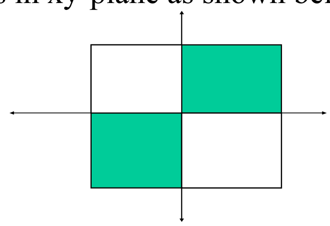
Example 3: $f(x, y) = \sqrt{x^2 + y^2 - 4}$

Domain of f consists of the region in xy-plane where $x^2 + y^2 \geq 4$. It means that the points of the domain lie outside the circle with radius 2. As shown in the figure



Example 4: $f(x, y) = \ln xy$

For the real values of logarithmic function, $xy > 0$ which is possible: When $x < 0,\ y < 0$ (3rd quadrant) and when $x > 0,\ y > 0$ (1st quadrant) Domain of f consists of region lying in first and third quadrants in xy-plane as shown below.



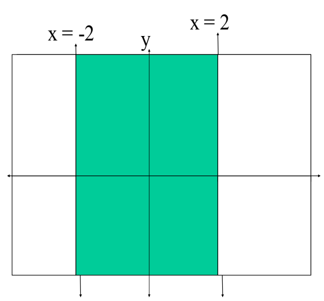
Example 5: $f(x, y, z) = e^{xyz}$

Domain of f consists of the entire region of three dimensional space.

Example 6: $f(x, y) = \dfrac{\sqrt{4 - x^2}}{y^2 + 3}$

Here we take $4 - x^2 \geq 0$ *for real values of* $f(x, y)$.

Domain of f consists of region in xy - plane where $x^2 \leq 4$ which implies that $-2 \leq x \leq 2$.



Example 7: $f(x, y, z) = \sqrt{25 - x^2 - y^2 - z^2}$

Here we take $25 - x^2 - y^2 - z^2 \geq 0$ for real values of $f(x, y)$. So, $x^2 + y^2 + z^2 \leq 5^2$

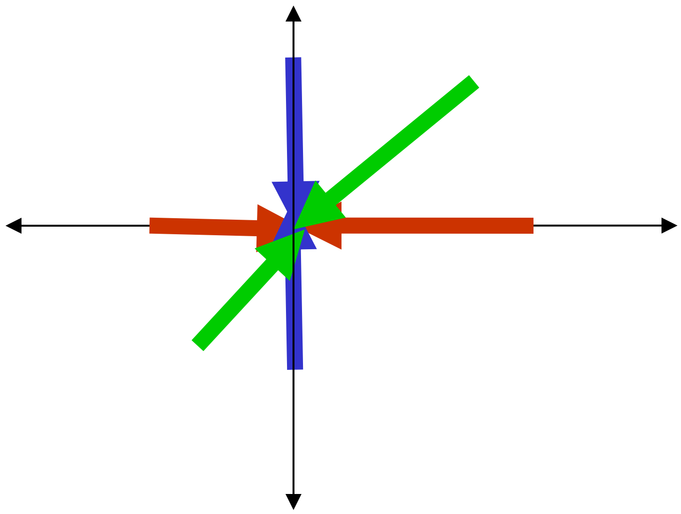
Domain of f consists of region in three dimensional space occupied by sphere centre at $(0, 0, 0)$ and radius 5.

Example 8: $f(x, y) = \dfrac{x^3 + 2x^2y - xy - 2y^2}{x+2y}$

$f(0, 0)$ is not defined but we see that limit exits.

Approaching to (0,0) through x-axis	$f(x, y)$	Approaching to (0,0) through y-axis	$f(x, y)$
$(0.5, 0)$	0.25	(0,0.1)	-0.1
$(0.25, 0)$	0.0625	(0,0.001)	-0.001
(0.1,0)	0.01	(0,0.00001)	0.00001
(-0.25,0)	0.0625	(0,-0.001)	0.001
(-0.1,0)	0.01	(0,-0.00001)	0.00001

Approaching to (0,0) through $y = x$	$f(x, y)$
(0.5,0.5)	-0.25
(0.1,0.1)	-0.09
(0.01,0.01)	-0.0099
(-0.5,-0.5)	0.75
(-0.1,-0.1)	0.11
(-0.01,-0.01)	0.0101



Example 9:

$$f(x,y) = \frac{xy}{x^2+y^2}$$

$f(0, 0)$ is not defined and we see that limit also does not exist.

Approaching to (0,0) through x-axis (y = 0)	f (x,y)	Approaching to (0,0) through y = x	f (x,y)
(0.5,0)	0	(0.5,0.5)	0.5
(0.1,0)	0	(0.25,0.25)	0.5
(0.01,0)	0	(0.1,0.1)	0.5
(0.001,0)	0	(0.05,0.05)	0.5
(0.0001,0)	0	(0.001,0.001)	0.5
(-0.5,0)	0	(-0.5,-0.5)	0.5
(-0.1,0)	0	(-0.25,-0.25)	0.5
(-0.01,0)	0	(-0.1,-0.1)	0.5
(-0.001,0)	0	(-0.05,-0.05)	0.5
(-0.0001,0)	0	(-0.001,-0.001)	0.5

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0 \text{ (along } y = 0)$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0.5 \text{ (along } y = x)$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ does not exist.}$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Let (x, y) approach $(0, 0)$ along the line $y = x$.

$$f(x, y) = \frac{xy}{x^2+y^2}$$

$$f(x, x) = \frac{xx}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2} \qquad x \neq 0$$

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \frac{1}{2} \qquad \textit{Along the line } y = x$$

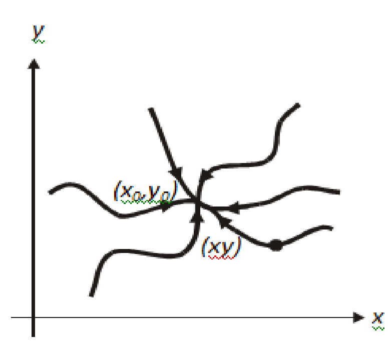
Now let (x, y) approach $(0, 0)$ along x-axis. On x-axis, $y = 0$.

$$f(x, 0) = \frac{x \times 0}{x^2+0^2} = \frac{0}{x^2} = 0 \qquad x \neq 0$$

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0 \qquad \textit{Along the line } x-\textit{axis}.$$

|Therefore $f(x, y)$ assumes two different values, as (x, y) approaches $(0,0)$ along two different paths. So $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

We can approach a point in space through infinite paths some of them are shown in the figure below:



Rule for Non-Existence of a Limit

If in $\lim\limits_{(x,y)\to(a,b)} f(x,y),$ we get two or more different values, as (x,y) approaches (a,b) along two different paths, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

The paths along which (a,b) is approached may be straight *lines or plane curves through* (a,b)

Example 10

$$\lim_{(x,y)\to(2,1)} \frac{x^3+2x^2y-x-2y^2}{x+2y} = \frac{\lim\limits_{(x,y)\to(2,1)}\left(x^3+2x^2y-x-2y^2\right)}{\lim\limits_{(x,y)\to(2,1)}\left(x+2y\right)}$$

$$=\frac{\left(2^3+2(2)^2(1)-(2)-2(1)^2\right)}{\left(2+2(1)\right)} = \frac{8+8-2-2}{4}=3$$

Example 11

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

We set $x=r\cos\theta,\ \ y=r\sin\theta$, then

$$\frac{xy}{\sqrt{x^2+y^2}} = \frac{(r\cos\theta)(r\sin\theta)}{\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}}$$

$$=\frac{\left(r^2\cos\theta\sin\theta\right)}{r\sqrt{\cos^2\theta+\sin^2\theta}}=\frac{\left(r\cos\theta\sin\theta\right)}{\sqrt{1}}$$

$$= r\cos\theta\sin\theta, \qquad r>0$$

Since $r=\sqrt{x^2+y^2}$, so $r\to 0$ as $(x,y)\to(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}=\lim_{r\to 0}\ r\cos\theta\sin\theta = 0\times\cos\theta\sin\theta=0$$

Note that $\left|\cos\theta\sin\theta\right|<1$ for all values of θ.

RULES FOR LIMIT

If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=L_1$ *and* $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y)=L_2$, then

(a) $\lim\limits_{(x,y)\to(x_0,y_0)} cf(x,y)=cL_1$ (if c is constant)

(b) $\lim\limits_{(x,y)\to(x_0,y_0)} \{f(x,y)+g(x,y)\}=L_1+L_2$

(c) $\lim\limits_{(x,y)\to(x_0,y_0)} \{f(x,y)-g(x,y)\}=L_1-L_2$

(d) $\lim\limits_{(x,y)\to(x_0,y_0)} \{f(x,y)g(x,y)\}=L_1L_2$

(e) $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)}=\dfrac{L_1}{L_2}$ (if $L_2 = 0$)

$$\lim_{(x,y)\to(x_0,y_0)} c = c \quad (c \text{ is a constant}),\quad \lim_{(x,y)\to(x_0,y_0)} x_0 = x_0\,,\quad \lim_{(x,y)\to(x_0,y_0)} y_0 = y_0$$

Similar rules are for the function of three variables.

Overview of lecture# 5

In this lecture we recall you all the limit concept which are prerequisite for this course and you can find all these concepts in the chapter # 16 (topic # 16.2)of your Calculus By Howard Anton.

LECTURE No. 6

GEOMETRY OF CONTINUOUS FUNCTIONS

Geometry of continuous functions in one variable or Informal definition of continuity of function of one variable

A function is continuous if we draw its graph by a pen such that the pen is not raised so that there is no gap in the graph of the function.

Geometry of continuous functions in two variables or Informal definition of continuity of function of two variables

The graph of a continuous function of two variables to be constructed from a thin sheet of clay that has been hollowed and pinched into peaks and valleys without creating tears or pinholes.

Continuity of functions of two variables

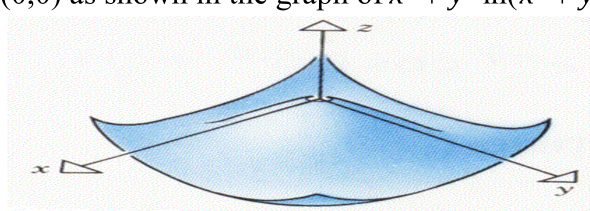
A function f of two variables is called continuous at the point (x_0, y_0) if f satisfies the following conditions:

1. $f(x_0, y_0)$ is defined.
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

The requirement that $f(x_0, y_0)$ must be defined at the point (x_0, y_0) eliminates the possibility of a hole in the surface $z = f(x_0, y_0)$ above the point (x_0, y_0).

Justification of three points involving in the definition of continuity

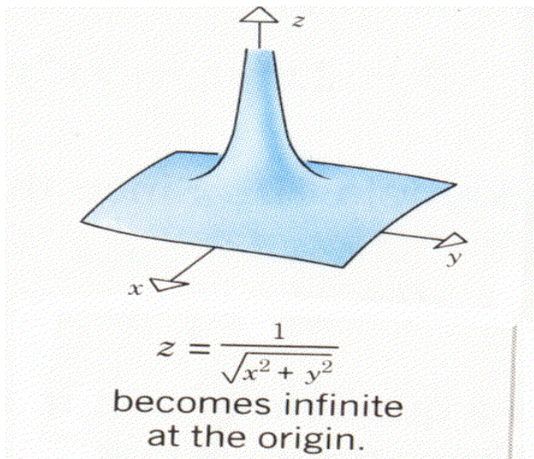
(1) Consider the function of two variables $x^2 + y^2 \ln(x^2 + y^2)$. Now as we know that the Log function is not defined at 0, it means that when $x = 0$ and $y = 0$, our function $x^2 + y^2 \ln(x^2 + y^2)$ is not defined. Consequently the surface $z = x^2 + y^2 \ln(x^2 + y^2)$ will have a hole just above the point (0,0) as shown in the graph of $x^2 + y^2 \ln(x^2 + y^2)$



$z = (x^2 + y^2)\ln(x^2 + y^2)$ has a hole at the origin.

(2) The requirement that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists ensures us that the surface $z = f(x,y)$ of the function $f(x, y)$ doesn't become infinite at (x_0, y_0) or doesn't oscillate widely.

Consider the function of two variables $\frac{1}{\sqrt{x^2+y^2}}$. Now as we know that the Natural domain of the function is whole the plane except origin. Because at origin, we have $x = 0$ and $y = 0$. In the defining formula of the function, we will have $\frac{1}{0}$ at that point which is infinity. Thus the limit of the function $\frac{1}{\sqrt{x^2+y^2}}$ does not exist at origin. Consequently the surface $z = \frac{1}{\sqrt{x^2+y^2}}$ will approach towards infinity when we approach towards origin as shown in the figure above.



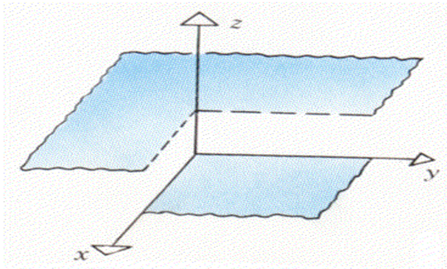
(3) The requirement that $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$ ensures us that the surface $z = f(x,y)$ of the function $f(x,y)$ doesn't have a vertical jump or step above the point (x_0, y_0).

Consider the function of two variables

$$f(x,y) = \begin{cases} 0 & if \;\; x \ge 0 and \; y \ge 0 \\ 1 & otherwise \end{cases}$$

Now as we know that the Natural domain of the function is whole the plane. But you should note that the function has one value "0" for all the points in the plane for which both *x* and *y* have nonnegative values. And value "1" for all other points in the plane. Consequently the surface

$$z = f(x,y) = \begin{cases} 0 & if \;\; x \ge 0 and \; y \ge 0 \\ 1 & otherwise \end{cases}$$ It has a jump as shown in the figure.



$z = \begin{cases} 0 & \text{if } x \geq 0, y \geq 0 \\ 1 & \text{otherwise} \end{cases}$
has a vertical jump at the origin.

Example 1: Check whether the limit at $(0,\ 0)$ exists or not for the function

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2}{x^2+y^2}$$

Solution: First we will calculate the Limit of the function along *x*-axis and we get

$$\lim_{(x,y)\to(0,0)} f(x,0) = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+0} = \lim_{(x,y)\to(0,0)} 1 = 1 \qquad \textbf{(Along } \boldsymbol{x}\textbf{-axis, } \boldsymbol{y = 0}\textbf{)}$$

Now we will find out the limit of the function along *y*-axis and we note that the limit is

$$\lim_{(x,y)\to(0,0)} f(0,y) = \lim_{(x,y)\to(0,0)} \frac{0^2}{0^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{y^2} = \lim_{(x,y)\to(0,0)} 0 = 0 \quad \textbf{(Along } \boldsymbol{y}\textbf{-axis, } \boldsymbol{x = 0}\textbf{)}$$

Now we will find out the limit of the function along the line $y = x$ and we note that

$$\lim_{(x,y)\to(0,0)} f(x,x) = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2} = \lim_{(x,y)\to(0,0)} \frac{1}{2} = \frac{1}{2} \quad \textbf{(Along } \boldsymbol{y = x}\textbf{)}$$

It means that limit of the function $f(x,y)$ at $(0,\ 0)$ doesn't exist because it has different values along different paths. Thus the function cannot be continuous at $(0,\ 0)$. And also note that the function is not defined at $(0,\ 0)$ and hence it doesn't satisfy two conditions of the continuity.

Example 2: Check the continuity of the function at $(0,\ 0)$

$$f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

Solution: First we will note that the function is defined on the point where we have to check the Continuity; that is, the function has value at $(0,\ 0)$. Next we will find out the Limit of the function at $(0,\ 0)$ and in evaluating this limit, we use the result $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ and note that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\text{Sin}(x^2+y^2)}{x^2+y^2}$$

$$=1 = f(0,0)$$

This shows that f is continuous at (0,0)

CONTINUITY OF FUNCTION OF THREE VARIABLES

A function f of three variables is called *continuous at a point* (x_0, y_0, z_0) if

1. $f(x_0,y_0,z_0)$ is defined.
2. $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)$ exists.
3. $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0, y_0, z_0)$.

EXAMPLE 3: Check the continuity of the function

$$f(x,y,z) = \frac{y+1}{x^2+y^2-1}$$

Solution: First of all, note that the given function is not defined on the cylinder $x^2+y^2-1=0$. Thus the function is not continuous on the cylinder $x^2+y^2-1=0$

However, $f(x,y,z)$ is continuous at **all other points** of its domain.

RULES FOR CONTINOUS FUNCTIONS

1) If g and h are continuous functions of one variable, then $f(x,y)=g(x)h(y)$ is a continuous function of x and y.

2) If g is a continuous function of one variable and h is a continuous function of two variables, then their composition $f(x,y)=g(h(x, y))$ is a continuous function of x and y.

3) A composition of continuous functions is continuous.

4) A sum, difference, or product of continuous functions is continuous.

5) A quotient of continuous function is continuous, expect where the denominator is zero.

EXAMPLE OF PRODUCT OF FUNCTIONS TO BE CONTINUED

In general, any function of the form $f(x,y)=Ax^m x^n$ (m and n non-negative integers) is continuous everywhere in the domain because it is the product of continuous functions Ax^m and x^n. The function of the form $f(x,y)=3x^2x^5$ is continuous every where in the domain because it is the product of continuous functions $g(x)=3x^2$ and $h(y)=y^5$.

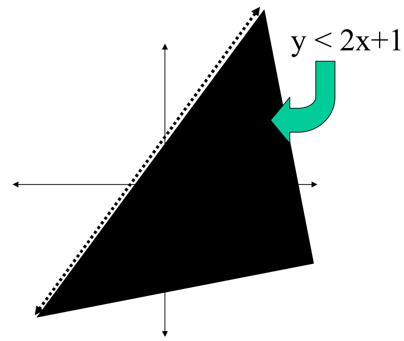
CONTINUOUS EVERYWHERE

A function f that is continuous at each point of a region R in 2-dimensional plane or 3-dimensional space is said to be *continuous on R.* A function that is continuous at every point in 2-dimensional plane or 3-dimensional space is called continuous *everywhere* or simply *continuous.*

EXAMPLES

(1) $f(x,y) = \ln(2x - y + 1)$

The function f is continuous in the whole region where $2x > y - 1,\ y < 2x+1$. And its region is shown in figure below.



(2) $f(x,y) = e^{1-xy}$

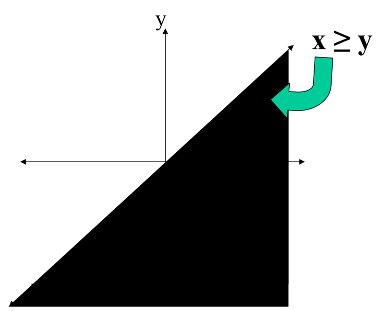
The function f is continuous in the whole region of xy-plane.

(3) $f(x,y) = \tan^{-1}(y-x)$

The function f is continuous in the whole region of xy- plane.

(4) $f(x,y) = \sqrt{y-x}$

The function is continuous where x ≥ y



Partial Derivative

Let f a function of x and y. If we hold y constant, say $y = y_0$ and view x as a variable, then $f(x, y_0)$ is a function of x alone. If this function is differentiable at $x = x_0$, then the value of this derivative is denoted by $f_x(x_0, y_0)$ and is called the Partial derivative of f with respect of x at the point (x_0, y_0).

Similarly, if we hold x constant, say $x = x_0$ and view y as a variable, then $f(x_0, y)$ is a function of y alone. If this function is differentiable at $y = y_0$, then the value of this derivative is denoted by $f_y(x_0, y_0)$ and is called the Partial derivative of f with respect of y at the point (x_0, y_0).

Example 4: Let $f(x,y)=2x^3y^2+2y+4x$ be a surface. Find the partial derivatives of f with respect to x and y at point (1, 2).

Solution: Treating y as a constant and differentiating with respect to x, we obtain

$$f_x(x,y)=6x^2y^2+4$$

Treating x as a constant and differentiating with respect to y, we obtain

$$f_y(x,y)=4x^3y+2$$

Substituting x = 1 and y = 2 in these partial-derivative formulas yields.

$$f_x(1,2)=6(1)^2(2)^2+4=28$$

$$f_y(1,2)=4(1)^3(2)+2=10$$

Example 5: Let $z=4x^2-2y+7x^4y^5$ be a surface. Find the partial derivatives of z with respect to x and y.

Solution : $z=4x^2-2y+7x^4y^5$ $\quad \dfrac{\partial z}{\partial x}=8x+28x^3y^5,$ $\quad \dfrac{\partial z}{\partial y}=-2+35x^4y^4$

Example 6: Let $z=f(x,y)=x^2\sin^2 y$ be a surface. Find the partial derivatives of z with respect to x and y.

Solution: $z=f(x,y)=x^2\sin^2 y$

Then to find the derivative of f with respect to x, we treat y as a constant.

Therefore, $\dfrac{\partial z}{\partial x}=f_x=2x\sin^2 y$

Then to find the derivative of f with respect to y, *we* treat x as a constant.

$$\frac{\partial z}{\partial y}=f_y=x^2\left(2\sin y\cos y\right)=x^2\sin 2y$$

Example 7: Let $z=\ln\left(\dfrac{x^2+y^2}{x+y}\right)$ be a surface. Find the partial derivatives of z with respect to x and y.

Solution: By using the properties of the *ln,* we can write it as

$$z=\ln(x^2+y^2)-\ln(x+y)$$

$$\frac{\partial z}{\partial x}=\frac{1}{x^2+y^2}\cdot 2x-\frac{1}{x+y}$$

$$=\frac{2x^2+2xy-x^2-y^2}{(x^2+y^2)(x+y)}$$

$$=\frac{x^2+2xy-y^2}{(x^2+y^2)(x+y)}$$

Similarly by symmetry,

$$\frac{\partial z}{\partial y}=\frac{y^2+2xy-x^2}{(x^2+y^2)(x+y)}$$

Example 8: Find the partial derivatives of $z = x^4 \sin\left(xy^3\right)$ with respect to x and y.

Solution:

$$z = x^4 \sin(xy^3)$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[x^4 \sin(xy^3)\right]$$

$$= x^4 \frac{\partial}{\partial x}\left[\sin(xy^3)\right] + \sin(xy^3)\frac{\partial}{\partial x}(x^4)$$

$$= x^4 \cos(xy^3)\ y^3 + \sin(xy^3)\ 4x^3$$

$$\frac{\partial z}{\partial x} = x^4 y^3 \cos(xy^3) + 4x^3 \sin(xy^3)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[x^4 \sin(xy^3)\right]$$

$$= x^4 \frac{\partial}{\partial y}\left[\sin(xy^3)\right] + \sin(xy^3)\frac{\partial}{\partial y}(x^4)$$

$$= x^4 \cos(xy^3)\ \ 3xy^2 + \sin(xy^3).0$$

$$= 3x^5 y^2 \cos(xy^3)$$

Example 9: Find the partial derivatives of $z = \cos(x^5\ y^4)$ with respect to x and y.

Solution:

$$z = \cos(x^5 y^4)$$

$$\frac{\partial z}{\partial x} = -\sin(x^5 y^4)\frac{\partial}{\partial x}(x^5 y^4)$$

$$= -5x^4 y^4 \sin(x^5 y^4)$$

$$\frac{\partial z}{\partial y} = -\sin(x^5 y^4)\frac{\partial}{\partial y}(x^5 y^4)$$

$$= -4x^5 y^3 \sin(x^5 y^4)$$

Example 10: Find the partial derivatives of $w = x^2 + 3y^2 + 4z^2 - x\,y\,z$ with respect to x, y and z.

Solution:

$$\mathrm{w = x^2 + 3y^2 + 4z^2 - xyz}$$

$$\frac{\partial \mathrm{w}}{\partial \mathrm{x}} = 2\mathrm{x} - \mathrm{yz}$$

$$\frac{\partial \mathrm{w}}{\mathrm{dy}} = 6\mathrm{y} - \mathrm{xz}$$

$$\frac{\mathrm{dw}}{\mathrm{dz}} = 8\mathrm{z} - \mathrm{xy}$$

LECTURE No.7

GEOMETRIC MEANING OF PARTIAL DERIVATIVE

Geometric meaning of partial derivative

$$z = f(x, y)$$

Partial derivative of f with respect to x is denoted by $\frac{\partial z}{\partial x}$ *or* f_x *or* $\frac{\partial f}{\partial x}$.

Partial derivative of f with respect to y is denoted by $\frac{\partial z}{\partial y}$ *or* f_y *or* $\frac{\partial f}{\partial y}$.

Partial Derivatives

Let $z = f(x, y)$ be a function of two variables x and y defined on a certain domain D.
For a given change Δx in x, keeping y as constant, the change Δz in z, is given by

$$\Delta z = f(x+\Delta x, y) - f(x, y)$$

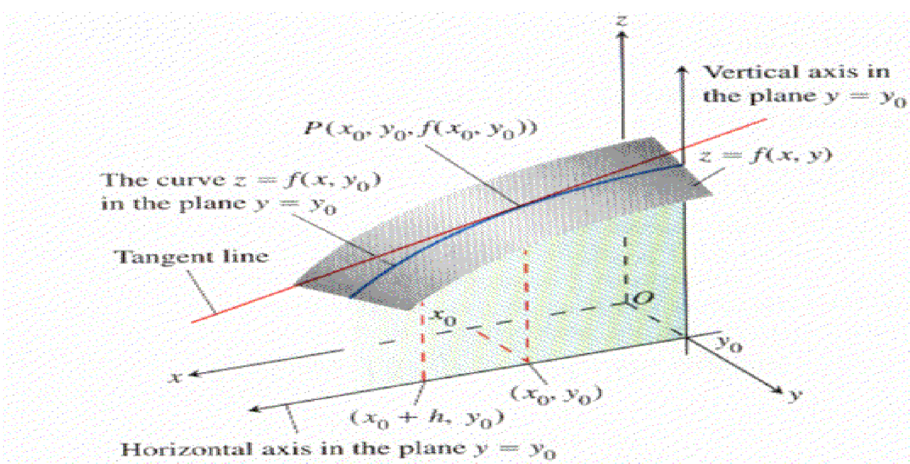
If the ratio $\frac{\Delta z}{\Delta x} = \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$ approaches to a finite limit as $\Delta x \to 0$, then this limit is called Partial derivative of f with respect to x.
Similarly for a given change Δy in y, keeping x as constant, the change Δz in z, is given by

$$\Delta z = f(x, y+\Delta y) - f(x, y)$$

If the ratio $\frac{\Delta z}{\Delta y} = \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$ approaches to a finite limit as $\Delta y \to 0$, then this limit is called Partial derivative of f with respect to y.

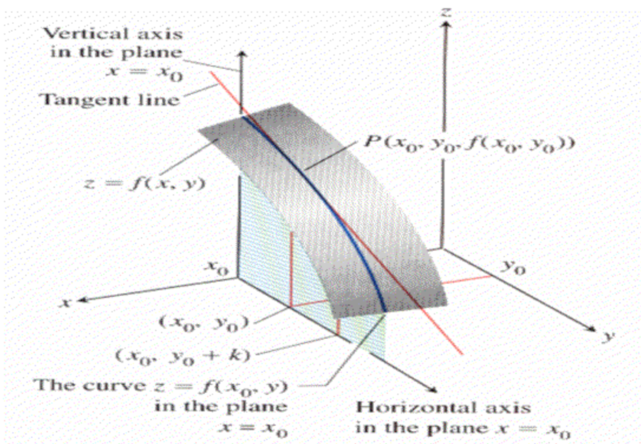
Geometric Meaning of Partial Derivatives

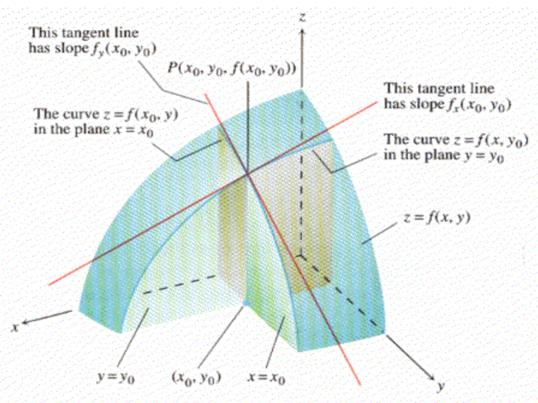
Suppose $z = f(x, y)$ is a function of two variables x and y. The graph of f is a surface. Let P be a point on the graph with the coordinates $(x_0, y_0, f(x_0, y_0))$.



If a point starting from P, changes its position on the surface such that y is constant, then the locus of this point is the curve of intersection of $z = f(x, y)$ and y = constant. On this curve, $\frac{\partial z}{\partial x}$ is a derivative of $z = f(x, y)$ with respect to x with y constant.

Thus, $\dfrac{\partial z}{\partial x}$ = slope of the tangent to this curve at P. Similarly, $\dfrac{\partial z}{\partial y}$ is the gradient of the tangent at P to this curve of intersection of z = *f(x, y)* and *x* = constant. As shown in the figure below (left). Also together these tangent lines are shown in figure below (right).





<u>Partial Derivatives of Higher Orders</u>

The partial derivatives f_x and f_y of a function *f* of two variables *x* and *y*, being functions of *x* and *y*, may possess derivatives. In such cases, the second order partial derivatives are defined as below:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}(f_x)=(f_x)_x= f_{xx}=f_{x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}(f_x)=(f_x)_y= f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}(f_y)=(f_y)_x= f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}(f_y)=(f_y)_y= f_{yy}=f_{y^2}$$

Thus there are four second order partial derivatives for a function *z* = *f(x, y).* The partial derivatives f_{xy} and f_{yx} are called **Mixed Second partials** and are not equal in general. Partial derivatives of order more than two can be defined in a similar manner.

Example 1: Find $\dfrac{\partial^2 z}{\partial x\partial y}$ and $\dfrac{\partial^2 z}{\partial y\partial x}$ for $z=arc\sin\left(\dfrac{x}{y}\right)$

Solution : $z=arc\sin\left(\dfrac{x}{y}\right)$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(arc\sin\left(\frac{x}{y}\right)\right)=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial x}\left(\frac{x}{y}\right)=\frac{y}{\sqrt{y^2-x^2}}\left(\frac{1}{y}\right)=\frac{1}{\sqrt{y^2-x^2}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(arc\sin\left(\frac{x}{y}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{y}{\sqrt{y^2-x^2}}\left(\frac{-x}{y^2}\right) = \frac{-x}{y\sqrt{y^2-x^2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{-1}{2}\left(y^2-x^2\right)^{-\frac{3}{2}}\frac{\partial}{\partial y}\left(y^2-x^2\right) = \frac{-1}{2\left(y^2-x^2\right)^{\frac{3}{2}}} \times 2y = \frac{-y}{\left(y^2-x^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{-1}{y\sqrt{y^2-x^2}} - \frac{x}{y}\left[-\frac{1}{2}\frac{-2x}{\left(y^2-x^2\right)^{\frac{3}{2}}}\right] = \frac{-y^2+x^2-x^2}{y\left(y^2-x^2\right)^{\frac{3}{2}}} = \frac{-y}{\left(y^2-x^2\right)^{\frac{3}{2}}}$$

Here, you can see that $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

Example 2: Find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ for $f(x,y) = x\cos y + y e^x$.

Solution: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x\cos y + y e^x\right) = \cos y + y e^x$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x\cos y + y e^x\right) = -x\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\cos y + y e^x\right) = 0 + y e^x = y e^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\cos y + y e^x\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-x\sin y + e^x\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-x\sin y + e^x\right) = -x\cos y$$

Laplace's Equation

For a function $w = f(x,y,z)$, the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is called Laplace's equation.

Example 3: Show that the function $f(x,y) = e^x \sin y + e^y \cos x$ satisfies the Laplace's equation.

Solution: $f(x,y) = e^x \sin y + e^y \cos x$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^x \sin y + e^y \cos x\right) = e^x \sin y - e^y \sin x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^x \sin y + e^y \cos x\right) = e^x \cos y + e^y \cos x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^x \sin y - e^y \sin x\right) = e^x \sin y - e^y \cos x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(e^x \cos y + e^y \cos x\right) = -e^x \sin y + e^y \cos x$$

Adding both partial second order derivatives, we have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(e^x \sin y - e^y \cos x\right) + \left(-e^x \sin y + e^y \cos x\right) = 0$$

Euler's Theorem

The Mixed Derivative Theorem

If *f(x, y)* and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point *(a, b)* and are all continuous at *(a, b)*, then

$$f_{xy}(a,\ b) = f_{yx}\ (a,\ b)$$

Advantage of Euler's theorem

$$w = xy + \frac{e^y}{y^2+1}$$

The symbol $\frac{\partial^2 w}{\partial x \partial y}$ tells us to differentiate first with respect to *y* and then with respect to *x*.

However, if we postpone the differentiation with respect to *y* and differentiate first with respect to *x*, we get the answer more quickly.

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left(xy + \frac{e^y}{y^2+1}\right) = y + 0 = y$$

and $$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial y}(y) = 1$$

Overview of lecture# 7

Chapter # 16 Partial derivatives

Page # 790 Article # 16.3

LECTURE No. 8

MORE ABOUT EULER THEOREM, CHAIN RULE

In general, the order of differentiation in an nth order partial derivative can be changed without affecting the final result whenever the function and all of its partial derivatives of order less than n are continuous.

For example, if f and its partial derivatives of the first, second and third orders are continuous on an open set, then at each point of the set,

$$f_{xyy} = f_{yxy} = f_{yyx}$$

or in another notation, $\dfrac{\partial^3 f}{\partial y^2 \partial x} = \dfrac{\partial^3 f}{\partial y \partial x \partial y} = \dfrac{\partial^3 f}{\partial x \partial y^2}$

Order of Differentiation

For a function $f(x, y) = y^2 x^4 e^x + 2$

If we are interested to find $\dfrac{\partial^5 f}{\partial y^3 \partial x^2}$, that is, differentiating in the order firstly w.r.t. x and then w.r.t. y , then the calculation will involve many steps making the job difficult. But if we differentiate this function with respect to y first, and then with respect to x secondly then the value of this fifth order derivative can be calculated in a few steps.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y^2 x^4 e^x + 2\right) = x^4 e^x \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial y} 2 = x^4 e^x (2y) + 0 = 2yx^4 e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(2yx^4 e^x\right) = 2x^4 e^x \frac{\partial}{\partial y}(y) = 2x^4 e^x (1) = 2x^4 e^x$$

$$\frac{\partial^3 f}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial y}\left(2x^4 e^x\right) = 0, \qquad \frac{\partial^4 f}{\partial x \partial y^3} = \frac{\partial}{\partial x}\left(\frac{\partial^3 f}{\partial y^3}\right) = \frac{\partial}{\partial x}(0) = 0$$

$$\frac{\partial^5 f}{\partial x^2 \partial y^3} = \frac{\partial}{\partial x}\left(\frac{\partial^4 f}{\partial x \partial y^3}\right) = \frac{\partial}{\partial x}(0) = 0$$

EXAMPLE 1: Let $f(x, y) = \dfrac{x+y}{x-y}$. Find f_x and f_y.

Solution: $f_x = \dfrac{\partial}{\partial x} f = \dfrac{\partial}{\partial x}\left(\dfrac{x+y}{x-y}\right) = \dfrac{(x-y)\dfrac{\partial}{\partial x}(x+y) - (x+y)\dfrac{\partial}{\partial x}(x-y)}{(x-y)^2} = \dfrac{-2y}{(x-y)^2}$

$$f_y = \frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\left(\frac{x+y}{x-y}\right) = \frac{(x-y)\dfrac{\partial}{\partial y}(x+y) - (x+y)\dfrac{\partial}{\partial y}(x-y)}{(x-y)^2} = \frac{2x}{(x-y)^2}$$

EXAMPLE 2: If $f(x,y)=x^3 e^{-y}+y^3\sec\sqrt{x}$ **,** then find the partial derivatives of *f(x, y)* with respect to *x* and *y*.

Solution:

$$f(x,y)=x^3 e^{-y}+y^3\sec\sqrt{x}$$

$$f_x=\frac{\partial}{\partial x}f=e^{-y}\left(3x^2\right)+y^3\sec\sqrt{x}\ \tan\sqrt{x}\times\left(\frac{\partial}{\partial x}\sqrt{x}\right)$$

$$=3x^2e^{-y}+y^3\sec\sqrt{x}\ \tan\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)$$

$$f_y=\frac{\partial}{\partial y}f=x^3\left(-e^{-y}\right)+\sec\sqrt{x}\ \times\left(3y^2\right)=-x^3e^{-y}+3y^2\sec\sqrt{x}$$

EXAMPLE 3: If $f(x,y)=x^2ye^{xy}$ **,** then find the partial derivatives of *f(x, y)* with respect to *x* and *y* at *(1, 1).*

Solution: $f(x,y)=x^2ye^{xy}$

$$f_x(x,y)=\frac{\partial}{\partial x}f=y\left[\frac{\partial}{\partial x}\left(x^2\right)e^{xy}+x^2\frac{\partial}{\partial x}\left(e^{xy}\right)\right]=y\left[2x\ e^{xy}+x^2e^{xy}\frac{\partial}{\partial x}(xy)\right]$$

$$f_x(x,y)=xye^{xy}\left[2+xy\right]$$

$$f_x(1,1)=(1)(1)e^{(1)(1)}\left[2+(1)(1)\right]=3e$$

$$f_y(x,y)=\frac{\partial}{\partial y}f(x,y)=x^2\left[\frac{\partial}{\partial y}(y)\ e^{xy}+y\frac{\partial}{\partial y}\left(e^{xy}\right)\right]$$

$$f_y(x,y)==x^2\left[1\times e^{xy}+ye^{xy}\frac{\partial}{\partial y}(xy)\right]=x^2e^{xy}\left[1+xy\right]$$

$$f_y(1,1)=(1)^2e^{(1)(1)}\left[1+(1)(1)\right]=2e$$

Example 4: If $f(x,y)=x^2Cos(xy)$ **,** then find the partial derivatives of *f(x, y)* with respect to *x* and *y* at $(\frac{1}{2},\ \pi)$.

Solution: $f(x,y)=x^2Cos(xy)$

$$f_x(x,y)=\frac{\partial}{\partial x}\left(x^2\right)\times Cos(xy)+x^2\frac{\partial}{\partial x}Cos(xy)=2xCos(xy)-x^2\ y\,Sin(xy)$$

$$f_x(\frac{1}{2},\pi)=2\left(\frac{1}{2}\right)Cos\left(\frac{1}{2}\times\pi\right)-\left(\frac{1}{2}\right)^2\pi\,Sin\left(\frac{1}{2}\times\pi\right)=0-\left(\frac{1}{4}\right)\pi\times1=-\frac{\pi}{4}$$

Now $f_y(x,y)=\frac{\partial}{\partial y}\left(x^2Cos(xy)\right)=x^2\frac{\partial}{\partial x}Cos(xy)=x^2\left(-Sin(xy)\right)\frac{\partial}{\partial y}(xy)=-x^3\,Sin(xy)$

$$f_y(\frac{1}{2},\pi)=-\left(\frac{1}{2}\right)^3Sin\left(\frac{1}{2}\times\pi\right)=-\left(\frac{1}{8}\right)Sin\left(\frac{\pi}{2}\right)=-\frac{1}{8}$$

EXAMPLE 5: Let $w = (4x-3y+2z)^5$. Find $\dfrac{\partial^4 w}{\partial z^2 \partial y \partial x}$.

Solution: $w = (4x-3y+2z)^5$

$$\frac{\partial w}{\partial x} = 5(4x-3y+2z)^4 \frac{\partial}{\partial x}(4x-3y+2z) = 20(4x-3y+2z)^4$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial y}\left(20(4x-3y+2z)^4\right)$$

$$= 20 \times 4(4x-3y+2z)^3 \frac{\partial}{\partial y}(4x-3y+2z) = -240(4x-3y+2z)^3$$

$$\frac{\partial^3 w}{\partial z \partial y \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial^2 w}{\partial y \partial x}\right) = \frac{\partial}{\partial z}\left(-240(4x-3y+2z)^3\right)$$

$$= -240 \times 3(4x-3y+2z)^2 \frac{\partial}{\partial z}(4x-3y+2z) = -1440(4x-3y+2z)^2$$

$$\frac{\partial^4 w}{\partial z^2 \partial y \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial^3 w}{\partial z \partial y \partial x}\right) = \frac{\partial}{\partial z}\left(-1440(4x-3y+2z)^2\right)$$

$$= -1440 \times 2(4x-3y+2z)\frac{\partial}{\partial z}(4x-3y+2z) = -5760(4x-3y+2z)$$

Chain Rule

I - Chain Rule in function of One Variable

The function $f(x)$ depends on one variable x, and x depends on single variable t.

Given that $w = f(x)$ and $x = g(t)$, we find $\dfrac{dw}{dt}$ as follows:

From $w = f(x)$, we get $\dfrac{dw}{dx}$, From $x = g(t)$, we get $\dfrac{dx}{dt}$

Then $$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$

Example 6: Let $w = x+4$, $x = Sin\,t$. Find $\dfrac{dw}{dt}$, using the chain rule.

Solution: $w = x+4$, $x = Sin\,t$

$$\frac{dw}{dx} = \frac{d}{dx}(x+4) = 1+0 = 1, \qquad \frac{dx}{dt} = \frac{d}{dx}(Sin\,t) = \cos t$$

By Chain Rule, $$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt} = (1)(\cos t) = \cos t$$

Chain Rule in function of one variable

y is a function of u, u is a function of v,

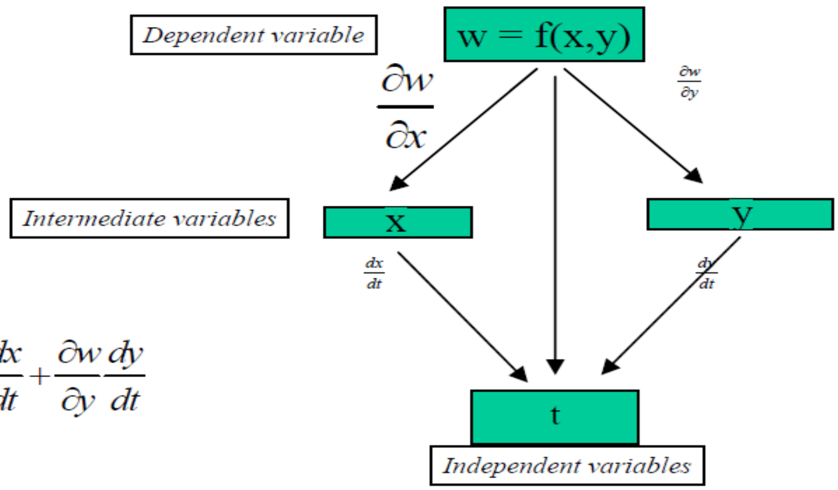
v is a function of w, w is a function of z,

z is a function of x. Ultimately, y is a function of x.So we can talk about $\frac{dy}{dx}$.

By the Chain Rule, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dw}\frac{dw}{dz}\frac{dz}{dx}$

II When the function *f* is a function of two variable *x* and *y*. And *x* and *y* are functions of one variable *t* .

$$w = f(x,y), \; x = g(t), y = f(t)$$



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

<u>**EXAMPLE BY SUBSTITUTION**</u>

Let $w = x\,y$, $x = \cos t$, $y = \sin t$. Find $\frac{dw}{dt}$ by Substitution method.

Solution : By subtitution, $w = x\,y$, $x = \cos t$, $y = \sin t$

$$w = \cos t \,\sin t = \frac{1}{2} \times 2\cos t\,\sin t = \frac{1}{2}\sin 2t$$

$$\frac{dw}{dt} = \frac{1}{2}(\cos 2t) \times 2 \;=\; \cos 2t$$

EXAMPLE 7 : Let $w = xy$, $x = \cos t$, and $y = \sin t$. Find $\frac{dw}{dt}$ by chain rule.

Solution : Given $w = xy$, $x = \cos t$, and $y = \sin t$

$$\frac{\partial w}{\partial x} = \frac{\partial (xy)}{\partial x} = y, \qquad \frac{\partial w}{\partial y} = \frac{\partial (xy)}{\partial y} = x, \qquad \frac{dx}{dt} = \frac{d\cos t}{dt} = -\sin t, \qquad \frac{dy}{dt} = \frac{d\sin t}{dt} = \cos t$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = (y)(-\sin t) + (x)(\cos t)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) = -\sin^2 t + \cos^2 t = \cos 2t$$

EXAMPLE 8: Let $z = 3x^2 y^3, \quad x = t^4, \quad y = t^2$. Find $\dfrac{dz}{dt}$.

Solution : Given $z = 3x^2 y^3, \quad x = t^4, \quad y = t^2$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(3x^2 y^3\right) = 3\left(2x\right) y^3 = 6xy^3, \qquad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(3x^2 y^3\right) = 3x^2\left(3y^2\right) = 9x^2 y^2$$

$$\frac{dx}{dt} = 4t^3, \qquad \frac{dy}{dt} = 2t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \left(6xy^3\right)\left(4t^3\right) + \left(9x^2y^2\right)\left(2t\right)$$

$$= \left(6\left(t^4\right)\left(t^2\right)^3\right)\left(4t^3\right) + \left(9\left(t^4\right)^2\left(t^2\right)^2\right)\left(2t\right) = 24t^{13} + 18t^{13} = 42t^{13}$$

EXAMPLE 9: Let $z = \sqrt{1 + x - 2xy^4}, \quad x = \ln t, \quad y = t$. Find $\dfrac{dz}{dt}$ by the Chain Rule.

Solution: Given $z = \sqrt{1 + x - 2xy^4}, \quad x = \ln t, \quad y = t$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\left(1 + x - 2xy^4\right)^{-\frac{1}{2}} \frac{\partial}{\partial x}\left(1 + x - 2xy^4\right) = \frac{1 - 2y^4}{2\sqrt{1 + x - 2xy^4}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}\left(1 + x - 2xy^4\right)^{-\frac{1}{2}} \frac{\partial}{\partial y}\left(1 + x - 2xy^4\right) = \frac{4xy^3}{\sqrt{1 + x - 2xy^4}}$$

$$\frac{dx}{dt} = \frac{d \ln t}{dt} = \frac{1}{t}, \qquad \frac{dy}{dt} = \frac{d}{dt}(t) = 1$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{1 - 2y^4}{2\sqrt{1 + x - 2xy^4}} \times \frac{1}{t} + \frac{4xy^3}{\sqrt{1 + x - 2xy^4}} \times 1$$

$$= \frac{1}{\sqrt{1 + x - 2xy^4}}\left(\frac{1 - 2y^4}{2t} + 4xy^3\right) = \frac{1}{\sqrt{1 + \ln t - 2t^4\left(\ln t\right)}}\left(\frac{1}{2t} - t^3 + 4t^3\left(\ln t\right)\right)$$

EXAMPLE 10: Let $z = \ln(2x^2 + y), \; x = \sqrt{t}, \; y = t^{\frac{2}{3}}$. Find $\dfrac{dz}{dt}$, using Chain Rule.

Solution : $\quad z = \ln(2x^2 + y) \qquad\qquad z = F(x, y)$

$$x = \sqrt{t}, \; y = t^{\frac{2}{3}} \qquad\qquad x = g(t), \; y = f(t)$$

$$\frac{\partial z}{\partial x} = \frac{\partial\left(\ln(2x^2 + y)\right)}{\partial x} = \frac{1}{2x^2 + y}\frac{\partial\left(2x^2 + y\right)}{\partial x} = \frac{1}{2x^2 + y}\left(4x\right) = \frac{4x}{2x^2 + y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial\left(\ln(2x^2 + y)\right)}{\partial y} = \frac{1}{2x^2 + y}\frac{\partial\left(2x^2 + y\right)}{\partial y} = \frac{1}{2x^2 + y}\left(0 + 1\right) = \frac{1}{2x^2 + y}$$

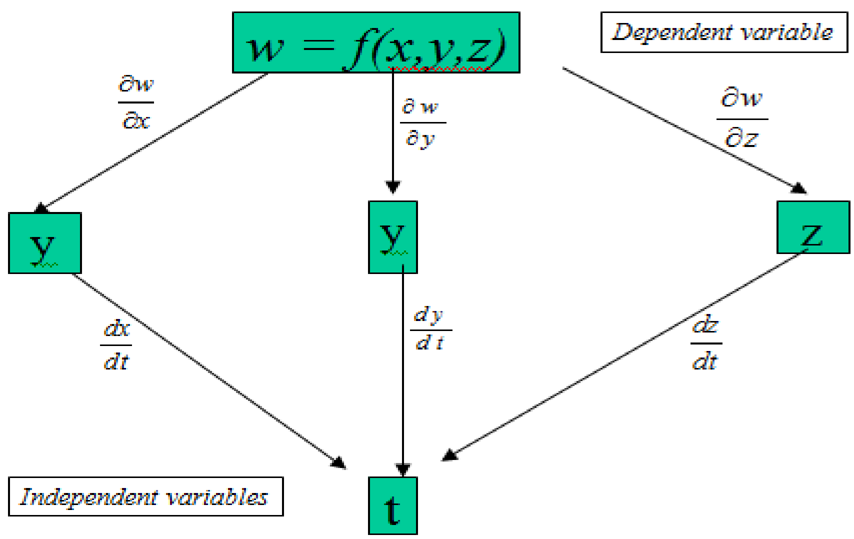
$$\frac{dx}{dt} = \frac{d\sqrt{t}}{dt} = \frac{1}{2\sqrt{t}}, \qquad \frac{dy}{dt} = \frac{dt^{\frac{2}{3}}}{dt} = \frac{2}{3}t^{\frac{2}{3}-1} = \frac{2}{3t^{\frac{1}{3}}}$$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\left(\frac{4x}{2x^2+y}\right)\left(\frac{1}{2\sqrt{t}}\right)+\left(\frac{1}{2x^2+y}\right)\left(\frac{2}{3t^{\frac{1}{3}}}\right)$$

$$=\left(\frac{4\sqrt{t}}{2\left(\sqrt{t}\right)^2+t^{\frac{2}{3}}}\right)\left(\frac{1}{2\sqrt{t}}\right)+\left(\frac{1}{2\left(\sqrt{t}\right)^2+t^{\frac{2}{3}}}\right)\left(\frac{2}{3t^{\frac{1}{3}}}\right)=\frac{2}{2t+t^{\frac{2}{3}}}+\frac{2}{3t^{\frac{1}{3}}\left(2t+t^{\frac{2}{3}}\right)}=\frac{6t^{\frac{1}{3}}+2}{3t^{\frac{1}{3}}\left(2t+t^{\frac{2}{3}}\right)}$$

III When the function *f* is a function of three variable *x, y* and *z*. And *x, y* and *z* are functions of one variable *t* .

w = f(x,y,z), x = g(t) ,y = f(t), z = h(t)



$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

<u>Overview of Lecture#8</u>
Book Calculus by Howard Anton
(Chapter # 16 - Topic # 16.4, Page # 799)

LECTURE No. 9

EXAMPLES

First of all, we revise the example which we did in our 8th lecture.
Consider $w = f(x, y, z)$, where $x = g(t)$, $y = f(t)$, $z = h(t)$, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example 1: Consider a function $w = x^2 + y + z + 4$, then find $\frac{dw}{dt}$.

Solution:

$$\mathbf{w = x^2 + y + z + 4}$$

$$\mathbf{x = e^t, \qquad y = \cos t, \qquad z = t + 4}$$

$$\frac{\partial w}{\partial x} = 2x, \quad \frac{\partial w}{\partial y} = 1, \quad \frac{\partial w}{\partial z} = 1$$

$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = -\mathrm{Sin}t, \quad \frac{dz}{dt} = 1$$

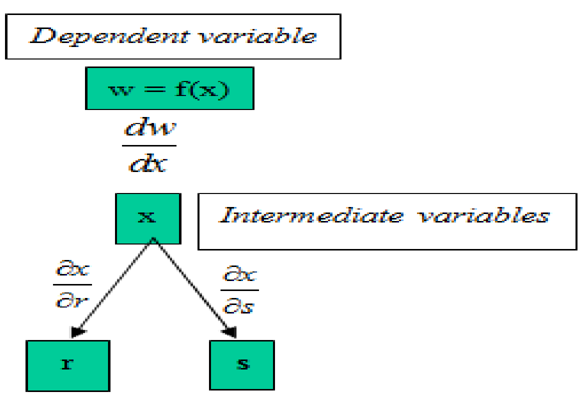
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= (2x)(e^t) + (1)\,.\,(-\mathrm{Sin}t) + (1)(1)$$

$$= 2(e^t)(e^t) - \mathrm{Sin}t + 1$$

$$= 2e^{2t} - \mathrm{Sin}t + 1$$

Consider $w = f(x)$, where $x = g(r, s)$. Now it is clear from the figure that "x" is intermediate variable and we can write $\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$ and $\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$



Example 2: If $w = \mathrm{Sin}\,x + x^2$, $x = 3r + 4s$, then find $\frac{\partial w}{\partial r}$ *and* $\frac{\partial w}{\partial s}$.

Solution: $w = \sin x + x^2, \quad x = 3r + 4s$

$$\frac{dw}{dx} = \cos x + 2x, \quad \frac{\partial x}{\partial r} = 3, \quad \frac{\partial x}{\partial s} = 4$$

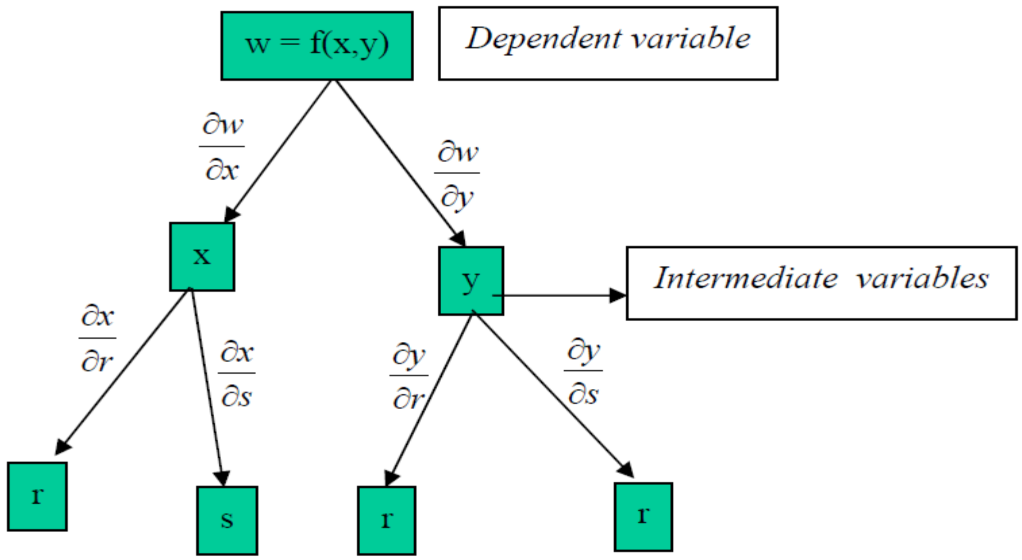
$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} = (\cos x + 2x)\cdot 3 = 3\cos x + 6x = 3\cos(3r + 4s) + 6(3r + 4s)$$

$$= 3\cos(3r+4s) + 18r + 24s$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s} = (\cos x + 2x)\cdot 4 = 4\cos x + 8x = 4\cos(3r+4s) + 8(3r+4s)$$

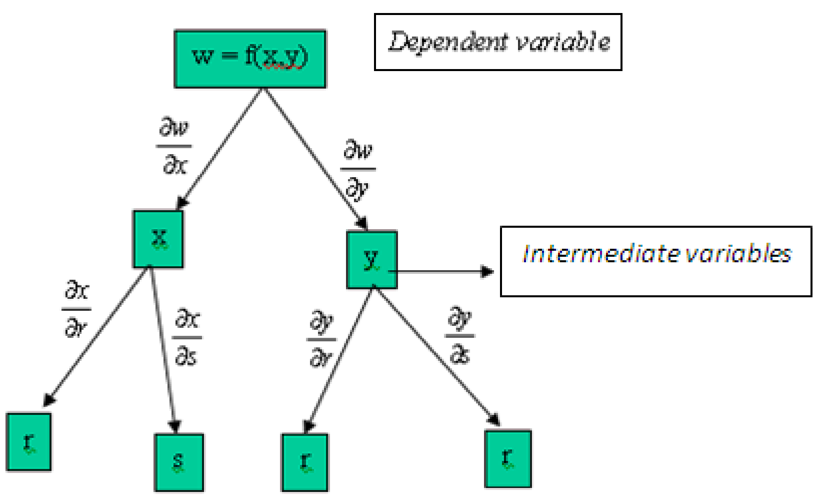
$$= 4\cos(3r+4s) + 24r + 32s$$

Consider the function $w = f(x, y)$, where $x = g(r, s)$, $y = h(r, s)$



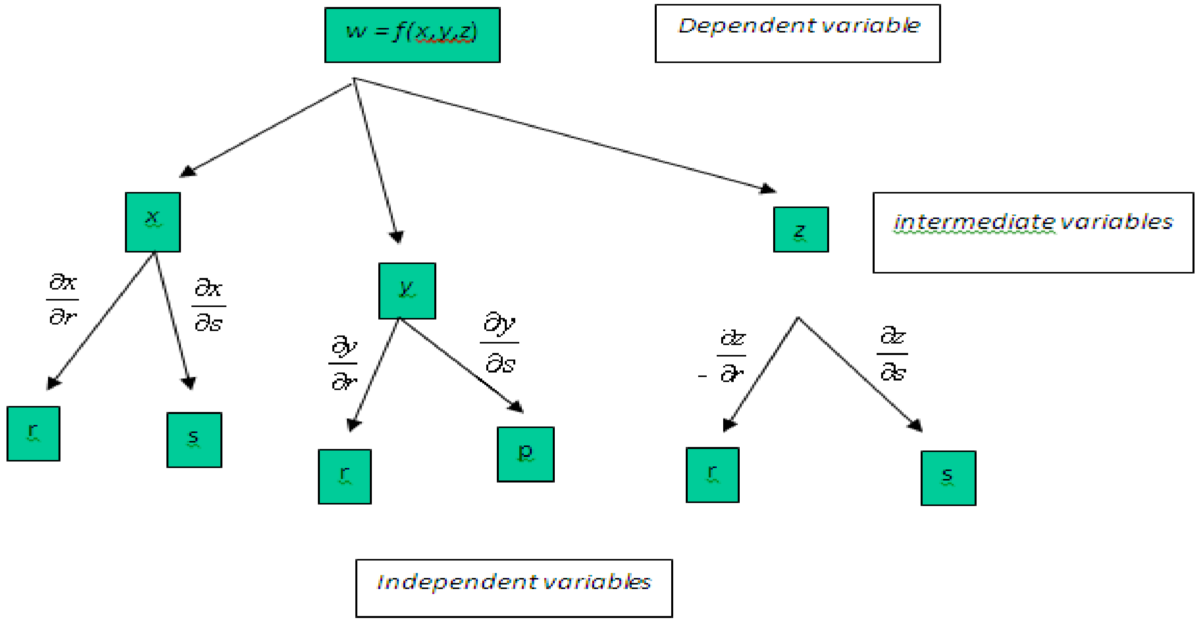
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

Similarly, if you differentiate the function w with respect to s we will get



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

Consider the function $w = f\left(x, y, z\right)$, where $x = g\left(r,\ s\right)$, $y = h\left(r, s\right)$, $z = k(r,\ s)$



Thus we have

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

Similarly if we differentiate with respect to s then we have,

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Example 3: Consider the function $w = x + 2y + z^2,\ x = \frac{r}{s},\ y = r^2 + \ln s,\ z = 2r.$

Solution:

First we calculate $\frac{\partial w}{\partial x} = 1,\ \frac{\partial w}{\partial y} = 2,\ \frac{\partial w}{\partial z} = 2z,\ \frac{\partial x}{\partial r} = \frac{1}{s},\ \frac{\partial y}{\partial r} = 2r,\ \frac{\partial z}{\partial r} = 2$

Since $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$

$$\frac{\delta w}{\delta r} = (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) = \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r$$

By putting the values from above, we get $\frac{\partial x}{\partial s} = -\frac{r}{s^2},\ \frac{\partial y}{\partial s} = \frac{1}{s},\ \frac{\partial z}{\partial s} = 0$

So we can calculate $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$

$$= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}$$

Remembering the Different Forms of the Chain Rule:
The best thing to do is to draw appropriate tree diagram by placing the dependent variable on top, the intermediate variables in the middle, and the selected independent variable at the bottom.

To find the derivative of dependent variable with respect to the selected independent variable, start at the dependent variable and read down each branch of the tree to the independent variable, calculating and multiplying the derivatives along the branch. Then add the products you found for the different branches.

The Chain Rule for Functions of Many Variables

Suppose $\omega = f(x, y, \ldots, \upsilon)$ is a differentiable function of the variables $x, y, \ldots, \upsilon$ (a finite set) and the $x, y, \ldots, \upsilon$ are differentiable functions of $p, q, , t$ (another finite set). Then ω is a differentiable function of the variables p through t and the partial derivatives of ω with respect to these variables are given by equations of the form

$$\frac{\partial \omega}{\partial p} = \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial p} + \ldots\ldots + \frac{\partial \omega}{\partial \upsilon}\frac{\partial \upsilon}{\partial p}.$$

The other equations are obtained by replacing p by q, …, t, one at a time. One way to remember last equation is to think of the right-hand side as the dot product of two vectors with components.

$$\left(\frac{\partial \omega}{\partial x}, \frac{\partial \omega}{\partial y} \ldots\ldots \frac{\partial \omega}{\partial \upsilon}\right) \quad \text{and} \quad \left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p} \ldots\ldots \frac{\partial \upsilon}{\partial p}\right)$$

Derivatives of ω with respect to the intermedaite variables

Derivatives of the intermedaite variables with respect to the selected independent variable

Example 4: For the function $w = \ln(e^r + e^s + e^t + e^u)$, find w_{rstu} where $w_{rstu} = \dfrac{\partial^4 w}{\partial u \partial t \partial s \partial r}$.

Solution: $w = \ln(e^r + e^s + e^t + e^u)$

$e^w = e^{\ln(e^r + e^s + e^t + e^u)}$ Take anti-log on both sides

$e^w = e^r + e^s + e^t + e^u \quad ----(1)$ since $e^{\ln x} = x$

Take derivative with respect to r,

$$\frac{\partial e^w}{\partial r} = \frac{\partial(e^r + e^s + e^t + e^u)}{\partial r} = (e^r + 0 + 0 + 0)$$

$$e^w w_r = e^r \qquad Since \quad \frac{\partial w}{\partial r} = w_r$$

$$w_r = \frac{e^r}{e^w} = e^{r-w} \quad ----(2)$$

Take derivative with respect to s,

$$\frac{\partial e^w}{\partial s} = \frac{\partial(e^r + e^s + e^t + e^u)}{\partial s} = (0 + e^s + 0 + 0)$$

$$e^w w_s = e^s \qquad Since \quad \frac{\partial w}{\partial s} = w_s$$

$$w_s = \frac{e^s}{e^w} = e^{s-w} \quad ----(3)$$

Similarly, by taking derivative of (1) with respect to u, we get

$$w_u = e^{u-w} \quad -----(4)$$

Similarly, by taking derivative of (1) with respect to t, we get

$$w_t = e^{t-w} \quad -----(5)$$

Now differentiate equation (2) with respect to s,

$$\frac{\partial w_r}{\partial s} = \frac{\partial e^{r-w}}{\partial s} = \frac{e^{r-w}\partial(r-w)}{\partial s} = e^{r-w}(0 - \frac{\partial w}{\partial s}) \qquad \text{since } r \text{ is kept constant}$$

$$w_{rs} = -e^{r-w}\, w_s = -e^{r-w}\, e^{s-w} \qquad by\,(3)$$

$$w_{rs} = -e^{r-w+s-w} = -e^{r+s-2w}$$

Now differentiate it with respect to t,

$$\frac{\partial w_{rs}}{\partial t} = -\frac{\partial e^{r+s-2w}}{\partial t} = -e^{r+s-2w}\,\frac{\partial(r+s-2w)}{\partial t}$$

$$= -e^{r+s-2w}(0 + 0 - \frac{\partial 2w}{\partial t}) \qquad \text{since } r \text{ is kept constant}$$

$$w_{rst} = -e^{r+s-2w}\,(-2\,w_t) = 2e^{r+s-2w}\,w_t = 2e^{r+s-2w}\,e^{t-w}$$

$$= 2e^{r+s+t-3w} \qquad by\ (5)$$

$$\frac{\partial w_{rst}}{\partial u} = \frac{\partial 2e^{r+s+t-3w}}{\partial u} = 2e^{r+s+t-3w}\,\frac{\partial(r+s+t-3w)}{\partial u}$$

$$w_{rstu} = 2e^{r+s+t-3w}(0 + 0 + 0 - \frac{\partial 3w}{\partial u})$$

$$= 2e^{r+s+t-3w}\,(-3w_u) = -6e^{r+s+t-3w}\,e^{u-w}$$

$$= -6e^{r+s+t+u-4w} \qquad by\ (4)$$

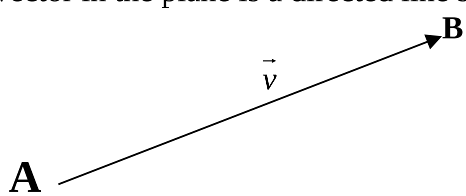
LECTURE No. 10

INTRODUCTION TO VECTORS

Some of things we measure are determined by their magnitude, but some times we need magnitude as well as direction to describe the quantities. For example, to describe a force, we need the direction in which that force is acting (Direction) as well as how large it is (Magnitude). Another example is the body’s velocity; we have to know where the body is headed as well as how fast it is.

Quantities that have direction as well as magnitude are usually represented by arrows that point the direction of the action and whose lengths give magnitude of the action in term of a suitably chosen unit.

A vector in the plane is a directed line segment.



$$\vec{v} = \overrightarrow{AB}$$

Vectors are usually described by the single **bold** face Roman letters or letter with an arrow. The vector defined by the directed line segment from point A to point B is written as $\overrightarrow{AB}$.

<u>Magnitude or Length of a Vector :</u>

Magnitude of the vector $\vec{v}$ is denoted by

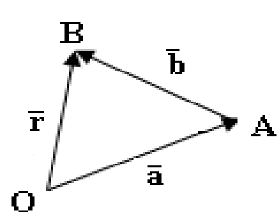
$$\left|\vec{v}\right| = \left|\overrightarrow{AB}\right|$$

which is the length of the line segment $\overrightarrow{AB}$

<u>Unit vector</u>: Any vector whose magnitude or length is 1 is a unit vector.

Unit vector in the direction of vector $\vec{v}$ is denoted by $\hat{v}$ and is given by $\hat{v} = \dfrac{\vec{v}}{|v|}$

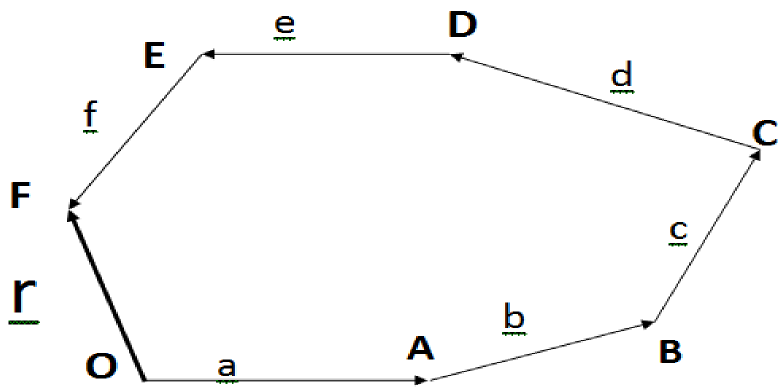
<u>Addition of Vectors</u>



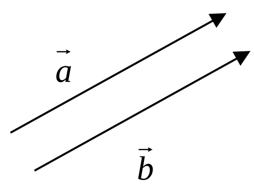
This diagram shows three vectors in two vectors; one vector $\overrightarrow{OA}$ is connected with tail of vector $\overrightarrow{AB}$. The tail of third vector $\overrightarrow{OB}$ is connected with the tail of $\overrightarrow{OA}$ and head is connected with the head of vector $\overrightarrow{AB}$. This third vector is called Resultant vector $\vec{r}$.

The resultant vector $\vec{r}$ can be written as $\vec{r} = \vec{a} + \vec{b}$

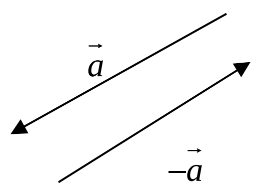
Similarly, $\vec{r} = \vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e} + \vec{f}$



Equal Vectors: Two vectors are equal or same vectors if they have same magnitude and direction. $|\vec{a}| = |\vec{b}|$

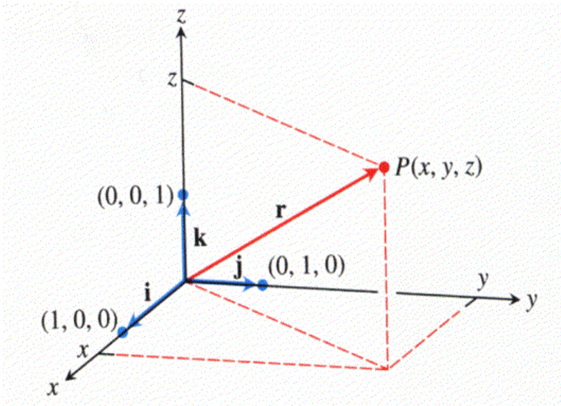


Opposite Vectors: Two vectors are opposite vectors if they have same magnitude and opposite directions.



Parallel Vectors: Two vectors $\vec{a}$ and $\vec{b}$ are parallel if one vector $\vec{a}$ is scalar multiple of the other $\vec{b}$.

$$\vec{b} = \lambda\,\vec{a} \qquad \text{where } \lambda \text{ is a non-zero scalar.}$$



$\underline{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$

Addition and subtraction of two vectors in rectangular component:

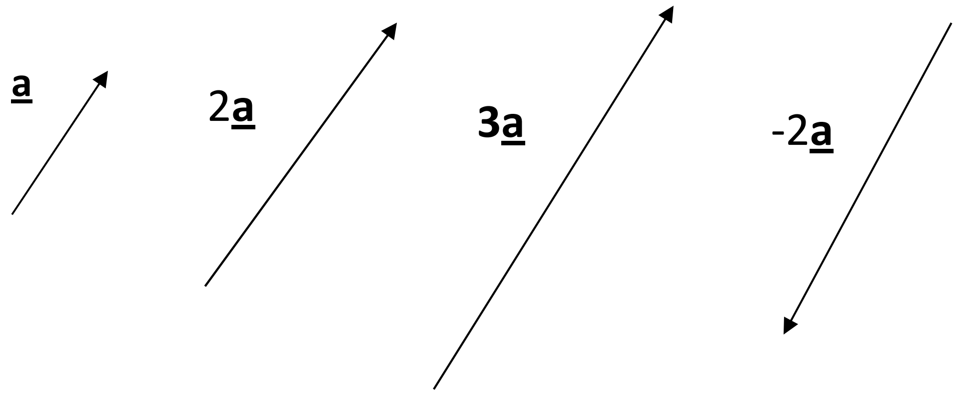
Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a} + \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) + (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$
$$= (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$$
$$\mathbf{a} - \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) - (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$
$$= (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} + (a_3 - b_3)\mathbf{k}$$

The ith component of first vector is added to (or subtracted from) the ith component of second vector, jth component of first vector is added to (or subtracted from) the jth component of second vector, similarly kth component of first vector is added to (or subtracted from) the kth component of second vector.

Multiplication of a Vector by a Scalar



Any vector $\vec{a}$ can be written as $\vec{a} = |\vec{a}|\ \hat{a}$

Scalar Product: Scalar product (dot product) ("$\vec{a}$ dot $\vec{b}$") of vector $\vec{a}$ and $\vec{b}$ is the number which is given by the formula:

$$\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}|\ \cos\ \theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

In words, $\vec{a}.\vec{b}$ is the length of $\vec{a}$ times the length of $\vec{b}$ times the cosine of the angle between $\vec{a}$ and $\vec{b}$.

Remark: This is known as commutative law. $\overrightarrow{a}.\overrightarrow{b} = \overrightarrow{b}.\overrightarrow{a}$

Some Results of Scalar Product

$$\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}|\ \cos\ \theta$$

1) If $\vec{a} \perp \vec{b}$, then it means that $\vec{a}$ is perpendicular to $\vec{b}$

So $\overrightarrow{a}.\overrightarrow{b} = 0$ since $\theta = 90^0,\ Cos\,90^0 = 0$

Also $\hat{i}\cdot\hat{j} = 0 = \hat{j}\cdot\hat{i},\quad \hat{j}\cdot\hat{k} = 0 = \hat{k}\cdot\hat{j},\quad \hat{k}\cdot\hat{i} = 0 = \hat{i}\cdot\hat{j}$

2) If $\vec{a} \parallel \vec{b}$ the it means $\vec{a}$ is parallel to $\vec{b}$.

So $\overrightarrow{a}.\overrightarrow{b} = |\overrightarrow{a}|.|\overrightarrow{b}|$ since $\theta = 0,\ Cos\,0 = 1$

If we replace $\vec{b}$ by $\vec{a}$, then

$$\overrightarrow{a}.\overrightarrow{a} = |\overrightarrow{a}|.|\overrightarrow{a}| = |\overrightarrow{a}|^2$$

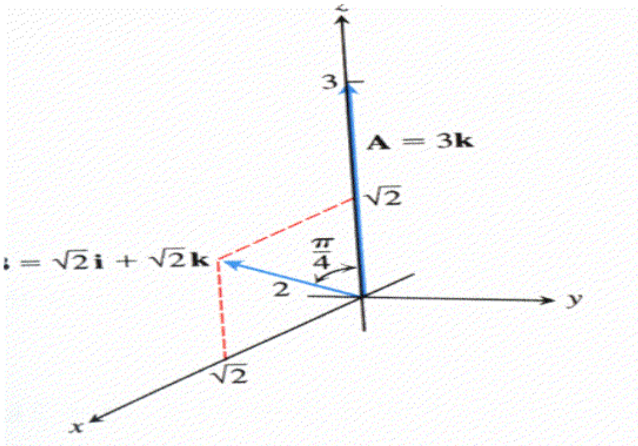
So $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

Example

If $\vec{a} = 3\hat{k}$ and $\vec{b} = \sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{k},\quad \theta = \dfrac{\pi}{4},$

then $\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}|\ \cos\ \theta = |3\hat{k}|\ |\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{k}|\ \cos\ \dfrac{\pi}{4}$

$$= (3)\ (2)\left(\frac{1}{\sqrt{2}}\right) = \frac{6}{\sqrt{2}} = 3\sqrt{2}$$



EXPRESSION FOR a . b IN COMPONENT FORM

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

$$\begin{aligned}\mathbf{a}.\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) . (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1\mathbf{i} . (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + a_2\mathbf{j} . (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &\quad + a_3\mathbf{k}. (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\end{aligned}$$

$$\begin{aligned}&= a_1b_1\mathbf{i} . \mathbf{i} + a_1b_2\mathbf{i}.\mathbf{j} + a_1b_3\mathbf{i} . \mathbf{k} + a_2b\mathbf{j} . \mathbf{i} + a_2b_2\mathbf{j} . \mathbf{j} \\ &\quad + a_2b_3\mathbf{j} . \mathbf{k} + a_3b_1\mathbf{k} . \mathbf{i} + + a_3b_2\mathbf{k} . \mathbf{j} + a_3b_3\mathbf{k} . \mathbf{k} \\ &= a_1b_1(1) + a_1b_2(0) + a_1b_3\ (0) + a_2b_1(0) + a_2b_2\ (1) \\ &\quad + a_2b_3(0) + a_3b_1(0) + \ + a_3b_2(0) + a_3b_3(1) \\ &= a_1b_1 + a_2b_2 + a_3b_3\end{aligned}$$

In dot product, the ith component of vector $\vec{a}$ will multiply with ith component of vector $\vec{b}$, jth component of vector $\vec{a}$ will multiply with jth component of vector $\vec{b}$ and kth component of vector $\vec{a}$ will multiply with kth component of vector $\vec{b}$.

Angle between Two Vectors

The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is

$$\theta = Cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}\right)$$

Since the values of arc-cosine lie in $[0, \pi]$, so the above equation automatically gives the angle made between $\vec{a}$ and $\vec{b}$.

Example : Find the angle between the vectors $\vec{a}=\hat{i}-2\hat{j}-2\hat{k}$ and $\vec{b}=6\hat{i}+3\hat{j}+2\hat{k}$.

Solution: $\vec{a}\,.\,\vec{b} = \left(\hat{i} - 2\hat{j} - 2\hat{k}\right).\left(6\hat{i} + 3\hat{j} + 2\hat{k}\right) = (1)(6) + (-2)(3) + (-2)(2)$

$= 6 - 6 - 4 = -4$

$\left|\vec{a}\right| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3,\qquad \left|\vec{b}\right| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$

$$\theta = Cos^{-1}\left(\frac{\vec{a}\,.\,\vec{b}}{\left|\vec{a}\right|.\left|\vec{b}\right|}\right) = Cos^{-1}\left(\frac{-4}{(3)(7)}\right) = Cos^{-1}\left(\frac{-4}{21}\right) \approx 1.76 \ \ radians$$

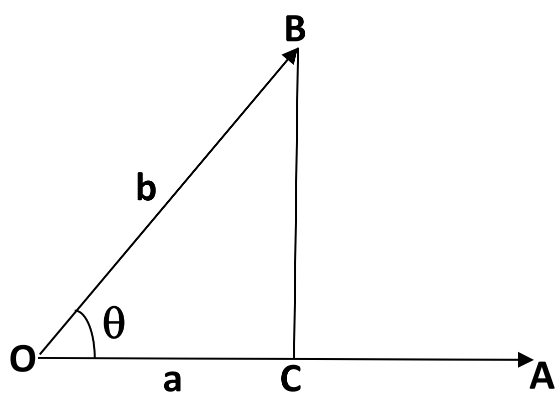
Perpendicular (Orthogonal)Vectors

The non-zero vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are perpendicular if and only if $\overrightarrow{a}\,.\,\overrightarrow{b} = 0$

This statement has two parts If $\overrightarrow{a}$ and $\overrightarrow{b}$ are per perpendicular, then $\overrightarrow{a}\,.\,\overrightarrow{b} = 0$. And if $\overrightarrow{a}\,.\,\overrightarrow{b} = 0$, then $\overrightarrow{a}$ and $\overrightarrow{b}$ are per perpendicular.

Vector Projection

Consider the Projection of a vector $\overrightarrow{b}$ on a vector $\overrightarrow{a}$ making an angle θ with each other



From right angle triangle *OCB*,

$$Cos\,\theta = \frac{base}{hypotenuse} = \frac{\left|\overrightarrow{OC}\right|}{\left|\vec{b}\right|}$$

$$\left|\overrightarrow{OC}\right| = \left|\vec{b}\right|\,Cos\,\theta$$

$$= \left|\vec{b}\right|\,\frac{\left|\vec{a}\right|}{\left|\vec{a}\right|}Cos\,\theta = \frac{\vec{b}\,.\,\vec{a}}{\left|\vec{a}\right|}$$

$$\text{Projection of } \vec{b} \text{ along } \vec{a} = \vec{b}.\frac{\vec{a}}{|\vec{a}|}\left(\frac{\vec{a}}{|\vec{a}|}\right) \qquad \text{where} \left(\frac{\vec{a}}{|\vec{a}|}\right) \text{ is the unit vector along } \vec{a}.$$

$$= \frac{\vec{b}.\vec{a}}{|\vec{a}||\vec{a}|}\vec{a} = \frac{\vec{b}.\vec{a}}{\vec{a}.\vec{a}}\vec{a}$$

The number $|\vec{b}|\, Cos\,\theta$ is called the scalar component of $\vec{b}$ in the direction of $\vec{a}$ because $|\vec{b}|\, Cos\,\theta = \vec{b}.\hat{a}$

Example : Find the vector projection of $\vec{b} = 6\hat{i} + 3\hat{j} + 2\hat{k}$ onto $\vec{a} = \hat{i} - 2\hat{j} - 2\hat{k}$.

Solution:

$$\text{Projection of } \vec{b} \text{ onto } \vec{a} = \frac{\vec{b}.\vec{a}}{\vec{a}.\vec{a}}\vec{a}$$

Here, $\vec{b}.\vec{a} = \left(6\hat{i} + 3\hat{j} + 2\hat{k}\right).\left(\hat{i} - 2\hat{j} - 2\hat{k}\right) = 6\times1 + 3(-2) + 2(-2) = 6 - 6 - 4 = -4$

$$\vec{a}.\vec{a} = \left(\hat{i} - 2\hat{j} - 2\hat{k}\right).\left(\hat{i} - 2\hat{j} - 2\hat{k}\right) = 1\times1 + (-2)(-2) + (-2)(-2) = 1 + 4 + 4 = 9$$

$$\text{Projection of } \vec{b} \text{ onto } \vec{a} = \frac{\vec{b}.\vec{a}}{\vec{a}.\vec{a}}\vec{a} = \frac{-4}{9}\left(\hat{i} - 2\hat{j} - 2\hat{k}\right) = -\frac{4}{9}\hat{i} + \frac{8}{9}\hat{j} + \frac{8}{9}\hat{k}$$

The scalar component of $\vec{b}$ in the direction of $\vec{a}$ is $|\vec{b}|\, Cos\,\theta$.

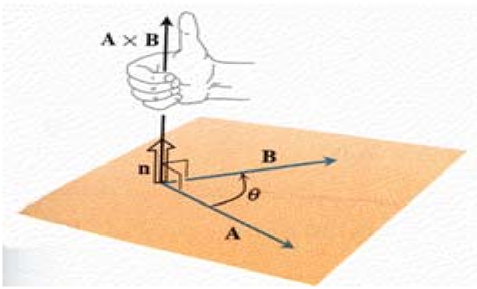
$$|\vec{b}|\, Cos\,\theta = \frac{\vec{b}.\vec{a}}{|\vec{a}|} = \frac{\left(6\hat{i} + 3\hat{j} + 2\hat{k}\right).\left(\hat{i} - 2\hat{j} - 2\hat{k}\right)}{\sqrt{(1)^2 + (-2)^2 + (-2)^2}} = \frac{6\times1 + 3(-2) + 2(-2)}{\sqrt{9}} = \frac{6 - 6 - 4}{3} = \frac{-4}{3}$$

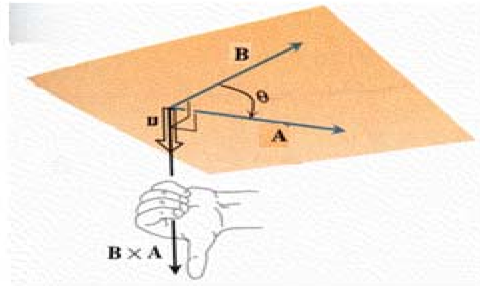
The Cross Product of Two Vectors in Space

Consider two non-zero vectors $\vec{a}$ and $\vec{b}$ in space. The vector product $\vec{a}\times\vec{b}$ ("$\vec{a}$ *cross* $\vec{b}$") to be the vector $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\, Sin\,\theta\, \hat{n}$ where $\hat{n}$ is the unit vector determined by the Right Hand rule.

Right-hand rule

We start with two nonzero nonparallel vectors **A** and **B** .We select a unit vector **n** perpendicular to the plane by the **right handed rule.** This means we choose **n** to be the unit vector that points the way your right thumb points when your fingers curl through the angle 0 from **A** to **B**.
The vector **A** $\times$ **B** is orthogonal to both **A** and **B.**





Some Results of Cross Product $\vec{a} \times \vec{b}$

As we know that $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}| \, Sin\theta \, \hat{n}$

1) If $\vec{a} \parallel \vec{b}$, then $\vec{a}\times\vec{b} = \vec{0}$ since $Sin 0^0 = 0$

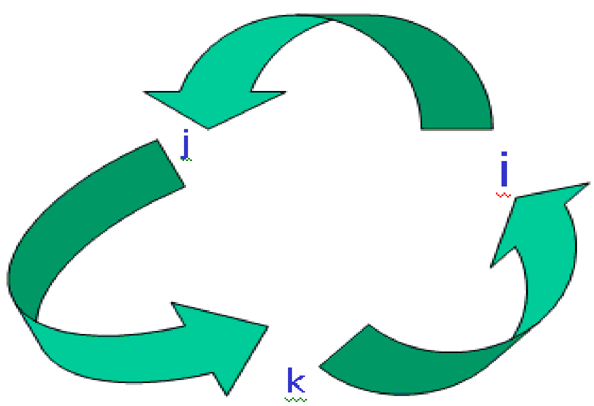
Similarly, $\vec{a}\times\vec{a} = \vec{0}$ and $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$

2) If $\vec{a} \perp \vec{b}$, then $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}| \, \hat{n}$ since $Sin 90^0 = 1$

$Similarly,\ \hat{i}\times\hat{j} = \hat{k}, \quad \hat{j}\times\hat{i} = -\hat{k}$

$\hat{j}\times\hat{k} = \hat{i}, \quad \hat{k}\times\hat{j} = -\hat{i}$

$\hat{k}\times\hat{i} = \hat{j}, \quad \hat{i}\times\hat{k} = -\hat{j}$

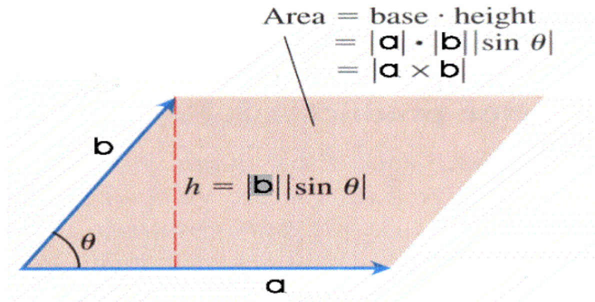


Note that the vector product is not commutative.

The Area of a Parallelogram

Because $\hat{n}$ is a unit vector and magnitude of $\vec{a} \times \vec{b}$ is

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \, Sin\theta \, |\hat{n}| = |\vec{a}||\vec{b}| \, Sin\theta \qquad Since\ |\hat{n}| = 1$$



This is the area of parallelogram which is determined by $\vec{a}$ and $\vec{b}$ where $\vec{a}$ is the base and $\left|\vec{b}\right| Sin\theta$ is the height of the parallelogram.

a × b from the components of a and b

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\vec{a}\times\vec{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)\times\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$

$$= a_1\hat{i}\times\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) + a_2\hat{j}\times\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) + a_3\hat{k}\times\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$$

$$= a_1b_1\hat{i}\times\hat{i} + a_1b_2\hat{i}\times\hat{j} + a_1b_3\hat{i}\times\hat{k} + a_2b_1\hat{j}\times\hat{i} + a_2b_2\hat{j}\times\hat{j} + a_2b_3\hat{j}\times\hat{k}$$
$$a_3b_1\hat{k}\times\hat{i} + a_3b_2\hat{k}\times\hat{j} + a_3b_3\hat{k}\times\hat{k}$$

$$= a_1b_1\times 0 + a_1b_2\hat{k} + a_1b_3\left(-\hat{j}\right) + a_2b_1\left(-\hat{k}\right) + a_2b_2\left(0\right) + a_2b_3\hat{i}$$
$$a_3b_1\hat{j} + a_3b_2\left(-\hat{i}\right) + a_3b_3\left(0\right)$$

$$= \left(a_2b_3 - a_3b_2\right)\hat{i} - \left(a_1b_3 - a_3b_1\right)\hat{j} + \left(a_1b_2 - a_2b_1\right)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: Let $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = -4\hat{i} + 3\hat{j} + \hat{k}$, then find $\vec{a}\times\vec{b}$.

Solution:

$$\left|\vec{a}\times\vec{b}\right| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \hat{i}\left(1-3\right) - \hat{j}\left(2+4\right) + \hat{k}\left(6+4\right) = -2\hat{i} - 6\hat{j} + 10\hat{k}$$

Over view of Lecture # 10

Chapter# 14: Article # 14.3, 14.4 Page # 679

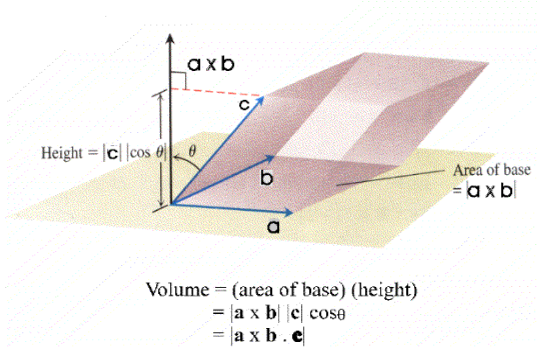
LECTURE No. 11

THE TRIPLE SCALAR OR BOX PRODUCT

The product $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$ is called the triple scalar product of $\vec{a},\vec{b}$ and $\vec{c}$ (in that order).

As $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} \;=\; \left|\vec{a}\times\vec{b}\right|\,\left|\vec{c}\right|\,\left|\cos\theta\right|$

So the absolute value of the product is the volume of the parallelepiped (parallelogram-sided box) determined by $\vec{a},\vec{b}$ and $\vec{c}$.



By treating the planes of $\vec{b}$ and $\vec{c}$ and of $\vec{c}$ and $\vec{a}$ as the base planes of the parallelepiped determined by $\vec{a},\vec{b}$ and $\vec{c}$.

We see that $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} \;=\; \left(\vec{b}\times\vec{c}\right)\cdot\vec{a} \;=\; \left(\vec{c}\times\vec{a}\right)\cdot\vec{b}$

Since the dot product is commutative, $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} \;=\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$

Example : Show that $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Proof : Consider

$$\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$$

$$\vec{b} = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$$

$$\vec{c} = c_1\,\hat{i} + c_2\,\hat{j} + c_3\,\hat{k}$$

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \vec{a}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \left(a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}\right)\cdot\left[\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}\hat{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\hat{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\hat{k}\right]$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

So, $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \begin{vmatrix} a_1 & a_1 & a_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Example : Let $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}, \quad \vec{b} = -2\hat{i} + 3\hat{k}, \quad \vec{c} = 7\hat{j} - 4\hat{k}.$ Find $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right).$

Solution:

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = 1\begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - 2\begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} + (-1)\begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix}$$

$$= 1(0-21) - 2(8-0) - (-14-0)$$

$$= -21 - 16 + 14 \quad = -23$$

When we solve $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$, then answer is -23. If we get negative value, then Absolute value makes it positive and also volume is always positive.

<u>Gradient of a Scalar Function</u>

$$\overline{\nabla} = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

where $\overline{\nabla}$ is called “del” operator.

Gradient ϕ is a vector operator defined as

$$\text{grad } \phi = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

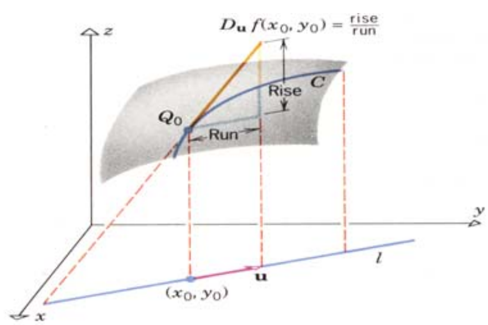
$$= \overline{\nabla}\phi$$

$\overline{\nabla}$ “del operator” is a vector quantity. Grad means gradient. Gradient is also vector quantity. $\overline{\nabla}\phi$ is vector and ϕ is scalar quantity.

<u>Directional Derivative</u>

If $f(x,y)$ is differentiable at (x_0, y_0) and if $\vec{u} = (u_1, u_2)$ is a unit vector, then the ***directional derivative*** of $f(x,y)$ at (x_0, y_0) in the direction of $\vec{u}$ is defined by

$$D_u\, f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$



It should be kept in mind that there are infinitely many directional derivatives of $z = f(x,y)$ at a point (x_0, y_0), one for each possible choice of the direction vector $\vec{u}$.

<u>Remarks (Geometrical Interpretation)</u>

The directional derivative $D_u\, f(x_0, y_0)$ can be interpreted algebraically as the instantaneous rate of change in the direction of $\vec{u}$ at (x_0, y_0) of $z = f(x,y)$ with respect to the distance parameters described above, or geometrically as the rise over the run of the tangent line to the curve C at the point Q_0 .

NOTE : Formula for the directional derivative can be written in the following compact form, using gradient notation: $D_u\, f\left(x,y\right) = \bar{\nabla} f\left(x,y\right)\cdot \hat{u}$

The dot product of the gradient of f with the unit vector $\hat{u}$ produces the directional derivative of f in the direction of $\vec{u}$.

Example : Find the directional derivative of $f(x,y) = 3x^2y$ at $\left(1,2\right)$ in the direction of $\vec{a} = 3\hat{i} + 4\hat{j}.$

Solution : Given $f(x,y) = 3x^2y, \quad (1,2), \quad \vec{a} = 3\hat{i} + 4\hat{j}$

$$f_x(x,y) = 6xy, \qquad f_x(1,2) = 6(1)(2) = 12$$

$$f_y(x,y) = 3x^2, \qquad f_y(1,2) = 3(1)^2 = 3$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3\hat{i} + 4\hat{j}}{\sqrt{3^2 + 4^2}} = \frac{3\hat{i} + 4\hat{j}}{\sqrt{25}} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$$

$$D_a\, f\left(x,y\right) = \bar{\nabla} f\left(1,2\right)\cdot \hat{a} = \left(f_x(1,2)\hat{i} + f_y(1,2)\hat{j}\right)\cdot\left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}\right)$$

$$= \left(12\hat{i} + 3\hat{j}\right)\cdot\left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}\right) = 12\left(\frac{3}{5}\right) + 3\left(\frac{4}{5}\right) = \frac{48}{5}$$

Example : Find the directional derivative of $f(x,y) = 2x^2 + y^2$ at $P_0\left(-1,1\right)$ in the direction of $\vec{u} = 3\hat{i} - 4\hat{j}.$

Solution : Given $f(x,y) = 2x^2 + y^2, \quad P_0\left(-1,1\right), \quad \vec{u} = 3\hat{i} - 4\hat{j}.$

$$f_x(x,y) = 4x, \qquad f_x(-1,1) = 4(-1) = -4$$

$$f_y(x,y) = 2y, \qquad f_y(-1,1) = 2(1) = 2$$

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{3\hat{i} - 4\hat{j}}{\sqrt{3^2 + 4^2}} = \frac{3\hat{i} - 4\hat{j}}{\sqrt{25}} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

$$D_a\, f\left(x,y\right) = \bar{\nabla} f\left(-1,1\right)\cdot \hat{u} = \left(-4\hat{i} + 2\hat{j}\right)\cdot\left(\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}\right)$$

$$= -4\left(\frac{3}{5}\right) - 2\left(\frac{4}{5}\right) = \frac{-20}{5} = -4$$

Remarks:

If $\vec{u} = u_1\hat{i} + u_2\hat{j}$ is a unit vector making an angle θ with the positive x-axis, then $u_1 = \cos\theta$ and $u_2 = \sin\theta$. So $D_u\, f\left(x_0,y_0\right) = f_x\left(x_0,y_0\right)u_1 + f_y\left(x_0,y_0\right)u_2$ can be written in the form $D_u\, f\left(x_0,y_0\right) = f_x\left(x_0,y_0\right)\cos\theta + f_y\left(x_0,y_0\right)\sin\theta$

Example: Find the directional derivative of e^{xy} at $(-2,0)$ in the direction of the unit vector $\vec{u}$ that makes an angle of $\frac{\pi}{3}$ with the positive x-axis.

Solution: Given $f(x,y)=e^{xy}$, $(-2,0)$, $\hat{u}=\cos\frac{\pi}{3}\hat{i}+\sin\frac{\pi}{3}\hat{j}.$

$$f_x(x,y)=ye^{xy}, \qquad f_x(-2,0)=(0)e^{(-2)(0)}=0$$

$$f_y(x,y)=xe^{xy}, \qquad f_y(-2,0)=(-2)e^{(-2)(0)}=-2$$

$$D_u\, f(x,y)=\bar{\nabla} f\,(-2,0)\cdot\hat{u}=\left(0\hat{i}-2\hat{j}\right)\cdot\left(\cos\frac{\pi}{3}\hat{i}+\sin\frac{\pi}{3}\hat{j}\right)$$

$$=\left(0\hat{i}-2\hat{j}\right)\cdot\left(\frac{1}{2}\hat{i}+\frac{\sqrt{3}}{2}\hat{j}\right)=0\left(\frac{1}{2}\right)-2\left(\frac{\sqrt{3}}{2}\right)$$

$$=-\sqrt{3}$$

Gradient of Function

If f is a function of x and y, then gradient of f is defined as

$$\nabla f(x,y)=f_x(x,y)\,\boldsymbol{i}+f_y(x,y)\,\boldsymbol{j}$$

Directional Derivative

Formula for the directional derivative can be written in the following compact form using the gradient

$$D_u\, f(x,y)=\nabla f(x,y)\cdot\hat{\mathrm{u}}$$

The dot product of the gradient f with the unit vector $\hat{\mathrm{u}}$ produces the directional derivative of f in the direction of $\hat{\mathrm{u}}$.

Example: Find the directional derivative of $f(x,y)=2xy-3y^2$ at $\mathrm{P}_0(5,5)$ in the direction of $\vec{u}=4\hat{i}+3\hat{j}.$

Solution:

Given $f(x,y)=2xy-3y^2$, $\mathrm{P}_0(5,5)$, $\vec{u}=4\hat{i}+3\hat{j}$

$$f_x(x,y)=2y, \qquad f_x(5,5)=2(5)=10$$

$$f_y(x,y)=2x-6y, \qquad f_y(5,5)=2x-6y=2(5)-6(5)=-20$$

$$\hat{u}=\frac{\vec{u}}{|\vec{u}|}=\frac{4\hat{i}+3\hat{j}}{\sqrt{4^2+3^2}}=\frac{4\hat{i}+3\hat{j}}{\sqrt{25}}=\frac{4}{5}\hat{i}+\frac{3}{5}\hat{j}$$

$$D_u\, f(x,y)=\bar{\nabla} f\,(5,5)\cdot\hat{u}=\left(10\hat{i}-20\hat{j}\right)\cdot\left(\frac{4}{5}\hat{i}+\frac{3}{5}\hat{j}\right)$$

$$=10\left(\frac{4}{5}\right)-20\left(\frac{3}{5}\right)=\frac{-20}{5}=-4$$

Example: Find the directional derivative of $f(x,y)=xe^y+\cos(xy)$ at the point $(2,0)$ in the direction of $\vec{a}=3\hat{i}-4\hat{j}.$

Solution:

Given $f(x,y)=xe^y+\cos(xy)$, $(2,0)$, $\vec{a}=3\hat{i}-4\hat{j}$

$f_x(x,y)=e^y-y\sin(xy)$, $f_x(2,0)=e^0-(0)\sin(2\times 0)=1-0=1$

$f_y(x,y)=xe^y-x\sin(xy)$, $f_y(2,0)=2e^0-(2)\sin(2\times 0)=2-0=2$

$$\hat{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{3\hat{i}-4\hat{j}}{\sqrt{3^2+4^2}}=\frac{3\hat{i}-4\hat{j}}{\sqrt{25}}=\frac{3}{5}\hat{i}-\frac{4}{5}\hat{j}$$

$$D_a f(x,y)=\bar{\nabla}f(2,0)\cdot\hat{a}$$

$$=(\hat{i}+2\hat{j})\cdot\left(\frac{3}{5}\hat{i}-\frac{4}{5}\hat{j}\right)$$

$$=1\left(\frac{3}{5}\right)-2\left(\frac{4}{5}\right)=\frac{-5}{5}=-1$$

Properties of Directional Derivatives

$$D_u f=\bar{\nabla}f\cdot\hat{u}=\left|\bar{\nabla}f\right|\cos\theta$$

1. The function f increases most rapidly when $\cos\theta=1$ or $\theta=0$ or when $\hat{u}$ is in the direction of $\bar{\nabla}f$. That is, at each point P in its domain, f increases most rapidly in the direction of gradient vector $\bar{\nabla}f$ at P. The derivative in this direction is

$$D_u f=\bar{\nabla}f\cdot\hat{u}=\left|\bar{\nabla}f\right|\cos 0=\left|\bar{\nabla}f\right|$$

2. The function f decreases most rapidly when $\cos\theta=-1$ or $\theta=\pi$ or when $\hat{u}$ is in the opposite direction of $\bar{\nabla}f$. That is, at each point P in its domain, f decreases most rapidly in the direction of gradient vector $-\bar{\nabla}f$ at P. The derivative in this direction is

$$D_u f=\bar{\nabla}f\cdot\hat{u}=\left|\bar{\nabla}f\right|\cos\pi=-\left|\bar{\nabla}f\right|$$

3. Any direction of $\hat{u}$ which is orthogonal to the gradient vector $\bar{\nabla} f$ is the direction of zero change in f because $\theta = \frac{\pi}{2}$ and $\cos\frac{\pi}{2} = 0$

$$D_u f = \bar{\nabla} f \cdot \hat{u} = \left|\bar{\nabla} f\right| \cos\frac{\pi}{2} = \left|\bar{\nabla} f\right|.0 = 0$$

Example: Find the directions of rapid increase, rapid decrease and no change for the function $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$.

Solution: $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

$$\bar{\nabla} f(x,y) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)\left(\frac{x^2}{2} + \frac{y^2}{2}\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(\frac{x^2}{2} + \frac{y^2}{2}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{x^2}{2} + \frac{y^2}{2}\right)$$

$$= \hat{i}\left(\frac{2x}{2} + 0\right) + \hat{j}\left(0 + \frac{2y}{2}\right)$$

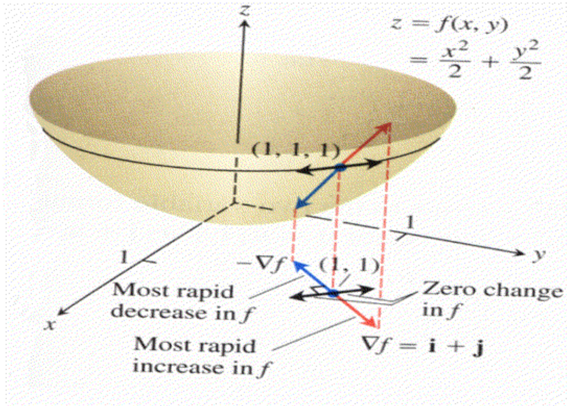
$$= x\hat{i} + y\hat{j}$$

$$\bar{\nabla} f(1,1) = \hat{i} + \hat{j}$$

(a) Its direction of rapid increase is $\hat{u} = \frac{\hat{i} + \hat{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

(b) The function f decreases most rapidly in the direction of gradient vector $-\bar{\nabla} f$ at $(1,1)$ which is $-\hat{u} = -\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

(c) The direction of zero change of the function f is orthogonal to gradient vector $-\bar{\nabla} f$ at $(1,1)$ which is $\hat{u} = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$, $\qquad -\hat{u} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$



$z = f(x, y)$
$= \frac{x^2}{2} + \frac{y^2}{2}$
(1, 1, 1)
1
1
$-\nabla f$
(1, 1)
Most rapid decrease in f
Zero change in f
Most rapid increase in f
$\nabla f = \mathbf{i} + \mathbf{j}$

LECTURE No. 12

TANGENT PLANES TO THE SURFACES

Normal line to the surfaces

If **C** is a smooth parametric curve in three dimensions, then tangent line to **C** at the point P_0 is the line through P_0 along the unit tangent vector to the C at the P_0. The concept of a tangent plane builds on this definition.

If $P_0(x_0,y_0,z_0)$ is a point on the Surface **S,** and if the tangent lines at P_0 to all the smooth curves that pass through P_0 and lies on the surface S all lie in a common plane, then we shall regard that plane to be the ***tangent plane*** to the surface S at P_0.

Its normal (the straight line through P_0 and perpendicular to the tangent) is called the ***surface normal*** of S at P_0.

Different forms of equation of straight line in two dimensional space

1. **Slope intercept form of the Equation of a line**

$$y = mx + c$$

where m is the slope and c is y intercept

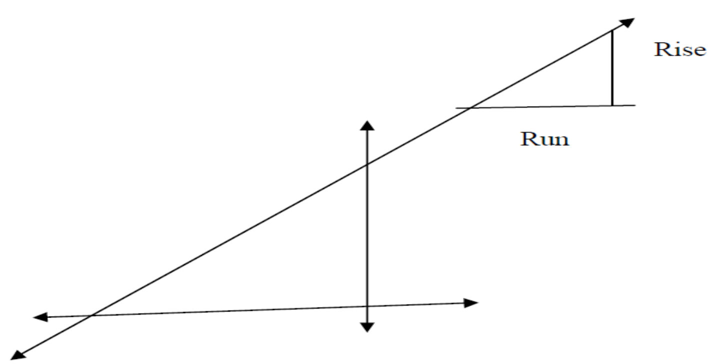
2. **Point-Slope Form**

Let m be the slope and $P_0(x_0, y_0)$ be the point of required line, then

$$y - y_0 = m\,(x - x_0)$$

$$m = \text{slope of line} = \frac{\text{Rise}}{\text{Run}} = \frac{\text{b}}{\text{a}}$$

$$y - y_0 = \frac{\text{b}}{\text{a}}(x - x_0)$$



3. **General Equation of straight line**

$$Ax + By + C = 0$$

Parametric equation of a line

Parametric equation of a line in two dimensional space passing through the point (x_0, y_0) and parallel to the vector $a\,\mathbf{i} + b\,\mathbf{j}$ is given by

$$x = x_0 + at, \qquad y = y_0 + bt$$

Eliminating t from both equations, we get

$$\frac{x - x_0}{a} = t, \qquad \frac{y - y_0}{b} = t$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}$$

$$y - y_0 = \frac{b}{a}(x - x_0)$$

Parametric vector form:

$$\mathbf{r}(t) = (x_0 + at)\,\mathbf{i} + (y_0 + bt)\,\mathbf{j}$$

Equation of line in three dimensional

Parametric equation of a line in three dimensional space passing through the point (x_0, y_0, z_0) and parallel to the vector $a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is given by

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct$$

Eliminating t from these equations we get

$$\frac{x - x_0}{a} = t, \qquad \frac{y - y_0}{b} = t, \quad \frac{z - z_0}{c} = t$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example: Find Parametric equations for the straight line through the point A (2,4,3) and parallel to the vector $\mathbf{v} = 4\mathbf{i} + 0\mathbf{j} - 7\mathbf{k}$.

Solution:

$$x_0 = 2,\ y_0 = 4,\ z_0 = 3$$

and $a = 4,\ b = 0,\ c = -7$

The required parametric equations of the straight line are

$$x = 2 + 4t,$$
$$y = 4 + 0t,$$
$$z = 3 - 7t$$

Different forms of the equation of curve

Curves in the plane are defined in different ways

(1) Explicit form: $y = f(x)$

Example: $y = \sqrt{9-x^2} \qquad -3 \le x \le 3$

(2) Implicit form: $F(x, y) = 0$

Example: $x^2 + y^2 = 9 \qquad -3 \le x \le 3, \quad 0 \le y \le 3$

(3) Parametric form: $x = f(t)$ and $y = g(t)$

Example:

$$x = 3\cos\theta, \quad y = 3\sin\theta \qquad 0 \le \theta \le \pi$$

$$x^2 + y^2 = 9\cos^2\theta + 9\sin^2\theta$$

$$= 9\left(\cos^2\theta + \sin^2\theta\right) = 9(1)$$

$$x^2 + y^2 = 9$$

(4) Parametric vector form: $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}, \qquad a \le t \le b.$

Example: $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} \qquad 0 \le t \le \pi$

<u>Equation of a plane</u>

A plane can be completely determined if we know its one point and direction of perpendicular (normal) to it.

Let a plane passing through the point $P_0(x_0, y_0, z_0)$ and the direction of normal to it is along the vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Let P (x, y, z) be any point on the plane, then the line lies on it so that $\mathbf{n} \perp \overline{P_0P}$

($\perp$ means "perpendicular to")

$$\overline{P_0P} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

Therefore, $\mathbf{n}.\overline{P_0P} = 0$

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}).\left((x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}\right) = 0$$

$$a(x - x_0) + \mathrm{b}(y - y_0) + c(z - z_0) = 0$$

which is the required equation of the plane.

NOTE : Here we use the theorem:

Let $\vec{a}$ and $\vec{b}$ be two vectors. If $\vec{a}$ and $\vec{b}$ are perpendicular,

then $\vec{a}\cdot\vec{b}=0$

Since $\mathbf{n}$ and $\overline{P_0P}$ are perpendicular vector, so $\mathbf{n}.\overline{P_0P} = 0$

REMARKS

Point normal form of equation of plane is

$$a(x - x_0) + \text{b}(y - y_0) + c(z - z_0) = 0$$

We can write this equation as

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$
$$ax + by + cz - ax_0 - by_0 - cz_0 = 0$$
$$ax + by + cz + d = 0$$
$$\text{where } d = -ax_0 - by_0 - cz_0$$

,which is the equation of plane

Example: An equation of the plane passing through the point (3, - 1, 7) and perpendicular to the vector $\mathbf{n} = 4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$.

A point-normal form of the equation is

$$\mathbf{4(x – 3) + 2 (y + 1) – 5 (z – 7) = 0}$$
$$\mathbf{4x + 2y – 5z + 25 = 0}$$

Which is the same form of the equation of plane **ax + by + cz + d = 0**

The general equation of straight line
is $ax + by + c = 0$
Let (x_1, y_1) and (x_2, y_2) be two points
on this line then
$ax_1 + by_1 + c = 0$
$ax_2 + by_2 + c = 0$
Subtracting above equation
$a(x_2 – x_1) + b (y_2 – y_1) = 0$

$\mathbf{v} = (x_2 – x_1)\mathbf{i} + (y_2 – y_1)\mathbf{j}$
is a vector in the direction of line
$\phi(x, y) = ax + by$
$\phi_x = a, \qquad \phi_y = b$
$\nabla\phi = a\mathbf{i} + b\mathbf{j} = \mathbf{n}$
$\nabla\phi \,.\, \mathbf{r} = 0$
Then n and **v** are perpendicular

The general equation of plane is
$ax + by + cz + d = 0$
For any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) lying on this plane we have

$ax_1 + by_1 + cz_1 + d = 0 \qquad (1)$
$a x_2 + by_2 + cz_2 + d = 0 \qquad (2)$

Subtracting equation (1) from (2) have

$a (x_2 - x_1) + b (y_2 - y_1) + c (z_2 - z_1) = 0$

$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}).\left[(X_2 - X_1)\, i + (Y_2 - Y_1)\, j + (Z_2 - Z_1)\, k\right]$

Here we use the definition of dot product of two vectors.

$\phi = ax + by + cz$
$\phi_x = a, \qquad \phi_y = b, \qquad \phi_z = c$
$\nabla\phi = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$
Where $\mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$

$\nabla\phi$ is always normal to the plane.

<u>Gradients and Tangents to Surfaces</u>

$f(x, y) = c$
$z = f(x, y), \qquad z = c$

If a differentiable function $f(x, y)$ has a constant value c along a smooth curve, having parametric equations:

$$x = g(t), \qquad y = h(t), \qquad \mathbf{r} = g(t)\,\mathbf{i} + h(t)\,\mathbf{j}$$

Differentiating both sides of $f(x, y) = c$ with respect to t,

$$\frac{d}{dt} f\left(g(t), h(t)\right) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0$$

$$\left(\frac{\partial f}{\partial x}\boldsymbol{i} \ + \ \frac{dh}{dt}\boldsymbol{j}\right)\cdot\left(\frac{dg}{dt}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}\right) = 0$$

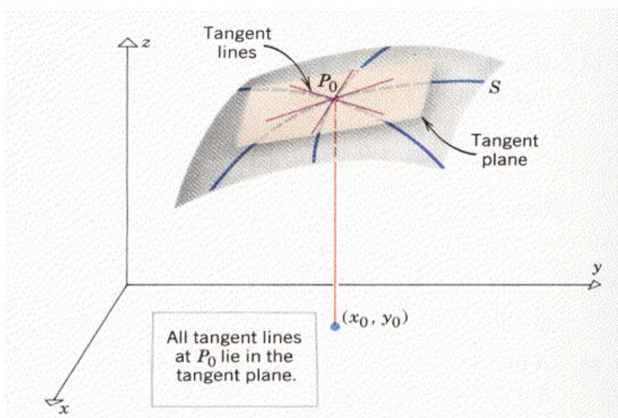
$$\nabla f \cdot \frac{d\boldsymbol{r}}{dt} = 0$$

∇f is normal to the tangent vector $\frac{d\boldsymbol{r}}{dt}$. So it is normal to the curve through (x_0, y_0).

Tangent Plane and Normal Line

Consider all the curves through the point **P_0(x0, y0, z0)** on a surface **f(x, y, z) = 0.** The plane containing all the tangents to these curves at the point P0(x0, y0, z0) is called the ***tangent plane to the surface at the point P_0***.

The straight lines perpendicular to all these tangent lines at P_0 is called the ***normal line to the surface at P_0*** if fx, fy, fz are all continuous at P_0 and not all of them are zero, then gradient f (i.e fx**i** + fy**j** + fz**k**) at P_0 gives the direction of this normal vector to the surface at P_0.



Tangent plane

Let P_0 (x_0, y_0, z_0) be any point on the Surface
f(x,y,z) = 0. If f(x,y,z) is differentiable
at $p_o(x_0,y_0,z_0)$ then the tangents plane at the
point P_0 (x_0,y_0,z_0) has the equation

Example: Find the equation of tangent plane to the surface $9x^2 + 4y^2 - z^2 = 36$ at point $P(2,3,6)$.

Solution:

$$9x^2 + 4y^2 - z^2 = 36 \qquad P\ (2,3,6).$$

$$\boldsymbol{f(x,y,z) = 9x^2 + 4y^2 - z^2 - 36}$$

$$f_x = 18x, \qquad f_y = 8y, \qquad f_z = -2z$$

$$f_x\,(P) = 36, \quad f_y\,(P) = 24, \quad f_z\,(P) = -12$$

Equations of Tangent Plane to the surface through P is

$36(x-2) + 24(y-3) - 12(z-6) = 0$
$3x + 2y - z - 6 = 0$

Example: Find the equation of tangent plane to the surface
$z = x\cos y - y e^x$ at point $(0,0,0)$.
Solution:

$z = x \cos y - ye^x$ (0,0,0).

$\cos y - ye^x - z = 0$

$f(x,y,z) = \cos y - ye^x - z$

$f_x(0,0,0) = (\cos y - ye^x)_{(0,0)} = 1 - 0.1 = 1$

$f_y(0,0,0) = (-x\sin y - e^x)_{(0,0)} = 0 - 1 = -1.$

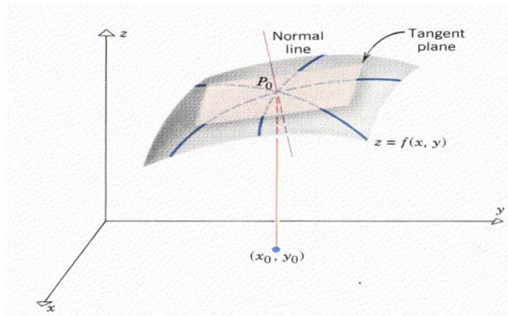
$f_z(0,0,0) = -1$

The tangent plane is

$f_x(0,0,0)(x-0) + f_y(0,0,0)(y-0) + f_z(0,0,0)(z-0) = 0$

$1(x-0) - 1(y-0) - 1(z-0) = 0,$

$x - y - z = 0.$



LECTURE No. 13

ORTHOGONAL SURFACE

In this Lecture we will study the following topics

- **Normal line**
- **Orthogonal Surface**
- **Total differential for function of one variable**
- **Total differential for function of two variables**

Normal line

Let $P_0(x_0,y_0,z_0)$ be any point on the surface $f(x,y,z)=0$ If $f(x,y,z)$ is differentiable at $P_0(x_0,y_0,z_0)$ then the normal line at the point $P_0(x_0,y_0,z_0)$ has the equation

$$x = x_0+f_x(P_0)t, \quad y = y_0+f_y(P_0)t, \quad z = z_0+f_z(P_0)t$$

Here $\mathbf{f_x}$ means that the function f(x,y,z) is partially differentiable with respect to x And $\mathbf{f_x(P_0)}$ means that the function **f(x,y,z**) is partially differentiable with respect to x at the point $\mathbf{P_0(x_0,y_0,z_0)}$

$\mathbf{f_y}$ means that the function f(x,y,z) is partially differentiable with respect to y And $\mathbf{fy(P_0)}$ means that the function **f(x,y,z)** is partially differentiable with respect to y at the point $\mathbf{P_0(x_0,y_0,z_0)}$

Similarly, **fz** means that the function f(x,y,z) is partially differentiable with respect to z And $\mathbf{fz(P_0)}$ means that the function **f(x,y,z)** is partially differentiable with respect to z at the point $\mathbf{P_0(x_0,y_0,z_0)}$

Example: Find the Equation of the tangent plane and normal of the surface f(x,y,z)= x2+y2+z2-4 at the point P(1,-2,3)

Solution:

$$f(x,y,z)=x^2+y^2+z^2-14, \qquad P(1,-2,3)$$

$$f_x = 2x, \quad f_y = 2y, \quad f_z = 2z$$

$$f_x(P_0)=2, \quad f_y(P_0)=-4, \quad f_z(P_0)=6$$

Equation of the tangent plane to the surface at P is

$$2(x-1)-4(y+2)+6(z-3)=0$$

$$x-2y+3z-14=0$$

Equation of the normal line of the surface through P is

$$\frac{(x-1)}{2}=\frac{(y+2)}{-4}=\frac{(z-3)}{6}$$

$$\frac{(x-1)}{1}=\frac{(y+2)}{-2}=\frac{(z-3)}{3}$$

Example : Find the equation of the tangent plane and normal plane

Solution:

$$4x^2 - y^2 + 3z^2 = 10 \quad P(2, -3, 1)$$

$$f(x,y,z) = 4x^2 - y^2 + 3z^2 - 10$$

$$f_x = 8x, \qquad f_y = -2y, \qquad f_z = 6z$$

$$f_x(P) = 16, \qquad f_y(P) = 6, \qquad f_z(P) = 6$$

Equations of Tangent Plane to the surface through P is

$$16(x-2) + 6(y+3) + 6(z-1) = 0$$

$$8x + 3y + 3z = 10$$

Equations of the normal line to the surface through P are

$$\frac{x-2}{16} = \frac{y+3}{6} = \frac{z-1}{6}$$

$$\frac{x-2}{8} = \frac{y+3}{3} = \frac{z-1}{3}$$

Example

$$z = \frac{1}{2} x^7 y^{-2}$$

$$f(x,y,z) = \frac{1}{2} x^7 y^{-2} - z$$

$$f_x = \frac{7}{2} x^6 . y^{-2}, \qquad f_y = - x^7 . y^{-3}, \qquad f_z = -1$$

$$f_x(2, 4, 4) = \frac{7}{2} (2)^6 (4)^{-2} = 14$$

$$f_y(2, 4, 4) = -(2)^7 (4)^{-3} = -2$$

$$f_z(2, 4, 4) = -1$$

Equation of Tangent at (2, 4, 4) is given by

$$f_x(2,4,4)(x-2) + f_y(2,4,4)(y-4) + f_z(2,4,4)(z-4) = 0$$

$$14(x-2) + (-2)(y-4) - (z-4) = 0$$

$$14x - 2y - z - 16 = 0$$

The normal line has equations

$$x = 2 + f_x(2,4,4)\,t, \quad y = 4 + f_y(2,4,4)\,t, \quad z = 4 + f_z(2,4,4)\,t$$

$$x = 2 + 14t, \qquad y = 4 - 2t, \qquad z = 4 - t$$

ORTHOGONAL SURFACES

Two surfaces are said to be orthogonal at a point of their intersection if their normals at that point are orthogonal. They are Said to intersect orthogonally if they are orthogonal at every point common to them.

CONDITION FOR ORTHOGONAL SURFACES

Let (x, y, z) be any point of intersection of

$$f(x, y, z) = 0 ---- \quad (1)$$

$$\text{and} \quad g(x, y, z) = 0 ----- (2)$$

Direction ratios of a line normal to (1) are f_x, f_y, f_z

Similarly, direction rations of a line normal to (2) are g_x, g_y, g_z

The two normal lines are orthogonal if and only if

$$f_x g_x + f_y g_y + f_z g_z = 0$$

Example

Show that given two surfaces are orthogonal or not

$$f(x,y,z) = x^2 + y^2 + z - 16$$

$$g(x,y,z) = x^2 + y^2 - 63z$$

$$f(x,y,z) = x^2 + y^2 + z - 16 \qquad ----(1)$$

$$g(x,y,z) = x^2 + y^2 - 63z \qquad ----(2)$$

Adding (1) and (2), $\quad x^2 + y^2 = \dfrac{63}{4}, \quad z = \dfrac{1}{4} \quad ----(3)$

$$f_x = 2x, \quad f_y = 2y, \quad f_z = 1$$

$$g_x = 2x, \quad g_y = 2y, \quad g_z = -63$$

$$f_x g_x + f_y g_y + f_z g_z = 4\left(x^2 + y^2\right) - 63 = 4\left(\frac{63}{4}\right) - 63 \qquad \text{using (3)}$$

$$= 0$$

Since they satisfy the condition of orthogonality, so they are orthogonal.

Differentials of a functions

For a function y = f(x)

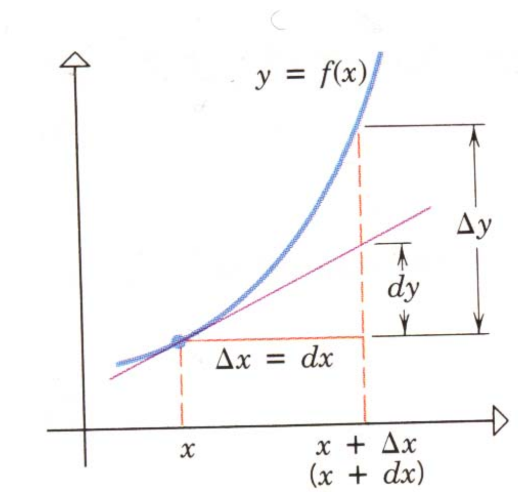
$$dy = f'(x)\, dx$$

is called the differential of functions f(x) dx the differential of x is the same as the actual change in x

i.e. $dx = \nabla x$ where as dy the

differential of y is the approximate change in the value of the functions which is different from the actual change ∇y in the value of the functions.

Distinction between the increments Δy and the differential dy



Approximation to the curve

If f is differentiable at x_0, then the tangent line to the curve $y = f(x)$ at x_0 is a reasonably good approximation to the curve $y = f(x)$ for value of x near x_0. Since the tangent line passes

through the point $(x_0, f(x_0))$ and has slope $f'(x_0)$, the point-slope form of its equation is

$$y - f'(x_0) = f(x_0)(x - x_0) \quad \text{or}$$

$$y = f(x_0) + f'(x_0)(x - x_0)$$

<u>EXAMPLE</u>

$$\mathbf{f(x) = \sqrt{x}}$$

$$\mathbf{x = 4 \text{ and } dx = \Delta x = 3 \quad y = \sqrt{3}}$$

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$$

$$= \sqrt{7} - \sqrt{4} \approx .65$$

If $y = \sqrt{x}$, then

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \quad \text{so} \quad dy = \frac{1}{2\sqrt{x}}\,dx$$

$$= \frac{1}{2\sqrt{x}}(3) = \frac{3}{4} = .75$$

Example **Using differentials approximation for the value of cos 61 °.**

Let $y = \cos x$ and $x = 60°$

then $dx = 61° - 60° = 1°$

$$\Delta y \approx dy = -\sin x\, dx = -\sin 60° (1°)$$

$$= \frac{\sqrt{3}}{2}\left(\frac{1}{180}\pi\right)$$

Now $y = \cos x$

$$y+\Delta y = \cos(x+\Delta x) = \cos(x+dx)$$

$$= \cos(60°+1°) = \cos 61°$$

$$\cos 61° = y+\Delta y = \cos x + \Delta y$$

$$\approx \cos 60° - \frac{\sqrt{3}}{2}\left(\frac{1}{180}\pi\right)$$

$$\cos 61° \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{1}{180}\pi\right)$$

$$= 0.5 - 0.01511 = 0.48489$$

$$\cos 61° \approx 0.48489$$

Example

A box with a square base has its height twice is width. If the width of the box is 8.5 inches with a possible error of $\pm$ 0.3 inches

.

Let x and h be the width and the height of the box respectively, then its volume V is given by

$$V = x^2 h$$

Since $h = 2x$, so (1) take the form

$$V = 2x^3$$

$$dV = 6x^2\, dx$$

Since $x = 8.5$, $dx = \pm 0.3$, so putting these values in (2), we have

$$dV = 6(8.5)^2 (\pm 0.3) = \pm 130.05$$

This shows that the possible error in the volume of the box is ± 130.05.

<u>TOTAL DIFFERENTIAL</u>

If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting differential in f is

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

This change in the linearization of

f is called the total differential of *f*.

EXACT CHANGE

Area = xy

x = 10, y = 8 Area = 80
x = 10.03 y = 8.02 Area = 80.4406
Exact Change in area = 80.4406 – 80
= 0.4406

Example

A rectangular plate expands in such a way that its length changes from 10 to 10.03 and its breadth changes from 8 to 8.02.

Let x and y the length and breadth of the rectangle respectively, then its area is

$$A = xy$$

$$dA = A_x\, dx + A_y dy = ydx + xdy$$

By the given conditions

$$x = 10,\ dx = 0.03,\ y = 8,\ dy = 0.02.$$

$$dA = 8(0.03) + 10(0.02) = 0.44$$

Which is an exact change.

Example

The volume of a rectangular parallelepiped is given by the formula V = xyz. If this solid is compressed from above so that z is decreased by 2% while x and y each is increased by 0.75% approximately

$$V = xyz$$

$$dV = V_x d\,x + V_y dy + V_z dx$$

$$dV = yzdx + xzdy + zydz \quad (1)$$

$$dx=\frac{0.75}{100}\,x,\ dy = \frac{0.75}{100}\,y,\ dz= -\frac{2}{100}\,z$$

Putting these values in (1), we have

$$dV = \frac{0.75}{100}\,xyz+\frac{0.75}{100}\,xyz - \frac{2}{100}\,xyz$$

$$= -\frac{0.5}{100}\,xyz = -\frac{0.5}{100}\,V$$

This shows that there is 0.5 % decrease in the volume.

Example

A formula for the area Δ of a triangle is $\Delta = \frac{1}{2}$ ab sin C. Approximately what error is made in computing Δ if a is taken to be 9.1 instead of 9, b is taken to be 4.08 instead of 4 and C is taken to be $30°3'$ instead of $30°$.

By the given conditions

$$a = 9,\ b = 4,\ C = 30°,$$

$$da = 9.1 - 9 = 0.1,$$

$$db = 4.08 - 4 = 0.08$$

$$dC = 30°3' - 3' = \left(\frac{3}{60}\right)^{\circ}$$

$$= \frac{3}{60} \times \frac{\pi}{180} \text{ radians}$$

Putting these values in (1), we have

$$\Delta = \frac{1}{2} \text{ ab sin C}$$

$$d\Delta = \frac{\partial}{\partial a}\left(\frac{1}{2}\text{ab sin C}\right)da + \frac{\partial}{\partial b}\left(\frac{1}{2}\text{ab sin C}\right)db + \frac{\partial}{\partial C}\left(\frac{1}{2}\text{ab sin C}\right)dC$$

$$d\Delta = \frac{1}{2}\text{ b sin Cda} + \frac{1}{2}\text{ a sin C db} + \frac{1}{2}\text{ ab cos CdC}$$

$$d\Delta = \frac{1}{2}4\sin 30^0(0.1) + \frac{1}{2}9\sin 30^0(0.08) + \frac{1}{2}36\cos 30^0\left(\frac{\pi}{3600}\right)$$

$$= 2\left(\frac{1}{2}\right)(0.1) + \frac{9}{2}\left(\frac{1}{2}\right)(0.08) + 18\left(\frac{\sqrt{3}}{2}\right)\left(\frac{3.14}{3600}\right) = 0.293$$

$$\%\text{age change in area} = \frac{0.293}{\Delta}\times 100 = \frac{0.293}{9}\times 100 = 3.25\%$$

LECTURE No.14

EXTREMEA OF FUNCTIONS OF TWO VARIABLES

In this lecture, we shall find the techniques for finding the highest and lowest points on the graph of a function or, equivalently, the largest and smallest values of the function.

The graphs of many functions form hills and valleys. The tops of the hills are relative maxima and the bottoms of the valleys are called relative minima. Just as the top of a hill on the earth's terrain need not be the highest point on the earth , so a relative maximum need not be the highest point on the entire graph .

Absolute maximum

A function f of two variables on a subset of $\mathbf{R}^2$ is said to have an D **absolute (global) maximum** value on D if there is some point (x_0, y_0) of D such that value of f on D

$$f(x_0, y_0) \geq f(x, y) \text{ for all } (x, y) \in D$$

In such a case $f(x_0, y_0)$ is the **absolute maximum**

Relative extremum and absolute extremum

If f has a relative maximum or a relative minimum at (x_0, y_0), then we say that f has a relative extremum at (x_0, y_0), and if f has an absolute maximum or absolute minimum at $(x_0 y_0)$, then we say that f has an absolute extremum at (x_0, y_0).

Absolute minimum

A function f of two variables on a subset D of $\mathbf{R}^2$ is said to have an **absolute (global) minimum** value on D if there is some point (x_0, y_0) of D such that

$$f(x_0, y_0) \leq f(x, y) \text{ for all } (x, y) \in D.$$

In such a case $f(x_0, y_0)$ is the **absolute minimum** value of f on D.

Relative (local) maximum

The function f is said to have a relative (local) maximum at some point (x0,y0) of its domain D if there exists an open disc *K centered* at (x0,y0) and of radius r

$$K=\{(x, y) \in \mathbf{R}^2 : (x - x_0)^2 + (y - y_0)^2 < r^2\}$$

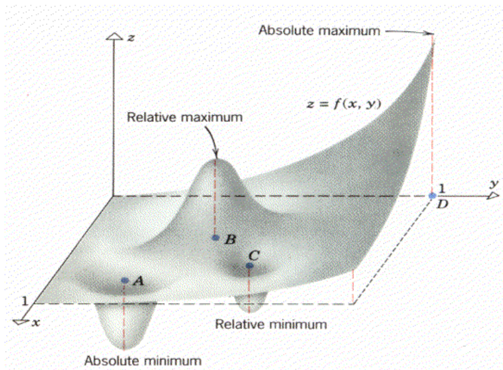
With $K \in D$ such that

$$f(x_0, y_0) \geq f(x, y) \text{ for all } (x, y)$$

Relative (Local) Minimum

The function f is said to have a **relative (local) minimum** at some point $\left(x_0, y_0\right)$ of D if there exists an open disc K centered at $\left(x_0, y_0\right)$ and of radius r with $K \subset D$ such that

$$f\left(x_0, y_0\right) \leq f\left(x, y\right) \qquad \text{for all } \left(x, y\right) \in K$$



Extreme Value Theorem

If f (x, y) is continuous on a closed and bounded set R, then f has both an absolute maximum and on absolute minimum on R .

Remarks

If any of the conditions the Extreme Value Theorem fail to hold, then there is no guarantee that an absolute maximum or absolute minimum exists on the region R. Thus, a discontinuous function on a closed and bounded set need not have any absolute extrema, and a continuous function on a set that is not closed and bounded also need not have any absolute extrema.

Extreme values or extrema of *f*

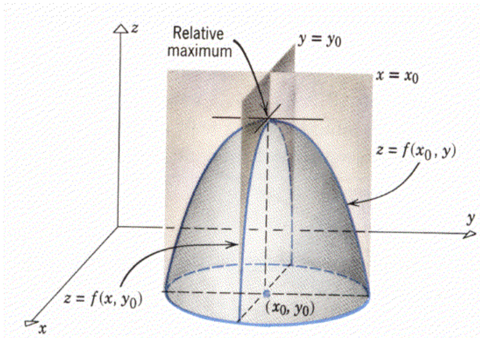
The maximum and minimum values of *f are* referred to as extreme values of extrema of f .Let a function f of two variables be defined on an open disc

$$K = \{(x, y): (x - x_0)^2 + (y - y_0)^2 < r^2\}.$$

Suppose $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist on K

If f has relative extrema at (x_0,y_0),then

$$\boldsymbol{f_x(x_0, y_0) = 0 = f_y(x_0, y_0).}$$



Saddle Point

A differentiable function f(x, y) has a saddle point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where f (x, y) > f (a, b) and domain points (x, y) where f (x, y) < f (a, b). The corresponding point (a, b, f (a, b)) on the surface z = f (x, y) is called a saddle point of the surface

Remarks

Thus, the only points where a function f(x,y) can assume extreme values are critical points and boundary points. As with differentiable functions if a single variable, not every critical point gives rise to o a local extremum. A differentiable function of a single variable might have a point of inflection. A differentiable function of two variable might have a saddle point.

EXAMPLE

Fine the critical points of the given function

$$\boldsymbol{f(x, y) = x^3 + y^3 - 3axy, a > 0.}$$

f_x, f_y exist at all points of the domain of f.

$f_x = 3x_2 - 3ay, \quad f_y = 3y^2 - 3ax$

For critical points $f_x = f_y = 0$.

Therefore, $x^2 - ay = 0 \quad (1)$

and $\quad ax - y^2 = 0 \quad (2)$

Substituting the value of x from (2) into (1), we have

$$\frac{y^4}{a^2} - ay = 0$$

$$y(y^3 - a^3) = 0$$

$y = 0, \qquad y = a$

and so

$x = 0, \qquad x = a.$

The critical points are (0, 0) and (a, a).

Overview of lecture # 14

Topic	Article #	page #
Extrema of Functions of Two Variables	16.9	833
Absolute maximum	16.9.1	833
Absolute manimum	16.9.2	833
Extreme Value Theorem	16.9.3	834
Exercise set	Q#1,3,5,7,9,11,13,15,17	841

Book

CALCULUS by HOWARD ANTON

LECTURE No. 15

EXAMPLES

Example: Find the critical point of $f(x,y) = \sqrt{x^2 + y^2}$.

$$\boldsymbol{f(x, y) = \sqrt{x^2 + y^2}}$$

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$$

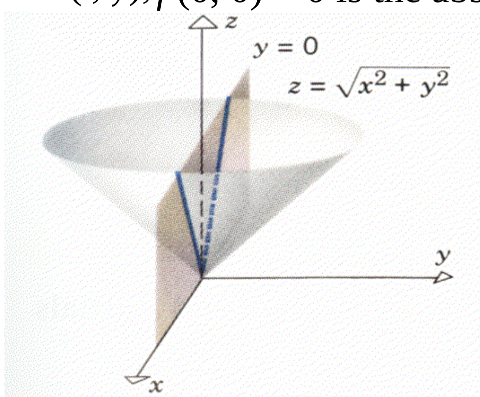
$$f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}$$

The partial derivatives exist at all points of the domain of f except at the origin which is in the domain of f . Thus (0, 0) is a critical point of f

Now $f_x(x, y) = 0$ only if $x = 0$ and
$f_y(x, y) = 0$ only if $y = 0$

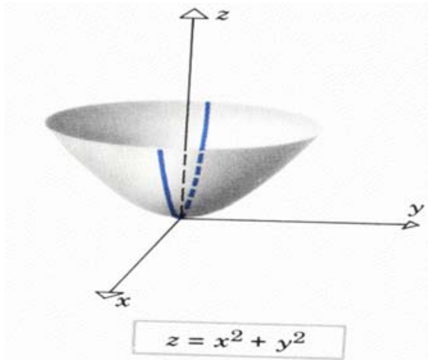
The only critical point is (0,0) and f(0,0)=0

Since $f(x, y) \geq 0$ for all (x, y), $f(0, 0) = 0$ is the absolute minimum value of f .



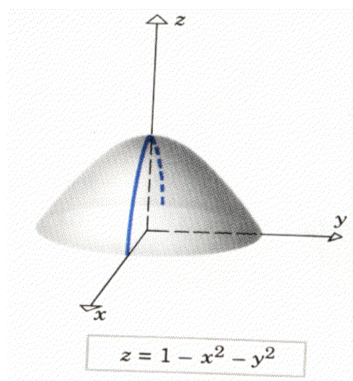
Example : Find the critical point of $f(x,y) = x^2 + y^2$.

$z = f(x, y) = x^2 + y^2$ (Paraboloid)
$f_x(x, y) = 2x$, $f_y(x, y) = 2y$
when $f_x(x, y) = 0$, $f_y(x, y) = 0$
we have (0, 0) as critical point.



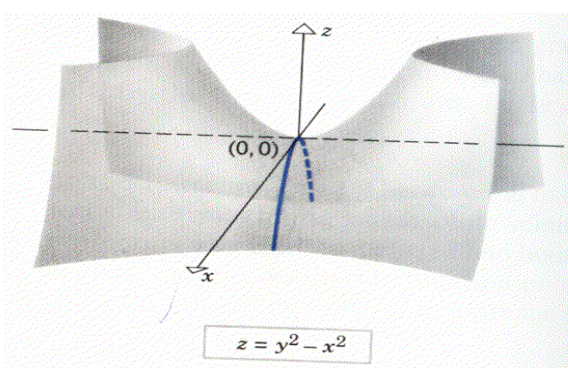
Example: Find the critical point of $z = g(x, y) = 1 - x^2 - y^2$.

$z = g(x, y) = 1 - x^2 - y^2$ (Paraboloid)
$g_x(x, y) = -2x,\ g_y(x, y) = -2y$
when $g_x(x, y) = 0,\ g_y(x, y) = 0$
we have (0, 0) as critical point.



Example: Find the critical point of $z = h(x, y) = y^2 - x^2$.

$z = h(x,y) = y^2 - x^2$ (Hyperbolic paraboloid)
$h_x(x, y) = -2x,\ h_y(x, y) = 2y$
when $h_x(x, y) = 0,\ h_y(x, y) = 0$
we have (0, 0) as critical point.



Example:

$$\mathbf{f(x, y) = \sqrt{x^2 + y^2}}$$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}} \quad f_y = \frac{y}{\sqrt{x^2 + y^2}}$$

The point (0,0) is critical point of f because the partial derivatives do not both exist. It is evident geometrically that $f_x(0,.0)$ does not exist because the trace of the cone in the plane y=0 has a corner at the origin.

The fact that $f_x(0,0)$ does not exist can also be seen algebraically by noting that $f_x(0,0)$ can be interpreted as the derivative with respect to x of the function

$$f(x, 0) = \sqrt{x^2 + 0} = |x| \quad \text{at } x = 0.$$

But $|x|$ is not differentiable at x = 0, so $f_x(0,0)$ does not exist. Similarly, $f_y(0,0)$ does not exist. The function f has a relative minimum at the critical point (0,0).

The Second Partial Derivative Test

Let f be a function of two variables with continuous second order partial derivatives in some circle centered at a critical point (x_0, y_0), and let

$$\mathbf{D} = f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0)$$

(a) If **D > 0** and $\mathbf{f_{xx}(x_0,y_0) > 0}$, then f has a **relative minimum** at (x_0,y_0).

(b) If **D > 0** and $\mathbf{f_{xx}(x_0,y_0) < 0}$, then f has a **relative maximum** at (x_0,y_0).

(c) If **D < 0**, then f has a **saddle point** at (x_0,y_0).

(d) If **D = 0**, then no conclusion can be drawn.

REMARKS

If a function f of two variables has an absolute extremum (either an absolute maximum or an absolute minimum) at an interior point of its domain, then this extremum occurs at a critical point.

Example:

$$f(x, y) = 2x^2 - 4x + xy^2 - 1$$

$$f_x(x, y) = 4x - 4 + y^2, \qquad f_{xx}(x, y) = 4$$

$$f_y(x, y) = 2xy, \qquad f_{yy}(x, y) = 2x, \quad f_{xy}(x, y) = f_{yx}(x, y) = 2y$$

For the critical points, we set the first partial derivatives equal to zero. Then

$$f_x(x,y)=0 \quad \Rightarrow \quad 4x-4+y^2=0 \qquad (1)$$

$$f_y(x,y)=0 \quad \Rightarrow \quad 2xy=0 \qquad (2)$$

$$\Rightarrow \quad x=0 \text{ or } y=0$$

When $x=0$, *then by* (1), $4(0)-4+y^2=0 \Rightarrow y^2=4 \quad \Rightarrow \quad y=\pm 2$

When $y=0$, *then by* (1), $4x-4+0^2=0 \Rightarrow 4x-4 \quad \Rightarrow \quad x=1$

So the critical points are $(1,\ 0)$, $(0,\ 2)$ and $(0,\ -2)$.

Now we check the nature of each point:

At $(1,0)$, $\quad f_{xx}(1,0)=4$

$$f_{yy}(1,0)=2(1)=2$$

$$f_{xy}(1,0)=2(0)=0$$

$$D=f_{xx}(1,0)\,f_{yy}(1,0)-\left[f_{xy}(1,0)\right]^2=4\times 2-0^2=8$$

Since D > 0 and $f_{xx}(1,0)$ is positive, so f has a relative minimum at $(1,0)$.

At $(0,-2)$, $\quad f_{xx}(0,-2)=4$

$$f_{yy}(0,-2)=2(0)=0$$

$$f_{xy}(0,-2)=2(-2)=-4$$

$$D=f_{xx}(0,-2)\,f_{yy}(0,-2)-\left[f_{xy}(0,-2)\right]^2=4\times 0-(-4)^2=0-16=-16$$

Since D < 0 , so f has a saddle point at $(0,-2)$.

At $(0,2)$, $\quad f_{xx}(0,2)=4, \quad f_{yy}(0,2)=2(0)=0, \quad f_{xy}(0,2)=2(2)=4$

$$D=f_{xx}(0,2)\,f_{yy}(0,2)-\left[f_{xy}(0,2)\right]^2=4\times 0-(4)^2=0-16=-16$$

Since D < 0 , so f has a saddle point at $(0,2)$.

Example:

$$f(x,y)=e^{-(x^2+y^2+2x)}$$

$$f_x(x,y)=-2(x+1)e^{-(x^2+y^2+2x)}, \qquad f_{xx}(x,y)=\left[(-2x-2)^2-2\right]e^{-(x^2+y^2+2x)}$$

$$f_y(x,y)=-2ye^{-(x^2+y^2+2x)}, \qquad f_{yy}(x,y)=\left[4y^2-2\right]e^{-(x^2+y^2+2x)}$$

$$f_{xy}(x,y)=-2y(-2x-2)e^{-(x^2+y^2+2x)}$$

For critical points,

put $\quad f_x(x,y)=0 \quad \Rightarrow -2(x+1)e^{-(x^2+y^2+2x)}=0$

$$\Rightarrow -2e^{-(x^2+y^2+2x)}\neq 0, \quad x+1=0 \Rightarrow x=-1$$

put $\quad f_y(x,y)=0 \qquad \Rightarrow\ -2ye^{-(x^2+y^2+2x)}=0$

$$\Rightarrow\ -2e^{-(x^2+y^2+2x)}\neq 0,\quad y=0$$

The critical point is $(-1,0)$.

$$f_{xx}(-1,0)=\left[\left(-2\left(-1\right)-2\right)^2-2\right]e^{-\left((-1)^2+0^2+2(-1)\right)}$$

$$=\left[0-2\right]e^{-(1+-2)}=-2e^1=-2e$$

$$f_{yy}(-1,0)=\left[4\left(0\right)^2-2\right]e^{-\left((-1)^2+0^2+2(-1)\right)}=-2e$$

$$f_{xy}(-1,0)=-2y\left(-2x-2\right)e^{-(x^2+y^2+2x)}=-2\left(0\right)\left(-2\left(-1\right)-2\right)e^{-\left((-1)^2+0^2+2(-1)\right)}=0e=0$$

$$D=f_{xx}(-1,0)\,f_{yy}(-1,0)-\left[f_{xy}(-1,0)\right]^2$$

$$=\left(-2e\right)\left(-2e\right)-\left(0\right)^2=4e^2$$

Since D > 0 , so $f_{xx}(-1,0)$ has a maximum point at $(-1,0)$.

EXAMPLE $\boldsymbol{f(x,y) = 2x^4 + y^2 - x^2 - 2y}$

$f_x(x, y) = 8x^3 - 2x,\qquad f_y(x, y) = 2y - 2$

$f_{xx}(x, y) = 24x^2 - 2,\qquad f_{yy}(x,y) = 2,$

$f_{xy}(x, y) = 0$

For critical points

$f_x(x, y) = 0,$

$2x(4x^2 - 1) = 0,\quad x = 0,1/2,-1/2$

$f_y(x, y) = 0,$

$2y - 2 = 0,\qquad y = 1$

Solving above equation we have the critical points (0,1), $\left(-\frac{1}{2}, 1\right)\left(\frac{1}{2}, 1\right)$.

$f_{xx}(0,1) = -2,\qquad f_{yy}(0, 1) = 2,$

$f_{xy}(0, 1) = 0$

$D = f_x(0, 1)\, f_{yy}(0, 1) - f^2_{xy}(0, 1)$

$= (-2)(2) - 0 = -4 < 0$

This shows that (0, 1) is a saddle point.

$$f_{xx}\left(\frac{1}{2},1\right)=4,\qquad f_{yy}\left(\frac{1}{2},1\right)=2,\qquad f_{xy}\left(\frac{1}{2},1\right)=0$$

$$D=f_{xx}\left(\frac{1}{2},1\right)f_{yy}\left(\frac{1}{2},1\right)-\left[f_{xy}\left(\frac{1}{2},1\right)\right]^2=\left(4\right)\left(2\right)-\left(0\right)^2=8>0$$

Since $f_{xx}\left(\frac{1}{2},1\right)=4>0$, so f is minimum at $\left(\frac{1}{2},1\right)$.

Example

Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4.$$

$f_x(x, y) = 4y - 4x^3$, $f_y(x, y) = 4x - 4y^3$

For critical points

$f_x(x, y) = 0$

$4y - 4x^3 = 0$ (1)

$y = x^3$

$f_y(x, y) = 0$

$4x - 4y^3 = 0$ (2)

$x = y^3$

Solving (1) and (2), we have the critical points (0,0), (1, 1),(−1, −1).

Now $f_{xx}(x, y) = -12x^2$, $f_{xx}(0, 0) = 0$

$f_{yy}(x, y) = -12y^2$, $f_{yy}(0,0) = 0$

$f_{xy}(x, y) = 4$, $f_{xy}(0, 0) = 4$

$\mathbf{D = f_{xx}(0,0)\, f_y(0,0) - f^2_{xy}(0,0)}$

$= (0)(0) - (4)^2 = -16 < 0$

This shows that (0,0) is the saddle point.

$f_{xx}(x, y) = -12x^2$, $f_{xx}(1,1) = -12 < 0$

$f_{yy}(x,y) = -12y^2$, $f_y(1,1) = -12$

$f_{xy}(x, y) = 4$, $f_{xy}(1,1) = 4$

$\mathbf{D = f_{xx}(1,1)\, f_{yy}(1,1) - f^2_{xy}(1, 1)}$

$= (-12)(-12) - (4)^2 = 128 > 0$

This shows that f has relative maximum at (1,1).

$f_{xx}(x,y) = -12x^2$, $f_{xx}(-1, -1) = -12 < 0$

$f_y(x, y) = -2y^2$, $f_y(-1, -1) = -12$

$f_{xy}(x, y) = 4$, $f_{xy}(-1, -1) = 4$

$\mathbf{D = f_{xx}(-1,-1)\, f_{yy}(-1,-1) - f^2_{xy}(-1,-1)}$

$= (-12)(-12) - (4)^2 = 128 > 0$

This shows that f has relative maximum (−1, −1).

Overview of lecture #15 Book (Calculus by HOWARD ANTON)

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The Second Partial Derivative Test	16.9.5	836
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LECTURE No.16

EXTREME VALUED THEOREM

EXTREME VALUED THEOREM

If the function f is continuous on the closed interval [a, b], then f has an absolute maximum value and an absolute minimum value on [a, b]

Remarks

An absolute extremum of a function on a closed interval must be either a relative extremum or a function value at an end point of the interval. Since a necessary condition for a function to have a relative extremum at a point C is that C be a critical point, we may determine the absolute maximum value and the absolute minimum value of a continuous function f on a closed interval [a, b] by the following procedure:

1. Find the critical points of f on [a, b] and the function values at these critical points.
2. Find the values of f (a) and f (b).
3. The largest and the smallest of the above calculated values are the absolute maximum value and the absolute minimum value respectively

Example: Find the absolute extrema of $\mathbf{f(x)= x^3+ x^2-x+1}$ **on [-2,1/2]**
Solution: Since f is continuous on **[-2,1/2],** the extreme value theorem is applicable. For this

$$\mathbf{f'(x) = 3x^2+2x-1}$$

This shows that f(x) exists for all real numbers, and so the only critical numbers of **f** will be the values of x for which f (x)=0.

Setting $f'(x) = 0$, we have

$(3x - 1)(x + 1) = 0$

from which we obtain

$x = -1$ and $x = \frac{1}{3}$

The critical points of f are -1 and $\frac{1}{3}$, and each of these points is in the given closed interval $(-2, \frac{1}{2})$ We find the function values at the critical points and at the end points of the interval, which are given below.

$f(-2) = -1,\ f(-1) = 2,$

$f\left(\frac{1}{3}\right) = \frac{22}{27},\ f\left(\frac{1}{2}\right) = \frac{7}{8}$

The absolute maximum value of f on $(-2, \frac{1}{2})$ is therefore 2, which occurs at -1, and the absolute min. value of f on $(-2, \frac{1}{2})$ is -1, which occurs at the left end point -2.

Example:

Find the absolute extrema of

$$f(x) = (x - 2)^{2/3} \quad \text{on } [1, 5].$$

Since f is continuous on [1. 5], the extreme-value theorem is applicable.

Differentiating f with respect to x, we get

$$f'(x) = \frac{2}{3\,(x-2)^{1/3}}$$

There is no value of x for which $f'(x) = 0$. However, since $f'(x)$ does not exist at 2, we conclude that 2 is a critical point of f,

so that the absolute extrema occur either at 2 or at one of the end points of the interval. The function values at these points are given below.

$f(1) = 1$, $f(2) = 0$, $f(5)= \sqrt[3]{9}$

From these values we conclude that the absolute minimum value of f on [1,5] is 0, occurring at 2, and the absolute maximum value of f on [1, 5] is $\sqrt[3]{9}$,occurring at 5.

Example:

Find the absolute extrema of

$$h(x) = x^{2/3} \text{ on } [-2, 3].$$

$$h'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

$h'(x)$ has no zeros but is undefined at $x = 0$.

The values of h at this one critical point and at the endpoints $x = -2$ and $x = 3$ are

$h(0) = 0$

$h(-2) = (-2)^{2/3} = 4^{1/3}$

$h(3) = (3)^{2/3} = 9^{1/3}$.

The absolute maximum value is $9^{1/3}$ assumed at $x = 3$; the absolute minimum is 0, assumed at $x = 0$.

<u>How to Find the Absolute Extrema of a Continuous Function f of Two Variables on a Closed and Bounded Region R.</u>

Step 1.

Find the critical points of f that lie in the interior of R.

Setp 2.

Find all boundary points at which the absolute extrema can occur,

Step 3.

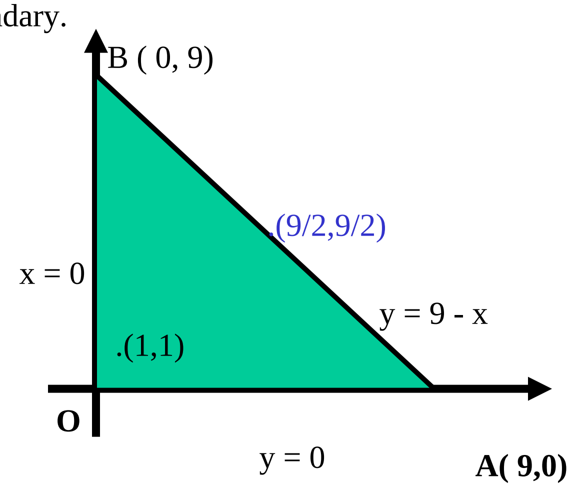
Evaluate f(x,y) at the points obtained in the previous steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

Example:

Find the absolute maximum and minimum value of

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

On the triangular plate in the first quadrant bounded by the lines x=0,y=0,y=9-x
Since f is a differentiable, the only places where f can assume these values are points inside the triangle having vertices at O(0,0), A(9,0)and B(0,9) where $f_x = f_y = 0$ and points of boundary.



<u>For interior points:</u>

We have $f_x=2-2x=0$ and $f_y=2-2y=0$
yielding the single point (1,1)

For boundary points we take take the triangle one side at time :

1. On the segment OA, y=0
   $U(x) = f(x, 0)=2+2x-x^2$

may be regarded as function of x defined on the closed interval $0\le x\le 9$ Its extreme values may occur at the endpoints x=0 and x=9 which corresponds to points (0, 0) and (9, 0) and U(x) has critical point where

$U'(x) = 2-2x=0$ Then x=1

On the segment OB, x=0 and

$V(y)=f(0,y)= 2+2y-y^2$

Using symmetry of function f, possible points are (0,0),(0,9) and (0,1)

3. The interior points of AB.
   With y = 9 - x, we have
   $f(x, y) = 2+2x+2(9-x)-x^2-(9-x)^2$
   $W(x) = f(x, 9-x) = -61 + 18x - 2x^2$
   Setting $w'(x)= 18 -4x = 0$, $x = 9/2$.
   At this value of x,$y = 9 - 9/2$
   Therefore we have $(\frac{9}{2},\frac{9}{2})$ as a critical point.

(x, y)	$(0, 0)$	$(9, 0)$	$(1, 0)$	$\left(\frac{9}{2}, \frac{9}{2}\right)$
$f(x, y)$	2	-61	3	$\frac{-41}{2}$

(x, y)	$(0, 9)$	$(0, 1)$	$(1, 1)$
$f(x, y)$	-61	3	4

The absolute maximum is 4 which f assumes at the point (1,1) The absolute minimum is -61 which f assumes at the points (0, 9) and (9,0)

EXAMPLE
Find the absolute maximum and the absolute minimum values of

$$f(x,y)=3xy-6x-3y+7$$

on the closed triangular region **r** with the vertices (0,0), (3,0) and (0,5) .

$$f(x, y) = 3xy - 6x - 3y + 7$$
$$f_x(x, y) = 3y - 6, \quad f_y(x, y) = 3x-3$$
For critical points
$$f_x(x, y) = 0$$
$$3y - 6 = 0$$
$$y = 2$$
$$f_y(x, y) = 0$$
$$3x - 3 = 0$$
$$x = 1$$

Thus, (1, 2) is the only critical point in the interior of R. Next, we want to determine the location of the points on the boundary of R at which the absolute extrema might occur. The boundary extrema might occur. The boundary each of which we shall treat separately.

(i) The line segment between (0, 0) and (3, 0):
On this line segment we have y=0 so (1) simplifies to a function of the single variable x,

$$u(x)=f(x, 0) = -6x + 7, 0 \leq x \leq 3$$

This function has no critical points because $u'(6)=-6$ is non zero for all x . Thus, the extreme values of u(x) occur at the endpoints x = 0 and x=3 , which corresponds to the points (0, 0) and (3,0) on R

(ii) The line segment between the (0,0) and (0,5)
On this line segment we have x=0 ,so single variable y,

$$v(y) = f(0, y) = -3y + 7, \quad 0 \le y \le 5$$

This function has no critical points because $v'(y)=-3$ is non zero for all y.Thus ,the extreme values of v(y) occur at the endpoints y = 0 and y=5 which correspond to the point (0,0) and (0,5) or R

(iii) The line segment between (3,0) and (0,5)

In the xy-plane, an equation for the line segment

$$y = -\frac{5}{3}x + 5, \qquad 0 \le x \le 3$$

So (1) simplifies to a function of the single variable x,

$$w(x) = f\left(x, -\frac{5}{3}x + 5\right)$$

$$= -5x^2 + 14x - 8, \quad 0 \le x \le 3$$

$w'(x) = -10x + 14$

$w'(x) = 0$

$10x + 14 = 0$

$x = \frac{7}{5}$

This shows that $x = \frac{7}{5}$ is the only critical point of w. Thus, the extreme values of w occur either at the critical point $x = \frac{7}{5}$ or at the endpoints x=0 and x=3.The endpoints correspond to the points (0, 5) and (3, 0) of R, and from (6) the critical point corresponds to $\left[\frac{7}{5}, \frac{8}{3}\right]$

(x, y)	(0, 0)	(3, 0)	(0, 5)	$\left(\frac{7}{5}, \frac{8}{3}\right)$	(1, 2)
f(x,y)	7	-11	-8	$-\frac{9}{5}$	1

Finally, table list the values of $f(x, y)$ at the interior critical point and at the points on the boundary where an absolute extremum can occur. From the table we conclude that the absolute maximum value of f is $f(0,0) = 7$ and the absolute minimum values is $f(3,0) = -11$.

OVER VIEW:
Maxima and Minima of functions of two variables.
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Exercise: 16.9
Q #26,27,28,29.

LECTURE No. 17

EXAMPLES

EXAMPLE

Find the absolute maximum and minimum values of f(x,y)=xy-x-3y on the closed triangular region R with vertices (0, 0), (0, 4), and (5, 0).

$$\boldsymbol{f(x,y) = xy - x - 3y} \qquad \textbf{(1)}$$

$f_x(x, y) = y - 1, \quad f_y(x, y) = x - 3$

For critical points

$f_x(x, y) = 0, \; y - 1 = 0$

$$y = 1 \qquad (2)$$

$f_y(x, y) = 0, \; 3x - 3 = 0$

$$x = 3 \qquad (3)$$

Thus, (3, 1) is the only critical point in the interior of R. Next, we want to determine the location of the points on the boundary of R at which the absolute extrema might occur. The boundary of R consists of three line segments, each of which we shall treat separately.

(i) The line segment between (0, 0) and (5,0)

On this line segment we have y = 0, so (1) simplifies to a function of the single variable x,

$$u(x) = f(x, 0) = - x, \; 0 \le x \le 5 \qquad (4)$$

The function has no critical points because the $u'(x)=-1$ is non zero for all x. Thus, the extreme values of u(x) occur at the endpoints x=0 and x=5 , which corresponds to the points (0,0) and (5,0) of R.

ii) The line segment between (0,0) and (0,4)

On this line segment we have x = 0, so (1) simplifies to a function of the single variable y,

$$v(y) = f(0, y) = -3y, \; 0 \le y \le 4. \qquad (5)$$

This function has no critical points because $v'(y) = -3$ is nonzero for all y. Thus, the extreme values of v(y) occur at the endpoints y =0 and y=4 ,which correspond to the point (0,0) and (0,4) or R.

iii) The line segment between (5,0) and (0,4)

In the xy-plan, an equation is

$$\mathbf{y} = -\frac{4}{5}x + 4, \; 0 \le x \le 5 \qquad (6)$$

so (1) simplifies to a function of the single variable x,

$$w(x) = f(x, -\frac{4}{5}x + 4)$$

$$= x(-\frac{4}{5}x + 4) - x - 3(-\frac{4}{5}x + 4)$$

$$= -\frac{4}{5}x^2 + \frac{27}{5}x - 12$$

$$w'(x) = -\frac{8}{5}x + \frac{27}{5}$$

$$by\ w'(x) = 0,\ we\ get\ x = \frac{27}{8}$$

This shows that $x = \frac{27}{8}$ is the only critical point of w. Thus, the extreme values of w occur either at the critical point $x = \frac{27}{8}$ or at the endpoints x = 0 and x = 5. The endpoints correspond to the points (0, 4) and (5, 0) of R, and from (6) the critical point corresponds to $\left[\frac{27}{8}, \frac{13}{10}\right]$

(x, y)	(0, 0)	(5, 0)	(0, 4)	(27/8, 13/10)	(3, 1)
$f(x, y)$	0	-5	-12	-231/80	-3

Finally, from the table below, we conclude that the absolute maximum value of f is f (0,0) = 0 and the absolute minimum value is $f(0, 4)$=-12

Example

Find three positive numbers whose sum is 48 and such that their product is as large as possible

Let x,y and z be the required numbers, then we have to maximize the product

f(x,y)=xy(48-x-y)

Since

$f_x=48y-2xy-y^2$, $f_y=48x-2xy-x^2$

solving $f_x=0$, $f_y=0$

we get x=16, y=16, z=16

Since x+y+z=48

$f_{xx}(x,y) = -2y$, $f_{xx}(16, 16) = -32 < 0$

$f_{xy}(x, y) = 48-2x-2y$, $f_{xy}(16, 16) = -16$

$f_{yy}(x, y) = -2x$, $f_{yy}(16, 16) = -32$

$D=f_{xx}(16,16)f_{yy}(16,16)-f^2_{xy}(16,16)$

$= (-32)(-32) - (16)^2 = 768 > 0$

For x = 16, y = 16 we have z = 16 since x + y + z = 48
Thus, the required numbers are 16, 16, 16.

Example

Find three positive numbers whose sum is 27 and such that the sum of their squares is as small as possible

Let x, y, z be the required numbers, then
we have to

$$f(x,y) = x^2 + y^2 + z^2$$
$$= x^2 + y^2 + (27 - x - y)^2$$

Since x+y+z = 27

$f_x = 4x+2y-54, \quad f_y = 2x+4y-54,$
$f_{xx} = 4, \quad f_{yy} = 4, \quad f_{xy} = 2$

Solving $f_x = 0, \quad f_y = 0$
We get x = 9, y = 9, z = 9
Since x + y + z = 27

$D = f_{xx}(9, 9)\, f_{yy}(9, 9) - [f_{xy}(9, 9)]^2$
$= (4)(4) - 2^2 = 12 > 0$

This shows that f is minimum
x = 9, y = 9, z = 9, so the required numbers are 9, 9, 9.

Example

Find the dimensions of the rectangular box of maximum volume that can be inscribed in a sphere of radius 4.

Solution:

The volume of the parallelepiped with dimensions x, y, z is

$$V = xyz$$

Since the box is inscribed in the sphere of radius 4, so equation of sphere is $x^2 + y^2 + z^2 = 4^2$ from this equation we can write $z = \sqrt{16 - x^2 - y^2}$ and putting this value of "z" in above equation we get $V = xy\sqrt{16 - x^2 - y^2}$ .Now we want to find out the maximum value of this volume, for this we will calculate the extreme values of the function "V". For extreme values we will find out the critical points and for critical points we will solve the equations $V_x=0$ and $V_y=0$ .Now we have

$$V_x = y\sqrt{16 - x^2 - y^2} + \frac{xy(-2x)}{2\sqrt{16 - x^2 - y^2}}$$

$$\Rightarrow V_x = y\left\{\frac{-2x^2 - y^2 + 16}{\sqrt{16 - x^2 - y^2}}\right\} \quad Now \quad V_x = 0 \Rightarrow y\left\{\frac{-2x^2 - y^2 + 16}{\sqrt{16 - x^2 - y^2}}\right\} = 0$$

$$\Rightarrow -2x^2 - y^2 + 16 = 0 \Rightarrow 2x^2 + y^2 = 16 \ldots\ldots\ldots\ldots\ldots\ldots(a)$$

Similarly we have

$$V_y = x\sqrt{16-x^2-y^2} + \frac{xy(-2y)}{2\sqrt{16-x^2-y^2}}$$

$$\Rightarrow V_y = x\left\{\frac{-x^2-2y^2+16}{\sqrt{16-x^2-y^2}}\right\} \; Now \; V_y = 0 \Rightarrow x\left\{\frac{-x^2-2y^2+16}{\sqrt{16-x^2-y^2}}\right\} = 0$$

$$\Rightarrow -x^2-2y^2+16=0 \Rightarrow x^2+2y^2=16......................(b)$$

Solving equations (a) and (b) we get the $x = \frac{4}{\sqrt{3}}$ and $y = \frac{4}{\sqrt{3}}$

Now $V_{xx} = \frac{xy(2x^2+3y^2-48)}{(16-x^2-y^2)^{\frac{3}{2}}}$ (We obtain this by using quotient rule of differentiation)

$$V_{xx}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}) = -\frac{16}{\sqrt{3}} \prec 0$$

Also we have to calculate $V_{yy} = \frac{xy(3x^2+2y^2-48)}{(16-x^2-y^2)^{\frac{3}{2}}}$ and $V_{yy}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}) = -\frac{16}{\sqrt{3}} \prec 0$ Also

note that $V_{xy}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}) = -\frac{8}{\sqrt{3}}$ Now as we have the formula for the second order partial derivative is $f_{xx}.f_{yy} - (f_{xy})^2$ and putting the values which we calculated above we note that $f_{xx}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}).f_{yy}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}) - (f_{xy}(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}))^2 = +\frac{320}{3} \succ 0$ Which shows that the function V has maximum value when $x = \frac{4}{\sqrt{3}}$ and $y = \frac{4}{\sqrt{3}}$. So the dimension of the rectangular box are $x = \frac{4}{\sqrt{3}}, y = \frac{4}{\sqrt{3}}$ and $z = \frac{4}{\sqrt{3}}$.

Example: A closed rectangular box with volume of 16 ft^3 is made from two kinds of materials. The top and bottom are made of material costing Rs. 10 per square foot and the sides from material costing Rs.5 per square foot. Find the dimensions of the box so that the cost of materials is minimized

Let x, y, z, and C be the length, width, height, and cost of the box respectively. Then it is clear form that

C=10(xy+xy)+5(xz+xz)+5(yz+yz)---------------(1)

C=20xy+10(x+y)z

The volume of the box is given by

xyz=16----------------------------------(2)

Putting the value of z from (2) in (1), we have

$$C = 20xy + 10(x+y)\frac{16}{xy}$$

$$C = 20xy + \frac{160}{y} + \frac{160}{x}$$

$$C_x = 20y - \frac{160}{x^2}, \; C_y = 20x - \frac{160}{y^2}$$

For critical points

$C_x = 0$

$20y - \frac{160}{x^2} = 0$ and $C_y = 0$

$20x - \frac{160}{y^2} = 0$

Solving these equations, we have $x = 2$, $y = 2$. Thus the critical point is (2, 2).

$C_{xx}(x, y) = \frac{320}{x^3}$

$C_{xx}(2, 2) = \frac{320}{8} = 40 > 0$

$C_{yy}(x, y) = \frac{320}{y^3}$

$C_{yy}(2, 2) = \frac{320}{8} = 40$

$C_{xy}(x, y) = 20$

$C_{xy}(2, 2) = 20$

$C_{xx}(2,2)\, C_{yy}(2,2) - C^2_{xy}(2,2) = (40)(40)-(20)^2 = 1200 > 0$

This shows that S has relative minimum at x = 2 and y = 2. Putting these values in (2), we have z = 4, so when its dimensions are $2 \times 2 \times 4$.

Example

Find the dimensions of the rectangular box of maximum volume that can be inscribed in a sphere of radius a.

Solution:

The volume of the parallelepiped with dimensions x, y, z is

$$V = xyz$$

Since the box is inscribed in the sphere of radius 4, so equation of sphere is $x^2 + y^2 + z^2 = 4^2$ from this equation we can write $z = \sqrt{a^2 - x^2 - y^2}$ and putting this value of "z" in above equation we get $V = xy\sqrt{a^2 - x^2 - y^2}$ .Now we want to find out the maximum value of this volume, for this we will calculate the extreme values of the function "V". For extreme values we will find out the critical points and for critical points we will solve the equations $V_x = 0$ and $V_y = 0$ .Now we have

$$V_x = y\sqrt{a^2 - x^2 - y^2} + \frac{xy(-2x)}{2\sqrt{a^2 - x^2 - y^2}}$$

$$\Rightarrow V_x = y\left\{\frac{-2x^2 - y^2 + a^2}{\sqrt{a^2 - x^2 - y^2}}\right\} \quad Now \;\; V_x = 0 \Rightarrow y\left\{\frac{-2x^2 - y^2 + a^2}{\sqrt{a^2 - x^2 - y^2}}\right\} = 0$$

$$\Rightarrow -2x^2 - y^2 + 16 = 0 \Rightarrow 2x^2 + y^2 = a^2 \,......................(a)$$

Similarly we have

$$V_y = x\sqrt{a^2 - x^2 - y^2} + \frac{xy(-2y)}{2\sqrt{a^2 - x^2 - y^2}}$$

$$\Rightarrow V_y = x\left\{\frac{-x^2 - 2y^2 + a^2}{\sqrt{a^2 - x^2 - y^2}}\right\} \; Now \;\; V_y = 0 \Rightarrow x\left\{\frac{-x^2 - 2y^2 + a^2}{\sqrt{a^2 - x^2 - y^2}}\right\} = 0$$

$$\Rightarrow -x^2 - 2y^2 + a^2 = 0 \Rightarrow \; x^2 + 2y^2 = a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(b)$$

Solving equations (a) and (b) we get the $x = \frac{a}{\sqrt{3}} \;\; and \;\; y = \frac{a}{\sqrt{3}}$

Now $V_{xx} = \frac{xy(2x^2 + 3y^2 - 3a^2)}{(a^2 - x^2 - y^2)^{\frac{3}{2}}}$ (We obtain this by using quotient rule of differentiation)

$$V_{xx}\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) = -\frac{a^2}{\sqrt{3}} \prec 0$$

Also we have to calculate $V_{yy} = \frac{xy(3x^2 + 2y^2 - 3a^2)}{(a^2 - x^2 - y^2)^{\frac{3}{2}}}$ and $V_{yy}\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) = -\frac{a^2}{\sqrt{3}}$ Also note that $V_{xy}\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) = -\frac{2a}{\sqrt{3}}$ Now as we have the formula for the second order partial derivative is $f_{xx}.f_{yy} - (f_{xy})^2$ and putting the values which we calculated above we note that $f_{xx}\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right).f_{yy}\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right) - \left(f_{xy}\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)\right)^2 = +\frac{20a^2}{3} \succ 0$ Which shows that the function V has maximum value when $x = \frac{a}{\sqrt{3}} \;\; and \;\; y = \frac{a}{\sqrt{3}}$. So the dimension of the rectangular box are $x = \frac{a}{\sqrt{3}}, y = \frac{a}{\sqrt{3}} \;\; and \;\; z = \frac{a}{\sqrt{3}}$.

Example: Find the points of the plane x + y + z = 5 in the first octant at which $f(x,y,z) = xy^2z^2$ has maximum value.

Solution: Since we have $f(x,y,z) = xy^2z^2$ and we are given the plane x + y + z = 5 from this equation we can write x = 5 – y – z . Thus our function "f' becomes $f((5 - y - z),y,z) = (5 - y - z)y^2z^2$ Say this function u(y,z) That is $u(y,z) = (5 - y - z)y^2z^2$ Now we have to find out extrema of this function. On simplification we get

$u(y,z) = 5y^2z^2 - y^3z^2 - y^2z^3$

$u_y = 10yz^2 - 3y^2z^2 - 2yz^3$

$\quad = yz^2(10 - 3y - 2z)$

$u_z = 10y^2z - 2y^3z - 3y^2z^2$

$\quad = y^2z(10 - 2y - 3z)$

$u_y = 0, \qquad u_z = 0$

$y = 0, \qquad z = 0$

$10 - 3y - 2z = 0$

$10 - 2y - 3z = 0$

On solving above equations we get $-10 + 5z = 0 \Rightarrow z = 2$ and $10 - 3y - 4 = 0 \Rightarrow y = 2$

$u_{yy} = 10z^2 - 6yz^2 - 2z^3$
$u_{zz} = 10y^2 - 2y^3 - 6y^2z$
$u_{yz} = 20yz - 6y^2z - 6yz^2$
at
$y = 2, \quad z = 2$
$u_{yy}(2,2) = 40 - 48 - 16 = -24 < 0$
$y_{zz}(2,2) = 40 - 16 - 48 = -24$
$u_{yz}(2,2) = 80 - 48 - 48 = -16$
$D = u_{yy}\, u_{zz} - (uyz)^2$
$= (-24)(-24) - (-16)^2$
$= 576 - 256$
$= 320 > 0$
For $y = 2$ and $z = 2$
We have $x = 5 - 2 - 2 = 1$

Example: Find all points of the plane $x+y+z=5$ in the first octant at which $f(x,y,z)=xy^2z^2$ has a maximum value.
Solution:

$f(x,y,z) = xy^2z^2 = xy^2(5 - x - y)^2$
Since $x+y+z = 5$
$f_x = y^2(5-3x-y)(5-x-y)$,
$f_y = 2xy(5-x-2y)(5-x-y)$
Solving $f_x = 0$, $f_y = 0$, we get
$x=1, y=2, z=2 \therefore x+y+z = 5$

$f_{xx}=-y^2(5-3x-y) - 3y^2(5-x-5)$
$f_{xy}=2y(5-x-y)(5-3x-y)-y^2(5-3x-y)$
$\quad - y^2(5 - x - y)$
$f_{yy}=2x(5-x-y)(5-x-2y)-2xy(5-x-2y)$
$\quad - 4xy(5 - x - y)$
$f_{xx}(1, 2, 2) = -24 < 0$
$f_{yy}(1, 2, 2) = -16$
$f_{xy}(1, 2, 2) = -8$
$f_{xx} f_{yy} - (f_{xy})^2 = (-24)(-16)-(-8)^2$
$= 320 > 0$
Hence "f" has maximum value when x = 1 and y = 2. Thus the points where the function has maximum value is x = 1,y = 2 and z = 2.

LECTURE No. 18

REVISON OF INTEGRATION

Example 1: Consider the following integral $\int_0^1 (xy+y^2)dx$

Integrating with respect to x, keeping y constant, we get

$$\int_0^1 (xy+y^2)dx = y\int_0^1 xdx + y^2\int_0^1 1dx$$

$$= y\left|\frac{x^2}{2}\right|_0^1 + y^2\left|x\right|_0^1 = y(\frac{1}{2}-0) + y^2(1-0)$$

$$\int_0^1 (xy+y^2)dx = \frac{y}{2} + y^2$$

Example 2: Consider the following integral $\int_0^1 (xy+y^2)dy$

Integrating with respect to y, keeping x constant, we get

$$\int_0^1 (xy+y^2)dy = x\int_0^1 ydy + \int_0^1 y^2dy$$

$$= x\left|\frac{y^2}{2}\right|_0^1 + \left|\frac{y^3}{3}\right|_0^1 = \frac{x}{2}(1^2-0^2) + \frac{1}{3}(1^3-0^3)$$

$$\Rightarrow \int_0^1 (xy+y^2)dy = \frac{x}{2} + \frac{1}{3}$$

Double Integral

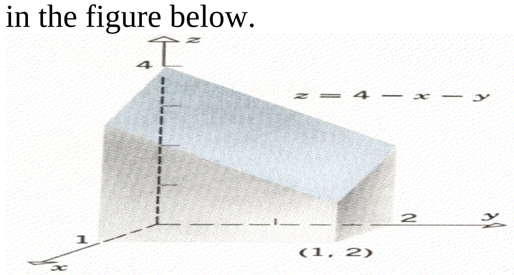
Symbolically, the double integral of two variables x and y over the certain region R of the $xy-$plane is denoted by $\iint_R f(x,y)dxdy$.

Example: Use a double integral to find out the solid bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = \{(x,y): 0 \le x \le 1, 0 \le y \le 2\}$

Solution: We have to find the region "R"out the volume "V" over that is,

$$V = \iint_R (4-x-y)dA$$

And the solid is shown in the figure below.



$$V = \iint_R (4-x-y)dA = \int_0^2\int_0^1 (4-x-y)dxdy = \int_0^2 \left| 4x - \frac{x^2}{2} - xy \right|_{x=0}^{x=1} dy$$

$$= \int_0^2 \left(\frac{7}{2} - y\right)dy = \left|\frac{7}{2}y - \frac{y^2}{2}\right|_0^2 = 5$$

Example 3: Evaluate the double integral $\int_0^1\int_0^1 (xy+y^2)dxdy$

Solution: $\int_0^1\int_0^1 (xy+y^2)dxdy = \int_0^1 \left(y\left|\frac{x^2}{2}\right|_0^1 + y^2 \left|x\right|_0^1 \right)dy = \int_0^1 \left(\frac{y}{2} + y^2\right)dy$

$$= \left|\frac{y^2}{4} + \frac{y^3}{3}\right|_0^1 = \frac{1}{4}\left(1^2 - 0^2\right) + \frac{1}{3}\left(1^3 - 0^3\right) = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

Example 4: Evaluate the double integral $\int_0^1\int_0^1 (xy+y^2)dydx$

Solution: First we will integrate the given function with respect to y and our integral becomes

$$\int_0^1\int_0^1 (xy+y^2)dydx = \int_0^1 \left(x\left|\frac{y^2}{2}\right|_0^1 + \left|\frac{y^3}{3}\right|_0^1 \right)dx$$

$$= \int_0^1 \left(\frac{x}{2}\left(1^2 - 0^2\right) + \frac{1}{3}\left(1^3 - 0^3\right)\right)dx = \int_0^1 \left(\frac{x}{2} + \frac{1}{3}\right)dx$$

$$\int_0^1\int_0^1 (xy+y^2)dydx = \left|\frac{x^2}{4} + \frac{x}{3}\right|_0^1 = \frac{1}{4}\left(1^2 - 0^2\right) + \frac{1}{3}\left(1 - 0\right) = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

Remarks: The example 3 and example 4 show that

$$\int_0^1\int_0^1 (xy+y^2)dxdy = \int_0^1\int_0^1 (xy+y^2)dydx = \frac{7}{12}$$

Iterated or Repeated Integral

The expression $\int_c^d \left[\int_a^b f(x,y)dx\right]dy$ is called iterated or repeated integral. Often the brackets are omitted and this expression is written as $\int_c^d\int_a^b f(x,y)dxdy = \int_c^d \left[\int_a^b f(x,y)dx\right]dy$ in which $\int_a^b f(x,y)dx$ yields a function of y , which is then integrated over the interval $c \le y \le d$.

Similarly $\int_a^b\int_c^d f(x,y)dydx=\int_a^b\left[\int_c^d f(x,y)dy\right]dx$ in which $\int_c^d f(x,y)dy$ yields a function of x which is then integrated over the interval $a\le x\le b$.

Example: Evaluate the integral $\int_0^1\int_0^2 (x+3)dydx$.

Solution: Here we will first integrate with respect to y and get a function of x then we will integrate that function with respect to x to get the required answer. So

$$\int_0^1\int_0^2 (x+3)dydx=\int_0^1 (x+3)\left|y\right|_0^2 dx$$

$$=\int_0^1 (x+3)\left(2-0\right)dx=\int_0^1 2(x+3)dx=2\left|\frac{x^2}{2}+3x\right|_0^1$$

$$=2\left(\frac{1}{2}\left(1^2-0^2\right)+3\left(1-0\right)\right)=2\left(\frac{1}{2}+3\right)=7$$

Now if we change the order of integration, so we get $\int_0^2\int_0^1 (x+3)dxdy$, then we have

$$\int_0^2\int_0^1 (x+3)dxdy=\int_0^2\left|(\frac{x^2}{2}+3x)\right|_0^1 dy$$

$$=\int_0^2\left(\frac{1}{2}\left(1^2-0^2\right)+3\left(1-0\right)\right)dy=\int_0^2\left(\frac{1}{2}+3\right)dy=\int_0^2\frac{7}{2}dy=\frac{7}{2}\left|y\right|_0^2=\frac{7}{2}\left(2-0\right)=7$$

Now you note that the values of the integral remain same if we change the order of integration. Actually we have a stronger result which we state as a theorem.

Theorem: Let R be the rectangle defined by the inequalities **a < x < b** and **c < y < d.** If $f\left(x,\ y\right)$ is continuous on this rectangle, then

$$\iint_R f(x,y)dA=\int_c^d\int_a^b f(x,y)dxdy=\int_a^b\int_c^d f(x,y)dydx.$$

Remark: This powerful theorem enables us to evaluate a double integral over a rectangle by calculating an iterated integral. Moreover, the theorem tells us that the **"order of integration in the iterated integral does not matter".**

Example: Evaluate the integral $\int_0^{\ln 2}\int_0^{\ln 3} e^{x+y}dxdy$

Solution: First we will integrate the function with respect to x . Note that we can write e^{x+y} as $e^x.e^y$

So we have, $\int_0^{\ln 2} e^y\left|e^x\right|_0^{\ln 3} dy=\int_0^{\ln 2} e^y(e^{\ln 3}-e^0)dy=\int_0^{\ln 2} e^y(3-1)dy=2\int_0^{\ln 2} e^y dy$

Here we use the fact that "e" and "ln" are inverse function of each other. So we have $e^{\ln 3} = 3$. Thus we get $\int_0^{\ln 2} e^y \left| e^x \right|_0^{\ln 3} dy = 2\int_0^{\ln 2} e^y dy = 2\left| e^y \right|_0^{\ln 2} = 2(2-1) = 2$

Example: Evaluate the integral $\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y} dydx$

Solution: First we will integrate the function with respect to y. Note that we can write e^{x+y} as $e^x.e^y$ So we have, $\int_0^{\ln 3} e^x \left| e^y \right|_0^{\ln 2} dx = \int_0^{\ln 3} e^x (e^{\ln 2} - e^0) dx = \int_0^{\ln 3} e^x (2-1) dx = \int_0^{\ln 3} e^x dx$

Since "e" and "ln" are inverse function of each other. So we have $e^{\ln 2} = 2$.

Thus we get $\int_0^{\ln 3} e^x \left| e^y \right|_0^{\ln 2} dx = \int_0^{\ln 3} e^x dx = \left| e^x \right|_0^{\ln 3} = (e^{\ln 3} - e^0) = (3-1) = 2$

Note that in both cases our integral has the same value.

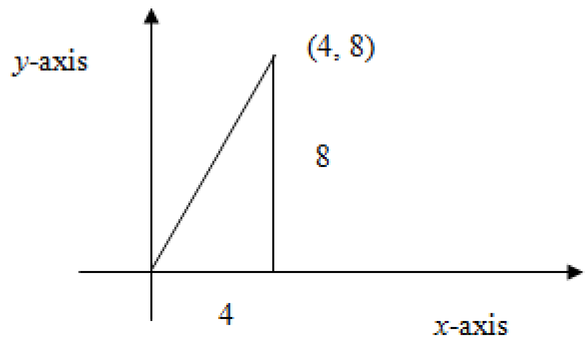
Overview:

Double integrals **Page # 854-857**
Exercise Set 17.1 (page 857): 1, 3, 5, 7, 9, 11, 13, 15, 17, 19

LECTURE No. 19

USE OF INTEGRALS

Area as anti-derivatives



$$\int_0^4 2x\,dx = 2\left|\frac{x^2}{2}\right|_0^4 = \left|x^2\right|_0^4$$

$$= 4^2 - 0^2 = 16$$

Area of triangle $= \frac{1}{2}\,\text{base}\times\text{altitude}$

$$= \frac{1}{2}\times 4\times 8 = 16$$

Volume as anti-derivatives

$$\text{Volume} = \int_0^2\int_0^3 5\,dydx = \int_0^2 5\,|y|_0^3\,dx = \int_0^2 5\,(3-0)\,dx = \int_0^2 15dx = 15|x|_0^2 = 15\,(2-0) = 30$$

Geometrically, $0 \le x \le 2,\ 0 \le y \le 3,\ 0 \le z \le 5$

$$\text{Volume} = 2 \times 3 \times 5 = 30$$

The following results are analogous to the result of the definite integrals of a function of single variable.

THEOREM

1) $\iint_R c\,f(x,y)\,dx\,dy = c\iint_R f(x,y)\,dx\,dy$ where c is a constant.

2) $\iint_R [\,f(x,y) + g(x,y)\,]\,dx\,dy = \iint_R f(x,y)\,dx\,dy + \iint_R g(x,y)\,dx\,dy$

3) $\iint_R [\,f(x,y) - g(x,y)\,]\,dx\,dy = \iint_R f(x,y)\,dx\,dy - \iint_R g(x,y)\,dx\,dy$

Example: Use double integral to find the volume under the surface $z = 3x^3 + 3x^2y$ and the rectangle $\{(x,y): 1 \le x \le 3,\ 0 \le y \le 2\}$.

Solution:

$$\text{Volume} = \int_0^2\int_1^3 \left(3x^3 + 3x^2y\right)\,dx\,dy = \int_0^2 \left|\frac{3x^4}{4} + \frac{3x^3y}{3}\right|_1^3 dy = \int_0^2 \left|\frac{3}{4}\left(3^4 - 1^4\right) + y\left(3^3 - 1^3\right)\right|_1^3 dy$$

$$= \int_0^2 \left|\frac{3}{4}(81-1) + y(27-1)\right| dy = \int_0^2 (60 + 26y)\,dy = \left|60y + \frac{26y^2}{2}\right|_0^2$$

$$= \left|60(2-0) + 13\left(2^2 - 0^2\right)\right|_0^2 = |120+52| = 172$$

Example: Use double integral to find the volume of solid in the first octant enclosed by the surface $z = x^2$ and the planes $x=2,\ y=0,\ y=3$ and $z=0$.

Solution: Volume = $\int_0^2\int_0^3 x^2\ dy\ dx = \int_0^2 x^2 \left|y\right|_0^3 dx = \int_0^2 x^2(3-0)\,dx$

$$= 3\int_0^2 x^2\ dx = 3\left|\frac{x^3}{3}\right|_0^2 = \left(2^3-0^3\right) = 8$$

SOME RESULTS:

1) $\iint_R f(x,y)\ dA \geq 0 \quad if \quad f(x,y) \geq 0$ on R.

2) $\iint_R f(x,y)\ dA \geq \iint_R c\,g(x,y)\ dA \quad if \quad f(x,y) \geq g(x,y)$

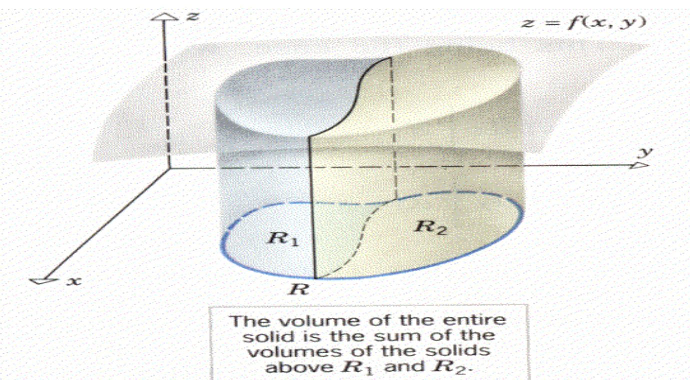
If $f(x,y)$ is nonnegative on a region R, then subdividing R into two regions R_1 and R_2 has the effect to subdividing the solid between R and $z=f(x,y)$ into two solids, the sum of whose volumes is the volume of the entire solid.

$$\iint_R f(x,y)\ dA = \iint_{R_1} f(x,y)\ dA + \iint_{R_2} f(x,y)\ dA$$

The volume of the solid S can also be obtained, using cross sections perpendicular to y-axis.

$$Vol(S) = \int_c^d A(y)\ dy \qquad ---------(1)$$

Where $A(y)$ represents the area of the cross section perpendicular to $y-$axis, taken at the point y.



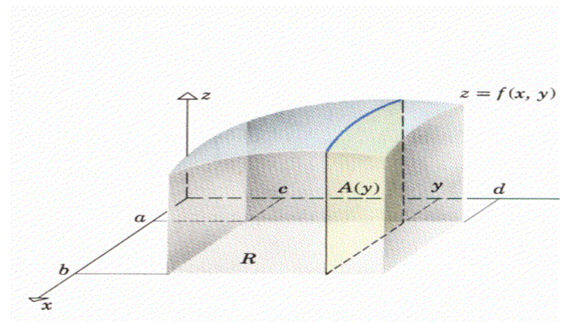
<u>**How to compute cross sectional area**</u>

For each fixed y in the interval $c \leq y \leq d$, the function $f(x,y)$ is a function of x alone, and $A(y)$ may be viewed as the area under the graph of this function along the interval $a < x < b$,

Thus $\qquad A(y) = \int_a^b f(x,y)\,dx$

Substituting this expression in (1), we get

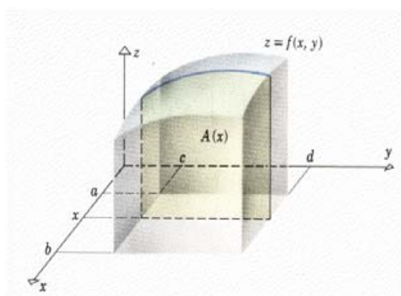
$$\mathrm{Vol}(S) = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy = \int_c^d \int_a^b f(x,y)\,dx\,dy \quad --------(2)$$



Similarly, the volume of the solid S can also be obtained, perpendicular to $x-$axis.

$$\mathrm{Vol}(S) = \int_a^b A(x)\,dx \quad ------(3)$$

Where $A(x)$ represents the area of the cross section perpendicular to $x-$axis, taken at the point x.



For each fixed x in the interval $a \le x \le b$, the function $f(x, y)$ is a function of y alone, and $A(x)$ is given by $A(x) = \int_c^d f(x,y)\,dy$ Substituting this expression in (3), we get

$$Vol(S) = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx = \int_a^b \int_c^d f(x,y)\,dy\,dx \quad --------(4)$$

By equations (2) and (4), $\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$

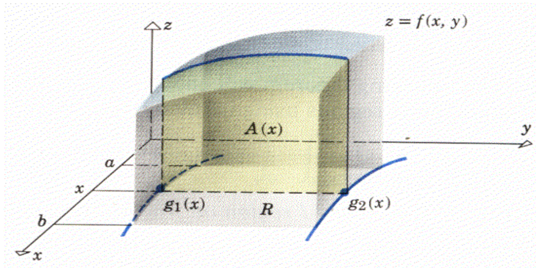
Double integral for non-rectangular region

Type I region is bounded the left and right by the vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where

$$g_1(x) \le g_2(x) \quad \text{for } a \le x \le b$$

If R is a type I region on which $f(x, y)$ is continuous, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



By the method of cross section, the volume of S is also given by

$$\text{Vol}(S) = \int_a^b A(x)\,dx \quad ----- (5)$$

where $A(x)$ is the area of the cross section at the fixed point x and this cross section area extends from $g_1(x)$ to $g_2(x)$ in the $y-$direction,

So $$A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

Using it in equation (5), we get $\text{Vol}(S) = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

The volume of S is also given by $\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

Type II region is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded in the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying

$$h_1(y) \le h_2(y) \quad \text{for } c \le y \le d.$$

If R is a type II region on which $f(x, y)$ is continuous,

then $\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

Similarly, the partial definite integral $\int_c^d f(x,y)\,dy$ with respect to y is evaluated by holding x fixed and integrating with respect to y. The integral of the form $\int_c^d f(x,y)\,dy$ produces a function of x.

LECTURE No.20

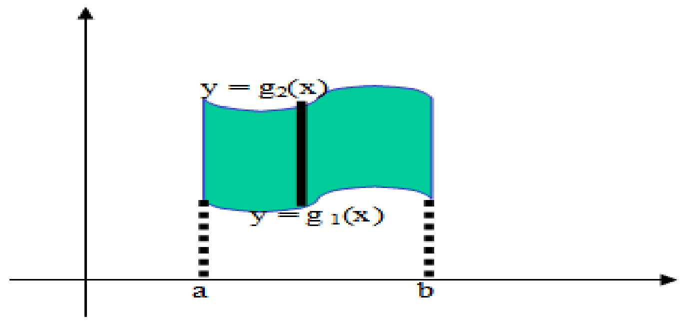
DOUBLE INTEGRAL FOR NON-RECTANGULAR REGION

Double Integral for Non-rectangular Region

Type I region is bounded the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by curves

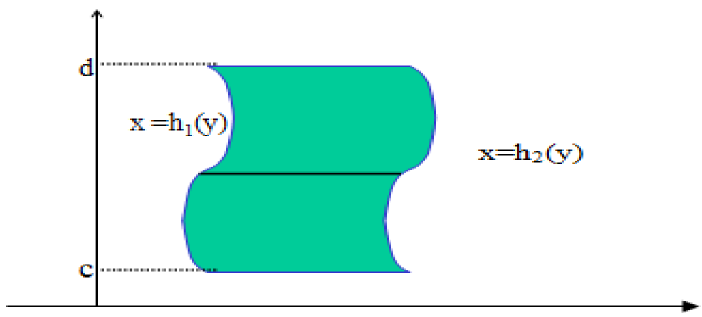
$y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \le g_2(x)$ for $a \le x \le b$

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



Type II region is bounded below and above by the horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by the continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(y) \le h_2(y)$ for $c \le y \le d$

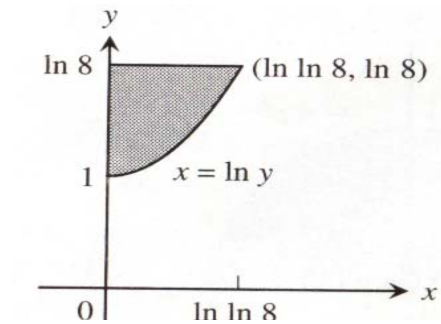
$$\iint_R f(x,y)\, dx\,dy = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dx\,dy$$



Write double integral of the function *f(x,y)* on the region whose sketch is given

$$\int_1^{\ln 8} \int_0^{\ln y} f(x,y)\,dx\,dy$$

$$\int_0^{\ln(\ln 8)} \int_{e^x}^{\ln 8} f(x,y)\,dy\,dx$$



Solution:

$$I=\int_{1}^{\ln 8}\int_{0}^{\ln y} f(x,y)\,dx\,dy$$

Here, limits of x are $0\le x\le \ln y$ $-----(1)$

limits of y are $1\le y\le \ln 8$ $-----(2)$

Take logrithm of each sides of (2), $0\le \ln y\le \ln(\ln 8)$ $---(3)$ $\because \ln 1=0$

Compare (1) *and* (3), $0\le x\le \ln y\le \ln(\ln 8)$ $-----(4)$

From (4), $0\le x\le \ln(\ln 8)$ and $x\le \ln y\le \ln(\ln 8)$

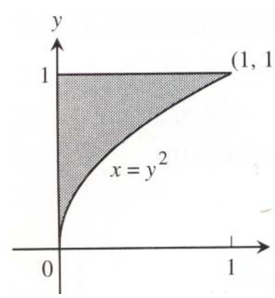
$e^x\le y\le \ln 8$

So, $$I=\int_{0}^{\ln(\ln 8)}\int_{e^x}^{\ln 8} f(x,y)\,dy\,dx$$

Write double integral of the function *f(x,y)*on the region whose sketch is given

$$\int_{0}^{1}\int_{0}^{y^2} f(x,y)\,dx\,dy$$

$$\int_{0}^{1}\int_{\sqrt{x}}^{1} f(x,y)\,dy\,dx$$



Solution : $I=\int_{0}^{1}\int_{0}^{y^2} f(x,y)\,dx\,dy$

Here, limits of x are $0\le x\le y^2$ $\Rightarrow$ $0\le \sqrt{x}\le y$ $---(1)$ by taking square root

limits of y are $0\le y\le 1$ $-----(2)$

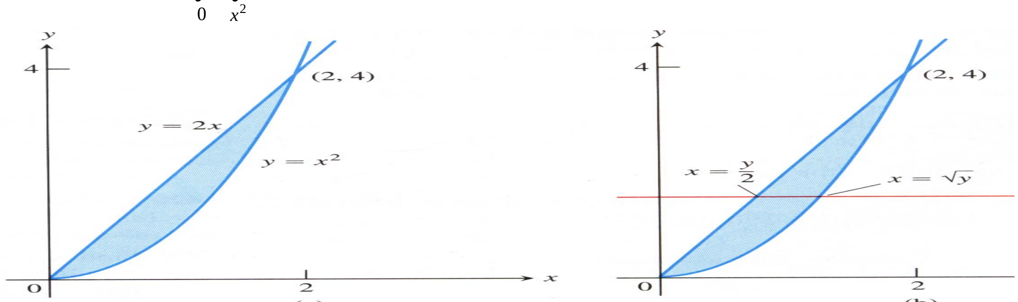
Compare (1) and (2), $0\le \sqrt{x}\le y\le 1$ $-----(3)$

From (3), $0\le \sqrt{x}\le 1$ and $\sqrt{x}\le y\le 1$

$0^2\le \left(\sqrt{x}\right)^2\le 1^2$ and $\sqrt{x}\le y\le 1$

$0\le x\le 1$ and $\sqrt{x}\le y\le 1$

EXAMPLE: Draw the region and evaluate an equivalent integral with the order of integration reversed. $\int_{0}^{2}\int_{x^2}^{2x}(4x+2)\,dy\,dx$



The region of integration is given by the inequalities $x^2 \le y \le 2x$ and $0 \le x \le 2$.

limits of y are $x^2 \le y \le 2x$ ----(1)

limits of x are $0 \le x \le 2$ *or* $0 \le 2x \le 4$ ----(2)

Compare (1) and (2), $0 \le x^2 \le y \le 2x \le 4$ ----(3)

From (3), $0 \le y \le 4$ and $x^2 \le y,\ y \le 2x$

$$x \le \sqrt{y},\ \frac{y}{2} \le x \quad or \quad \frac{y}{2} \le x \le \sqrt{y}$$

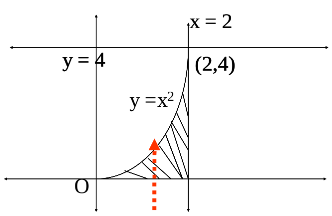
$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (4x+2)\,dx\,dy = \int_0^4 \left| 4\frac{x^2}{2} + 2x \right|_{\frac{y}{2}}^{\sqrt{y}} dy$$

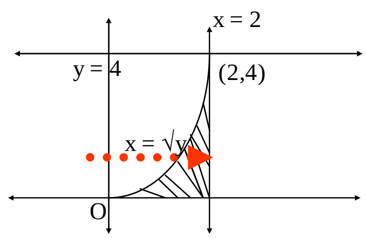
$$= \int_0^4 \left(2y - \frac{y^2}{2}\right) + \left(2\sqrt{y} - y\right) dy = \int_0^4 \left(2\sqrt{y} + y - \frac{y^2}{2}\right) dy$$

$$= \left| 2\frac{y^{\frac{3}{2}}}{\frac{3}{2}} + \frac{y^2}{2} - \frac{1}{2}\frac{y^3}{3} \right|_0^4 = \left| \frac{4}{3}y^{\frac{3}{2}} + \frac{y^2}{2} - \frac{y^3}{6} \right|_0^4 = \frac{4}{3}(8) + \frac{1}{2}(16) - \frac{1}{6}(64) = 8$$

EXAMPLE

Evaluate $I = \int_0^4 \int_{\sqrt{y}}^2 y\cos x^5\,dx\,dy$. The integral is over the region $0 \le y \le 4$, $x = \sqrt{y}$ *and* $x = 2$.





Solution : For reversing the limits of the integral,

limits of x are $\sqrt{y} \le x \le 2$ *or* $y \le x^2 \le 4$ ----(1)

limits of y are $0 \le y \le 4$ ----(2)

By (1) and (2), $0 \le y \le x^2 \le 4$ ----(3)

By (3), $0 \le y \le x^2$ and $0 \le x^2 \le 4$

$0 \le x \le 2$

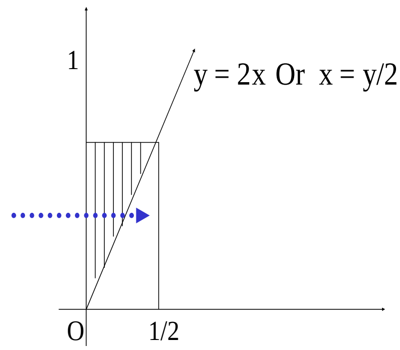
$$I = \int_0^2 \int_0^{x^2} y\cos x^5\,dy\,dx = \int_0^2 \left| \frac{y^2}{2} \right|_0^{x^2} \cos x^5\,dx = \int_0^2 \frac{1}{2}\left(\left(x^2\right)^2 - 0^2\right)\cos x^5\,dx$$

$$= \frac{1}{2}\int_0^2 x^4 \cos x^5\,dx = \frac{1}{2\times 5}\int_0^2 \cos x^5 \left(5x^4\right)dx = \frac{1}{10}\left|\sin x^5\right|_0^2 = \frac{1}{10}\sin 32$$

EXAMPLE: Evaluate $I = \int_0^{\frac{1}{2}} \int_{2x}^{1} e^{y^2}\, dy dx$.

Solution: The integral cannot be evaluated in the given order, since e^{y^2} has mo anti-derivative. So we shall change the order of integration. The region R in which integration is performed is given by $0 \le x \le \frac{1}{2}$, $y = 2x$ and $y = 1$.

The region is enclosed by $x = 0$, $x = \frac{y}{2}$ and $0 \le y \le 1$



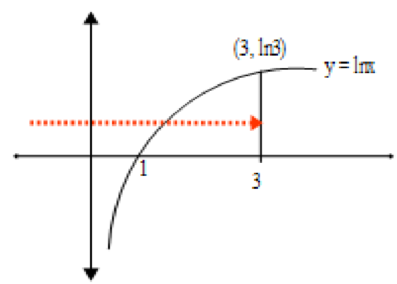
$$I = \int_0^1 \int_0^{\frac{y}{2}} e^{y^2}\, dx\, dy = \int_0^1 e^{y^2} \left| x \right|_0^{\frac{y}{2}} dy$$

$$= \int_0^1 \frac{y}{2} e^{y^2}\, dy = \frac{1}{2\times 2}\int_0^1 2y e^{y^2}\, dy$$

$$= \frac{1}{4}\left| e^{y^2} \right|_0^1 = \frac{1}{4}(e-1)$$

EXAMPLE: Evaluate $I = \int_1^3 \int_0^{\ln x} x\, dy dx$.

Solution: The region R in which integration is performed is given by



Limits of x are $1 \le x \le 3$

$\ln 1 \le \ln x \le \ln 3$

$0 \le \ln x \le \ln 3 \quad ---(1)$

Limits of y are $0 \le y \le \ln x \quad ---(2)$

By (1) and (2), $0 \le y \le \ln x \le \ln 3 \quad ---(3)$

From (3), $y \le \ln x \le \ln 3$ and $0 \le y \le \ln 3$

$e^y \le x \le 3$ and $0 \le y \le \ln 3$

$$I = \int_0^{\ln 3} \int_{e^y}^{3} x\, dx\, dy = \int_0^{\ln 3} \left| \frac{x^2}{2} \right|_{e^y}^{3} dy = \frac{1}{2}\int_0^{\ln 3} \left(9 - e^{2y}\right) dy$$

$$= \frac{1}{2}\left| 9y - \frac{e^{2y}}{2} \right|_0^{\ln 3} = \frac{1}{2}\left(9\ln 3 - \frac{1}{2}\left(e^{\ln 3^2} - e^0\right)\right) = \frac{1}{2}(9\ln 3 - 4) = \frac{9}{2}\ln 3 - 2$$

Over view of Lecture # 20

Book Calculus by Howard Anton

Chapter # 17 Article # 17.2

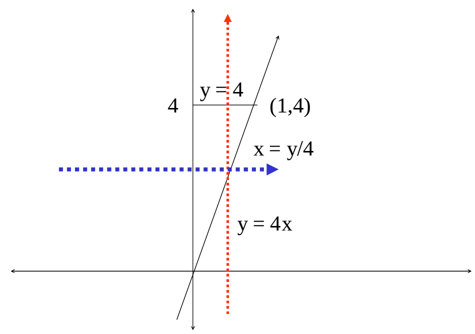
Page (858-863) Exercise set 17.2 (21, 22, 23, 25, 27, 35, 37, 38)

LECTURE No. 21

EXAMPLES

Example

$$\int_0^1 \int_{4x}^4 e^{-y^2}\, dy\, dx$$



Reversing the order of integration

$$\int_0^4 \int_0^{y/4} e^{-y^2}\, dx\, dy$$

$$= \int_0^4 \left| x\, e^{-y^2} \right|_0^{y/4} dy \quad = \int_0^4 \frac{y}{4}\, e^{-y^2}\, dy$$

$$= \frac{-1}{8} \int_0^4 e^{-y^2} (-2y)\, dy$$

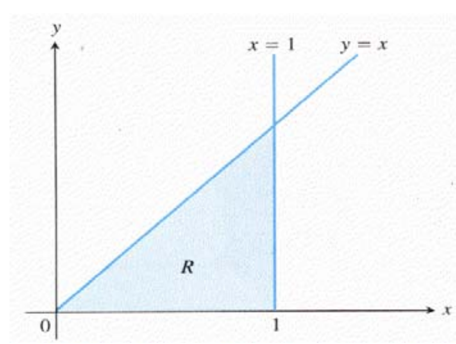
$$= -\frac{1}{8} \left| e^{-y^2} \right|_0^4 = -\frac{1}{8} \left| e^{-16} - c^0 \right|$$

$$= \frac{1}{8}\left(1 - \frac{1}{e^{16}}\right)$$

Example : Calculate $\iint_R \frac{\sin x}{x} dA$, where R is the triangle in *xy*-plane bounded by the *x*-axis, the line $y = x$ and the line $x=1$.

Solution : $\int_0^1 \left(\int_0^x \frac{\sin x}{x} dy \right) dx = \int_0^1 \left(\frac{\sin x}{x} \left| y \right|_0^x \right) dx = \int_0^1 \left(\frac{\sin x}{x} (x-0) \right) dx$

$$= \int_0^1 \sin x\, dx = -\left| \cos x \right|_0^1 = -(\cos 1 - \cos 0) \approx 0.46$$



Example

$$\int_0^2 \int_{y/2}^1 e^{x^2}\,dxdy$$

Since there is no elementary antiderivative of e^{x^2}, the integral

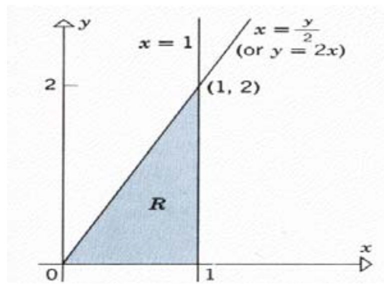
$$\int_0^2 \int_{y/2}^1 e^{x^2}\,dxdy$$

cannot be evaluated by performing the x-integration first.

To evaluate this integral , we express is as an equivalent iterated integral with the order if integration reversed . For the inside integration, y is fixed and x varies from he line $x = y/2$ to the line $x = 1$. For the outside integration, y varies from 0 to 2, so the given iterated integral is equal to a double integral over the triangular region R.

To reverse the order of integration, we treat R as a type I region, which enables us to write the given integral as

$$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\,dxdy$$



By changing the order of integration we get,

$$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\,dxdy = \int_0^1 \int_0^{2x} e^{x^2}\,dydx$$

$$= \int_0^1 [e^{x^2} y]_{y=0}^{2x}\,dx$$

$$= \int_0^1 2xe^{x^2}\,dx$$

$$= e^{x^2}\Big|_0^1 = e - 1$$

Example

Use a double integral to find the volume of the solid that is bounded above by the palne $z = 4-x-y$ and below by the rectangle $R = \{(x,y): 0 \le x \le 1,\ 0 \le y \le 2\}$

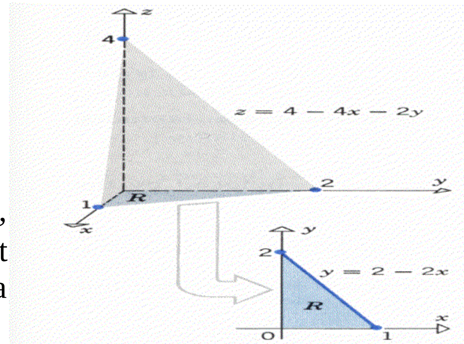
$$V = \iint_R (4-x-y)\,dA = \int_0^2 \int_0^1 (4-x-y)\,dx\,dy$$

$$= \int_0^2 \left| 4x - \frac{x^2}{2} - xy \right|_0^1 dy = \int_0^2 \left(\frac{7}{2} - y\right) dy = \left| \frac{7}{2}y - \frac{y^2}{2} \right|_0^2 = 5$$

Example
Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane z=4-4x-2y The tetrahedron is bounded above by the plane.

$$z=4-4x-2y \quad \text{------------------(1)}$$

and below by the triangular region R



Thus, the volume is given by

$$V = \iint_R (4 - 4x - 2y)\, dA$$

The region R is bounded by the x-axis, the y-axis, and the line y = 2 – 2x [set z = 0 in (1)], so that treating R as a type I region yields.

$$V = \iint_R (4-4x-2y)\, dA$$

$$= \int_0^1 \int_0^{2-2x} (4 - 4x - 2y)\, dy\, dx$$

$$= \int_0^1 \left[4y-4xy-y^2\right]_{y=0}^{2-2x} dx$$

$$= \int_0^1 (4-8x+4x^2)dx$$

$$= \frac{4}{3}$$

Example : Find the volume of the solid bounded by the cylinder $x^2+y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

The solid is bounded above by the plane z = 4 – y and below by the region R within the circle $x^2 + y^2 = 4$. The volume is given by

$$V = \iint_R (4 - y)\, dA$$

Treating R as a type I region we obtain

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\, dy\, dx$$

$$= \int_{-2}^{2} \left[4y - \frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 8\sqrt{4-x^2}\, dx$$

$$= 8\left| \frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2} \right|_{-2}^{2}$$

$$= 8\left| 2\sin^{-1}(1) - 2\sin^{-1}(-1) \right|$$

$$= 8[2(\frac{\pi}{2}) + 2(\frac{\pi}{2})]$$

$$= 8(2\pi) = 16\pi$$

Example

Use double integral to find the volume of the solid that is bounded above by the paraboiled $z=9x^2 + y^2$,below by the plane $z=0$ and laterally by the planes

$x = 0, \quad y = 0, \quad x = 3, \quad y = 2$

$$\text{Volume} = \int_0^3\int_0^2 (9x^2 + y^2)\, dy\, dx$$

$$= \int_0^3 \left[9x^2y + \frac{y^3}{3}\right]_0^2 dx$$

$$= \int_0^3 \left(18x^2 + \frac{8}{3}\right) dx$$

$$= \left| 6x^3 + \frac{8}{3}x \right|_0^3$$

$$= 6(27) + 8$$

$$= 170$$

LECTURE No. 22

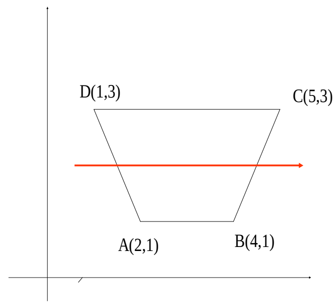
EXAMPLES

Example: Evaluate $\iint\limits_R xy\ dA$, where R is the region bounded by the trapezium with the vertices $(1,\,3)$, $(5,\,3)$, $(2,\,1)$ and $(4,\,1)$.

Solution: Slope of AD $= \dfrac{3-1}{1-2} = \dfrac{2}{-1} = -2$ Since Slope $= m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Equation of line AD

$$y - y_1 = m\left(x - x_1\right)$$
$$y - 1 = -2\left(x - 2\right)$$
$$y - 1 = -2x + 4$$
$$y - 5 = -2x$$
$$x = -\frac{y-5}{2}$$



Slope of BC $= \dfrac{3-1}{5-4} = \dfrac{2}{1} = 2$

Equation of line BC: $y - y_2 = m\left(x - x_2\right)$

$$y - 1 = 2\left(x - 4\right) \quad \text{or} \quad y + 7 = 2x \quad \Rightarrow \quad x = \frac{y+7}{2}$$

Limits of x are from $x = -\dfrac{y-5}{2}$ to $x = \dfrac{y+7}{2}$.

Limits of y are from 1 to 3.

$$\iint\limits_R xy\ dA = \int_1^3 \int_{-\frac{y-5}{2}}^{\frac{y+7}{2}} xy\ dx\ dy = \int_1^3 y\left|\frac{x^2}{2}\right|_{-\frac{y-5}{2}}^{\frac{y+7}{2}} dy = \frac{1}{2}\int_1^3 y\left|\left(\frac{y+7}{2}\right)^2 - \left(-\frac{y-5}{2}\right)^2\right| dy$$

$$= \frac{1}{2}\int_1^3 y\left\{\frac{y^2+49+14y}{4} - \frac{y^2+25-10y}{4}\right\} dy = \frac{1}{2}\int_1^3 y\left\{\frac{24+24y}{4}\right\} dy$$

$$= \int_1^3 \left(3y+3y^2\right) dy = \left|3\frac{y^2}{2} + 3\frac{y^3}{3}\right|_1^3 = 12 + 26 = 38$$

<u>EXAMPLE</u>: Use double integral to find the volume of the wedge cut from the cylinder $4x^2 + y^2 = 9$ by the plane $z = 0$ and $z = y + 3$

Solution: Since we can write $4x^2 + y^2 = 9$ as $\dfrac{x^2}{(3/2)^2} + \dfrac{y^2}{9} = 1$ This is eq of ellipse.

Now the Lower and upper limits for x are $x = \dfrac{-\sqrt{9-y^2}}{2}$ *and* $x = \dfrac{\sqrt{9-y^2}}{2}$

And upper and lower limits for y are -3 and 3 respectively. So the required volume is given by

$$\int_{-3}^{3}\int_{\frac{-\sqrt{9-y^2}}{2}}^{\frac{\sqrt{9-y^2}}{2}} (y+3)\,dx\,dy = \int_{-3}^{3}\left\{ y\left|x\right|_{\frac{-\sqrt{9-y^2}}{2}}^{\frac{\sqrt{9-y^2}}{2}} + 3\left|x\right|_{\frac{-\sqrt{9-y^2}}{2}}^{\frac{\sqrt{9-y^2}}{2}} \right\} dy$$

$$= \int_{-3}^{3}\left\{ y\left|\frac{\sqrt{9-y^2}}{2} - \left(-\frac{\sqrt{9-y^2}}{2}\right)\right| + 3\left|\frac{\sqrt{9-y^2}}{2} - \left(-\frac{\sqrt{9-y^2}}{2}\right)\right|\right\} dy$$

$$= \int_{-3}^{3}\left\{ y\sqrt{9-y^2} + 3\sqrt{9-y^2}\right\} dy = \frac{-1}{2}\int_{-3}^{3}\sqrt{9-y^2}\,(-2y)\,dy + 3\int_{-3}^{3}\sqrt{9-y^2}\,dy$$

$$= \frac{-1}{2}\times\frac{2}{3}\left|\left(9-y^2\right)^{\frac{3}{2}}\right|_{-3}^{3} + 3\left\{\left|\frac{y}{2}\sqrt{9-y^2}\right|_{-3}^{3} + \left|\frac{9}{2}\sin^{-1}\left(\frac{y}{3}\right)\right|_{-3}^{3}\right\}$$

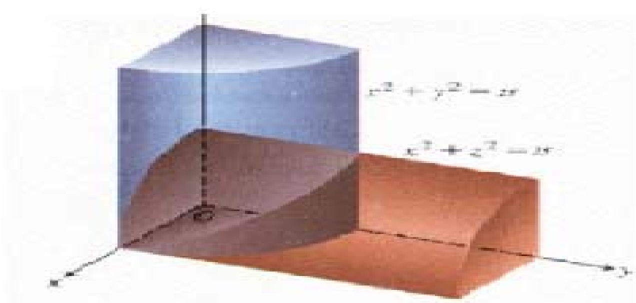
$$= 0 + 3\left\{\left|\frac{3}{2}\sqrt{0} + \frac{3}{2}\sqrt{0}\right| + \frac{9}{2}\left|\sin^{-1}(1) - \sin^{-1}(-1)\right|\right\} = \frac{27\pi}{2}$$

EXAMPLE: Use double integral to find the volume of solid common to the cylinders $x^2 + y^2 = 25$ and $x^2 + z^2 = 25$.

Solution: From $x^2 + y^2 = 25$ or $y^2 = 25 - x^2$ or $y = \pm\sqrt{25 - x^2}$

Radius of cylinder is 5, so limits of x is from 0 to 5.

From $x^2 + z^2 = 25 \Rightarrow z = \sqrt{25 - x^2}$ Only +ve value taken in first octant.



Volume $= 8\times$ Area of cylinder in first Octant

$$= 8\iint_R z\,dA = 8\int_0^5\int_0^{\sqrt{25-x^2}} \sqrt{25-x^2}\,dy\,dx = 8\int_0^5 \sqrt{25-x^2}\,\left|y\right|_0^{\sqrt{25-x^2}}\,dx$$

$$= 8\int_0^5 \sqrt{25-x^2}\left(\sqrt{25-x^2} - 0\right)dx = 8\int_0^5 \left(25 - x^2\right)dx$$

$$=8\left|25x - \frac{x^3}{3}\right|_0^5 = 8\left[25(5-0) - \frac{1}{3}\left(5^3-0^3\right)\right] = \frac{2000}{3}$$

AREA CALCUALTED AS A DOUBLE INTEGRAL

$$V = \iint_R 1\, dA = \iint_R dA \qquad (1)$$

However, the solid has congruent cross sections taken parallel to the xy-plan so that

$$\mathbf{V = area\ of\ base \times height = area\ of\ R \times 1 = area\ of\ R}$$

Combining this with (1) yields the area formula

$$\text{Area of } R = \iint_R dA \qquad (2)$$

EXAMPLE: Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{x^2}{2}$ and the line $y = 2x$.

Solution: The required area is between the parabola $y = \frac{x^2}{2}$ -------(1)

and the line $y = 2x$ -------(2)

By (1) and (2), $\frac{x^2}{2} = 2x \quad \Rightarrow \quad x^2 = 4x \quad \Rightarrow \quad x = 0,\ x = 4$

$$\text{Area of } R = \iint_R dA = \int_0^4 \int_{y=\frac{x^2}{2}}^{y=2x} dy\, dx = \int_0^4 \left|y\right|_{y=\frac{x^2}{2}}^{y=2x} dx = \int_0^4 \left\{2x - \frac{x^2}{2}\right\} dx$$

$$= \left|2\frac{x^2}{2} - \frac{1}{2}\times\frac{x^3}{3}\right|_0^4 = \left(4^2-0^2\right) - \frac{1}{6}\left(4^3-0^3\right) = \frac{16}{3}$$

EXAMPLE: Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

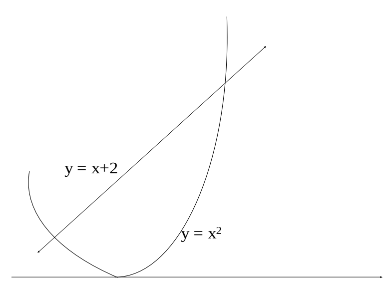
Solution: The required area is between the parabola $y = x^2$ -------(1)

and the line $y = x + 2$ -------(2)

By (1) and (2), $x^2 = x + 2$

$$x^2 - x - 2 = 0$$

$$x = 2,\ x = -1$$



$$\int_{-1}^{2} \int_{y=x^2}^{y=x+2} dy\, dx = \int_{-1}^{2} \left|y\right|_{y=x^2}^{y=x+2} dx$$

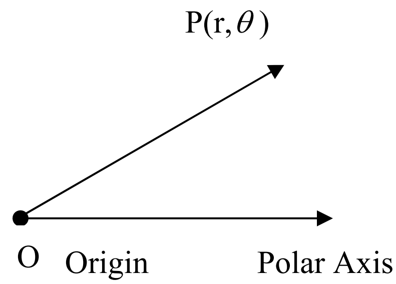
$$= \int_{-1}^{2} \left\{(x+2) - \left(x^2\right)\right\} dx = \left|\frac{x^2}{2}\right|_{-1}^{2} + \left|2x\right|_{-1}^{2} - \left|\frac{x^3}{3}\right|_{-1}^{2} = \frac{1}{2}(4-1) + 2(2+1) - \frac{1}{3}(8+1) = \frac{9}{2}$$

LECTURE No. 23

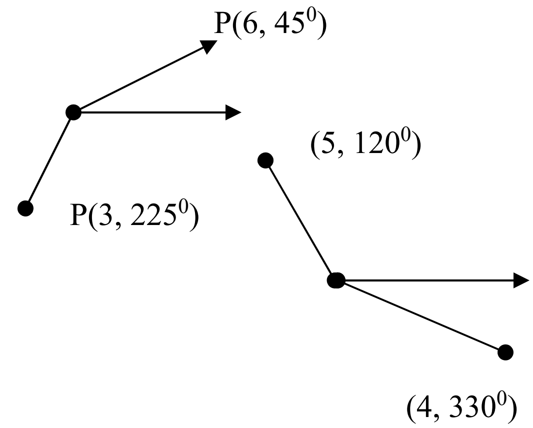
POLAR COORDINATE SYSTEMS

POLAR COORDIANTE SYSTEMS

To form a polar coordinate system in a plane, we pick a fixed point O, called the origin or pole, and using the origin as an endpoint we construct a ray, called the polar axis. After selecting a unit of measurement, we may associate with any point P in the plane a pair of polar coordinates (r, θ), where r is the distance from P to the origin and θ measures the angle from the polar axis to the line segment OP.



The number r is called the radial distance of P and θ is called a polar angle of P. In the points (6, 45°), (3, 225°), (5, 120°), and (4, 330°) are plotted in polar coordinate systems.



THE POLAR COORDINATES OF A POINT ARE NOT UNIQUE.

For example, the polar coordinates

(1, 315°), (1, – 45°), and (1, 675°)

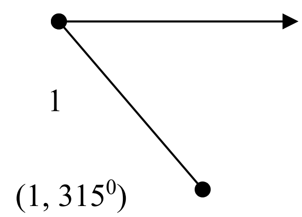
all represent the same point

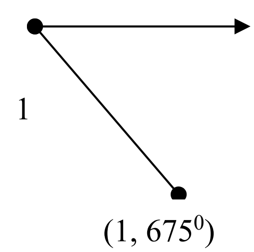
In general, if a point P has polar coordinate (r, θ), then for any integer n=0,1,2,3,…….

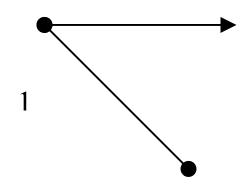
$(r, \theta + n \cdot 360^0)$ and $(r, \theta + n \cdot 360^0)$

are also polar coordinates of p

In the case where P is the origin, the line segment OP reduces to a point, since r = 0. Because there is no clearly defined polar angle in this case, we will agree that an arbitrary Polar angle θ may be used. Thus, for every θ may be used. Thus , for every θ, the point (0, θ) is the origin.





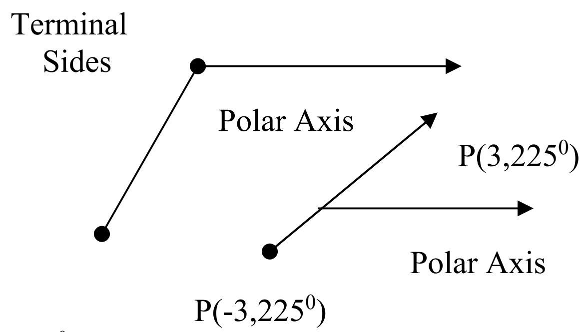


(1, -45^0)

NEGATIVE VALUES OF R

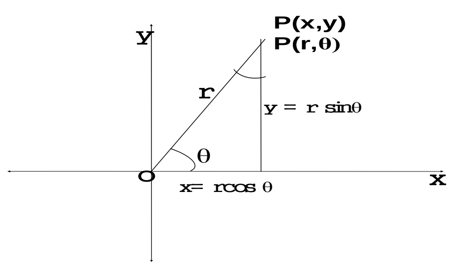
When we start graphing curves in polar coordinates, it will be desirable to allow negative values for r. This will require a special definition. For motivation, consider the point P with polar coordinates $(3, 225^0)$ We can reach this point by rotating the polar axis 225^0 and then moving forward from the origin 3 units along the terminal side of the angle. On the other hand, we can also reach the point P by rotating the polar axis 45^0 and then moving backward 3 units from the origin along the extension of the terminal side of the angle

This suggests that the point $(3, 225^0)$ might also be denoted by $(-3, 45^0)$ with the minus sign serving to indicate that the point is on the extension of angle's terminal side rather than on terminal side itself.

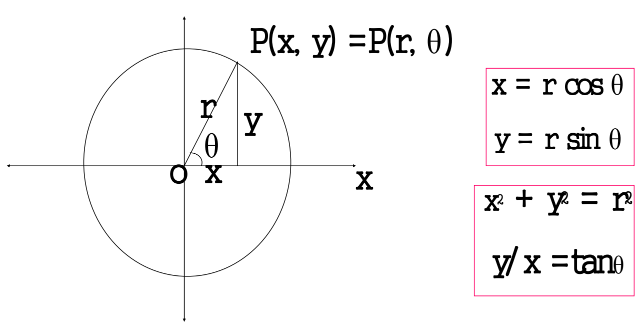


Since the terminal side of the angle $\theta + 180^0$ is the extension of the terminal side other angle θ,We shall define.
(-r, θ) and (r, $\theta + 180^0$) to be polar coordinates for the same point .
With r = 3 and $\theta = 45^0$ in (2) if follows that $(-3, 45^0)$ and $(3, 225^0)$ represent the same point.

RELATION BETWEEN POLAR AND RECTANGULAR COORDINATES



CONVERSION FORMULA FROM POLAR TO CARTESIAN COORDINATES AND VICE VERSA

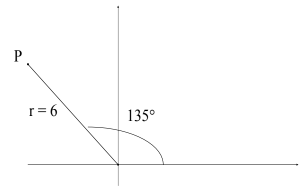


Example :
Find the rectangular coordinates of the point P whose polar coordinates are $(6, 135^0)$

Solution:

Substituting the polar coordinates,

r = 6 and $\theta = 135^0$ in x = r cosθ and y = r sinθ yields



$x = 6 \cos 135° = 6\,(-\sqrt{2}/2) = -3\sqrt{2}$

$y = 6 \sin 135° = 6\,(\sqrt{2}/2) = 3\sqrt{2}$

Thus, the rectangular coordinates of the point P are ($-3\sqrt{2}$, $3\sqrt{2}$)

Example: Find polar coordinates of the point P whose rectangular coordinates are $(-2, 2\sqrt{3})$

Solution: We will find polar coordinates (r, θ) of P such that r >0 and $0 \le \theta \le 2\pi$.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$$

$$\tan\theta = \frac{y}{x} = \frac{2\sqrt{3}}{-2} = -\sqrt{3} \Rightarrow \theta = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$$

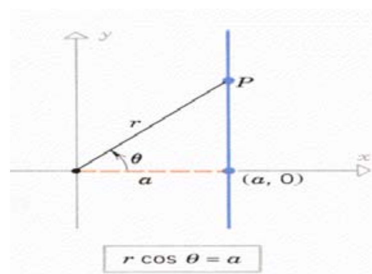
From this we have (-2, $2\sqrt{3}$) lies in the second quadrant of P. All other Polar coordinates of P have the form

$$(4, \frac{2\pi}{3} + 2n\pi) \quad \text{or} \quad (-4, \frac{5\pi}{3} + 2n\pi) \qquad \text{, where } n \text{ is integer}$$

<u>LINES IN POLAR COORDIANTES</u>

A line perpendicular to the x-axis and passing through the point with xy coordinates with (a, 0) has the equation x = a . To express this equation in polar coordinates we substitute x = r cos θ $\Rightarrow$ a = r cos θ --------(1)

This result makes sense geometrically since each point P (r, θ) on this line will yield the value a for r cos θ.

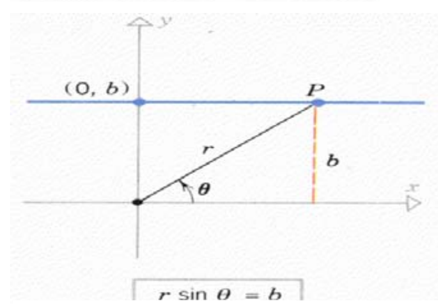


A line parallel to the x-axis that meets the y-axis it the point with xy-coordinates (0, b) has the equation y = b.

Substituting y = r sin θ yields.

r sin θ = b (2)

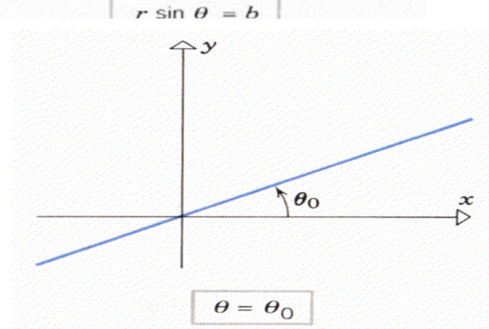
as the polar equation of this line. This makes sense geometrically since each point P (r, θ) on this line will yield the value b for r sin θ



For Any constant θ_0, the equation

$\theta = \theta_0$ (3)

is satisfied by the coordinates of all points of the form P (r, θ_0), regardless of the value of r. Thus, the equation represents the line through the origin making an angle of θ_0 (radians) with the polar axis.



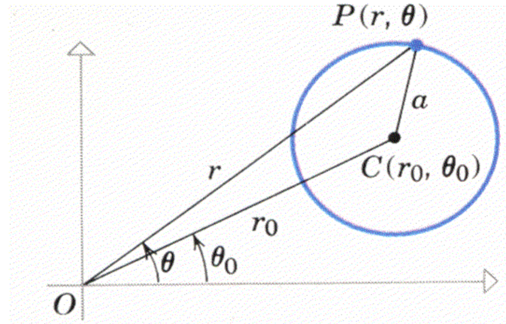
By substitution $x = r\cos\theta$ and $y = r\sin\theta$ in the equation $Ax + By + C = 0$. We obtain the general polar form of the line,

$$r(A\cos\theta + B\sin\theta) + C = 0$$

CIRCLES IN POLAR COORDINATES

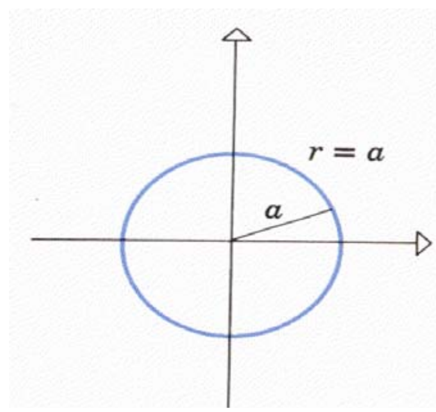
Let us try to find the polar equation of a circle whose radius is a and whose center has polar coordinates (r_0, θ_0). If we let $P(r, \theta)$ be an arbitrary point on the circle, and if we apply the law of cosines to the triangle OCP we obtain

$$r^2 - 2rr_0\cos(\theta - \theta_0) + r_0^2 = a^2 \qquad (1)$$



SOME SPECIAL CASES OF EQUATION OF CIRCLE IN POLAR COORDINATES

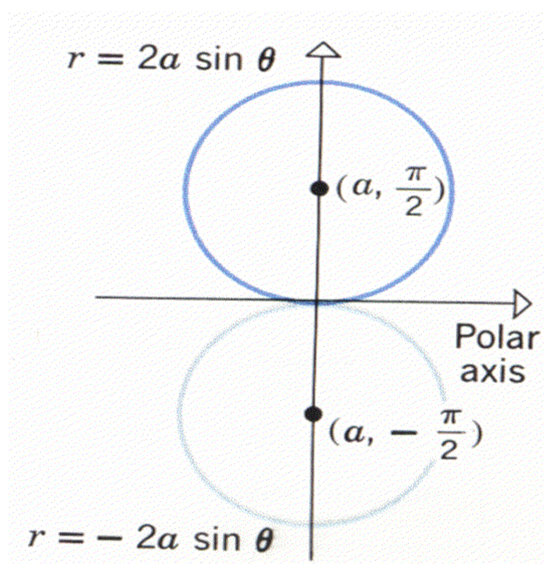
A circle of radius a, centered at the origin, has an especially simple polar equation. If we let $r_0 = 0$ in (1), we obtain $r^2 = a^2$ or, since $a \geq 0$, $r = a$. This equation makes sense geometrically since the circle of radius a, centered at the origin, consists of all points $P(r, \theta)$ for which $r = a$, regardless of the value of θ



If a circle of radius a has its center on the x-axis and passes through the origin, then the polar coordinates of the center are either

$(a, 0)$ or (a, π)

depending on whether the center is to the right or left of the origin



LECTURE No. 24

SKETCHING

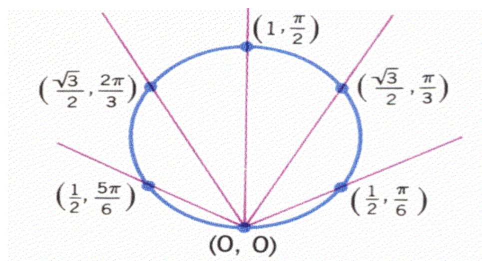
Draw graph of the curve having the equation r = sin θ

By substituting values for θ at increments of $\frac{\pi}{6}(30^0)$ and calculating r, we can construct The following table:

θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
r = sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

θ (radians)	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r = sin θ	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

Note that there are 13 pairs listed in Table, but only 6 points plotted in This is because the pairs from θ = π on yield duplicates of the preceding points. For example, (– ½, 7π/6) and (1/2, π/6) represent the same point. The points appear to lie on a circle.



This is indeed the case may be seen by expressing the given equation in terms of x and y. We first multiply the given equation through by r to obtain $r^2 = r \sin\theta$ which can be rewritten as

$$x^2 + y^2 = y \quad \text{or} \quad x^2 + y^2 - y = 0$$

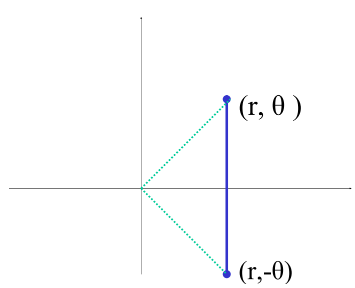
or on completing the square $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$. This is a circle of radius $\frac{1}{2}$ centered at the point $(0, \frac{1}{2})$ in the xy-plane.

Sketching of Curves in Polar Coordinates

1. SYMMETRY

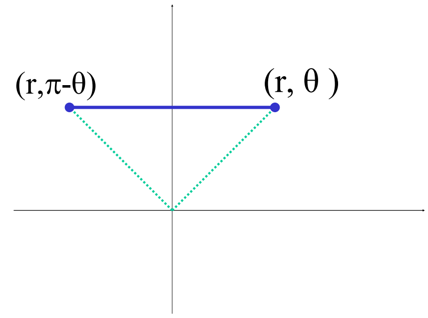
(i) Symmetry about the Initial Line

If the equation of a curve remains unchanged when (r, θ) is replaced by either (r, -θ) in its equation ,then the curve is symmetric about initial line.



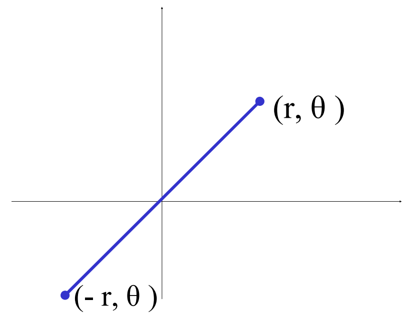
(ii) Symmetry about the y-axis

If when (r, θ) is replaced by either (r, $\pi-\theta$) in The equation of a curve and an equivalent equation is obtained ,then the curve is symmetric about the line perpendicular to the initial i.e, the y-axis



(ii) Symmetry about the Pole

If the equation of a curve remains unchanged when either (-r, θ) or is substituted for (r, θ) in its equation ,then the curve is symmetric about the pole. In such a case, the center of the curve.



2. Position of the Pole Relative to the Curve

See whether the pole on the curve by putting r = 0 in the equation of the curve and solving for θ.

3. Table Of Values

Construct a sufficiently complete table of values. This can be of great help in sketching the graph of a curve.

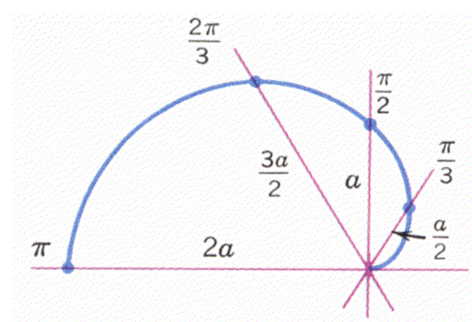
II Position Of The Pole Relative To The Curve.

When r = 0, θ = 0. Hence the curve passes through the pole.

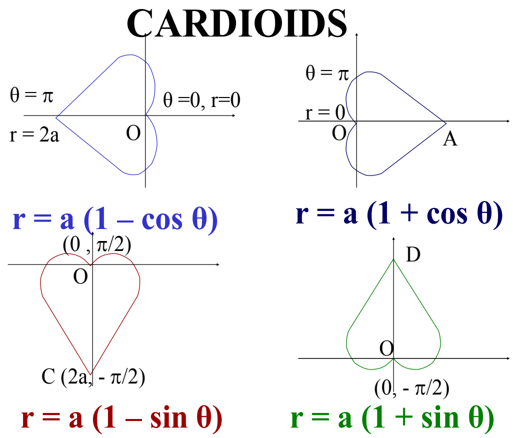
III. Table of Values

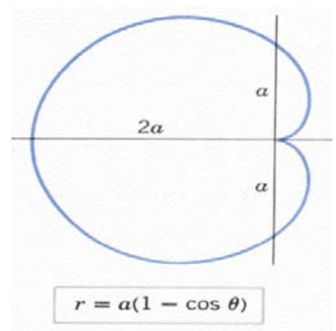
θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
$r=a(1-\cos\theta)$	0	a/2	a	3a/2	2a

As θ varies from 0 to π, $\cos\theta$ decreases steadily from 1 to -1, and $1-\cos\theta$ increases steadily from 0 to 2. Thus, as θ varies from 0 to π, the value of $r = a(1-\cos\theta)$ will increase steadily from an initial value of r = 0 to a final value of r = 2a.



On reflecting the curve in about the x-axis, we obtain the curve.

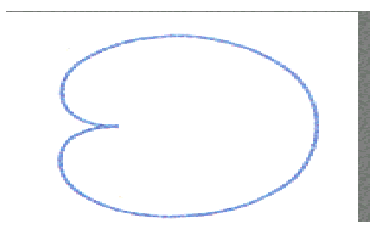


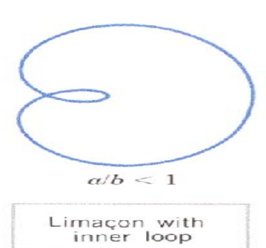


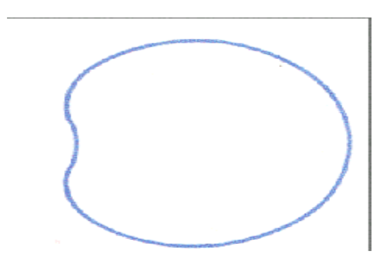
<u>CARDIDOIDS AND LIMACONS</u>

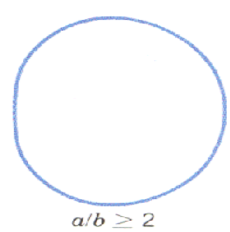
r=a+b sin θ, r = a–bsin θ
r=a+b cos θ, r = a–bcos θ

The equations of above form produce polar curves called limacons. Because of the heart-shaped appearance of the curve in the case a = b, limacons of this type are called cardioids. The position of the limacon relative to the polar axis depends on whether sin θ or cos θ appears in the equation and whether the + or – occurs.







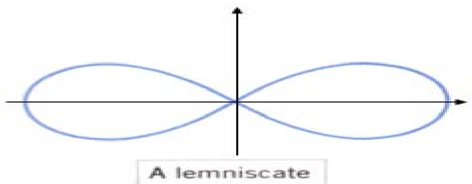


<u>LEMNISCATE</u>

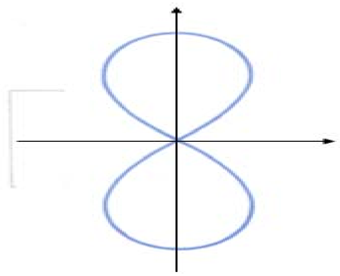
If a > 0, then equation of the form

$$r^2 = a^2 \cos 2\theta, \quad r^2 = -\, a^2 \cos 2\theta$$
$$r^2 = a^2 \sin 2\theta, \quad r^2 = -\, a^2 \sin 2\theta$$

represent propeller-shaped curves, called lemniscates (from the Greek word "lemnicos" for a looped ribbon resembling the Fig 8. The lemniscates are centered at the origin, but the position relative to the polar axis depends on the sign preceding the a^2 and whether sin 2θ or cos 2θ appears in the equation.



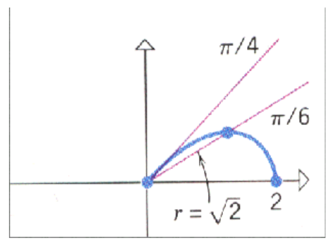
A lemniscate

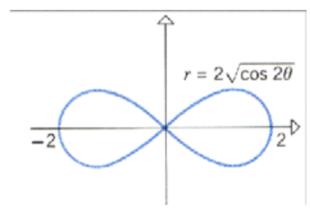


Example

$$\mathbf{r^2 = 4 \cos 2\theta}$$

The equation represents a lemniscate. The graph is **symmetric** about the **x-axis** and the **y-axis**. Therefore, we can obtain each graph by first sketching the portion of the graph in the range $0 \leq \theta < \pi/2$ and then reflecting that portion about the x- and y-axes. The curve passes through the origin when $\theta = \pi/4$, so the line $\theta = \pi/4$ is tangent to the curve at the origin. As θ varies from 0 to π/4, the value of cos2θ decreases steadily from 1 to 0, so that r decreases steadily from 2 to 0.For θ in the range $\pi/4 < \theta < \pi/2$, the quantity cos2θ is negative, so there are no real values of r satisfying first equation. Thus, there are no points on the graph for such θ. The entire graph is obtained by reflecting the curve about the x-axis and then reflecting the resulting curve about the y-axis.



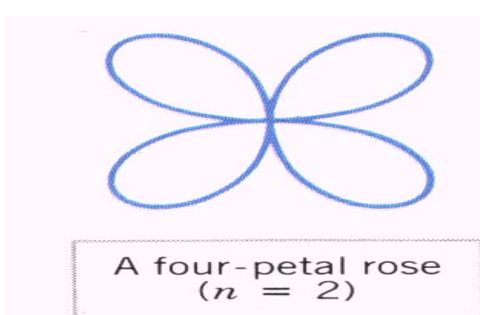


ROSE CURVES

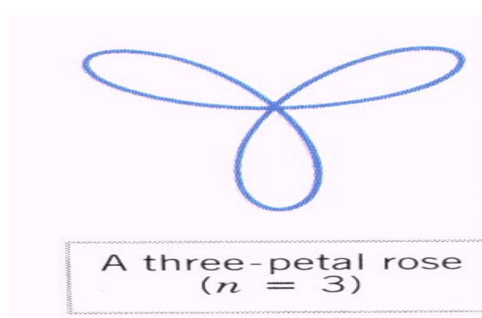
Equations of the form

$$\mathbf{r = a \sin n\theta \quad and \quad r = a \cos n\theta}$$

represent flower-shaped curves called roses. The rose has **n equally spaced petals or loops if n is odd** and **2n equally spaced petals if n is even**



A four-petal rose ($n = 2$)



A three-petal rose ($n = 3$)

The orientation of the rose relative to the polar axis depends on the sign of the constant a and whether $\sin\theta$ or $\cos\theta$ appears in the equation.

SPIRAL

A curve that "winds around the origin" infinitely many times in such a way that r increases (or decreases) steadily as θ increases is called a spiral. The most common example is the spiral of Archimedes, which has an equation of the form.

$$r = a\theta \qquad (\theta \geq 0) \quad \text{or} \quad r = a\theta \qquad (\theta \leq 0)$$

In these equations, θ is in radians and a is positive.

EXAMPLE

Sketch the curve **$r = \theta$ $(\theta \geq 0)$** in polar coordinates.

This is an equation of spiral with a = 1; thus, it represents an Archimedean spiral.

Since $r = 0$ when $\theta = 0$, the origin is on the curve and the polar axis is tangent to the spiral.

A reasonably accurate sketch may be obtained by plotting the intersections of the spiral with the x and y axes and noting that r increases steadily as θ increases. The intersections with the x-axis occur when

$\theta = 0, \ \pi, \ 2\pi, \ 3\pi, \ldots\ldots.$

at which points r has the values

$r = 0, \ \pi, \ 2\pi, \ 3\pi, \ldots..$

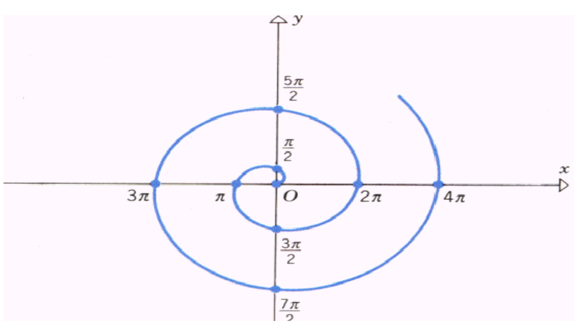
and the intersections with the y-axis occur when

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots\ldots$$

at which points r has the values

$$r = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots\ldots$$

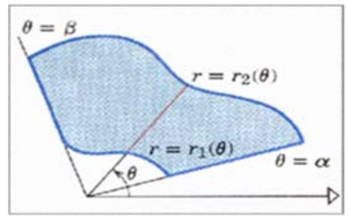
Starting from the origin, the Archimedean spirals $r = \theta$ $(\theta \geq 0)$ loops counterclockwise around the origin.



LECTURE No. 25

DOUBLE INTEGRALS IN POLAR COORDINATES

Double integrals in which the integrand and the region of integration are expressed in polar coordinates are important for two reasons: First, they arise naturally in many applications, and second, many double integrals in rectangular coordinates are more easily evaluated if they are converted to polar coordinates. The function $z = f(r,\theta)$ to be integrated over the region R as shown in the figure.



INTEGRALS IN POLAR COORDIATES

When we define the double integral of a function over a region *R* in the *xy*-plane, we begin by cutting R into rectangles whose sides are parallel to the coordinate axes. These are the natural shapes to use because their sides have either constant x-values or constant y-values. In polar coordinates, the natural shape is a "polar rectangle" whose sides have constant r and θ - values.

Suppose that a function $f(r,\theta)$ is defined over a region R that is bounded by the ray $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = r_1(\theta)$ and $r = r_2(\theta)$. Suppose also that $0 \le r_1(\theta) \le r_2(\theta) \le a$ for every value of θ between α and β. Then R lies in a fan-shaped region Q defined by the inequalities $0 \le r \le a$ and $\alpha \le \theta \le \beta$.

Then the double integral in polar coordinates is given as

$$\iint_R f(r,\theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r,\theta)\, dr\, d\theta$$

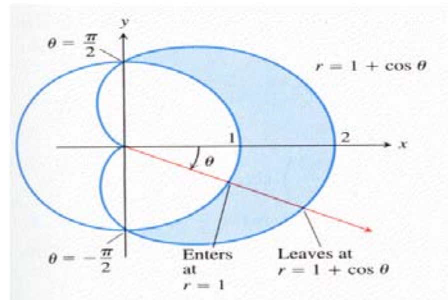
How to find limits of integration from sketch

Step 1. Since θ is held fixed for the first integration, draw a radial line from the origin through the region R at a fixed angle θ. This line crosses the boundary of R at most twice. The innermost point of intersection is one the curve $r = r_1(\theta)$ and the outermost point is on the curve $r = r_2(\theta)$. These intersections determine the r-limits of integration.

Step 2. Imagine rotating a ray along the positive x-axis one revolution counterclockwise about the origin. The smallest angle at which this ray intersects the region R is $\theta = \alpha$ and the largest angle is $\theta = \beta$. This yields the θ-limits of the integration.

EXAMPLE

Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardioids $r = 1 + cos\theta$ and outside the circle $r = 1$.



Solution:

Step 1. We sketch the region and label the bounding curves.

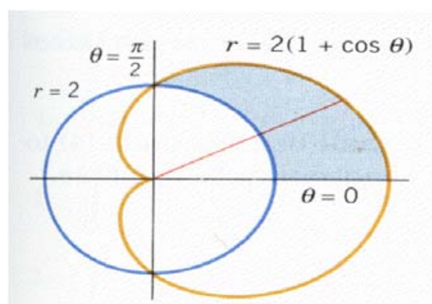
Step 2. The r-limits of integration. A typical ray from the origin enters R where r =1 and leaves where $r = 1 + cos\theta$.

Step 3. The θ-limits of integration. The rays from the origin that intersect R run from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

The integral is $\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r,\theta)\, r\, dr\, d\theta = 2\int_{0}^{\pi/2}\int_{1}^{1+\cos\theta} f(r,\theta)\, r dr\, d\theta$

EXAMPLE : Evaluate $\iint_R \sin\theta\, dA$ where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioids $r = 2(1 + cos\theta)$.

Solution:



$$\iint_R \sin\theta\, dA = \int_0^{\pi/2}\int_2^{2(1+\cos\theta)} (\sin\theta)\, r\, dr\, d\theta$$

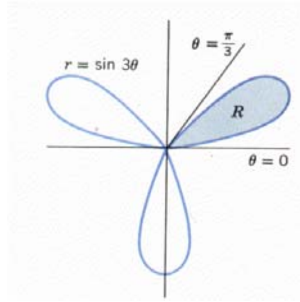
$$= \int_0^{\pi/2} \frac{1}{2} r^2 \sin\theta \Big]_{r=2}^{2(1+\cos\theta)} d\theta$$

$$= 2\int_0^{\pi/2} [(1+\cos\theta)^2 \sin\theta - \sin\theta]d\theta$$

$$= 2\left[-\frac{1}{3}(1+\cos\theta)^3 + \cos\theta\right]_0^{\pi/2} = 2\left[-\frac{1}{3} - \left(-\frac{5}{3}\right)\right] = \frac{8}{3}$$

EXAMPLE : Use a double polar integral to find the area enclosed by the three-petaled rose r = sin 3θ.

Solution: We calculate the area of the petal R in the first quadrant and multiply by three.



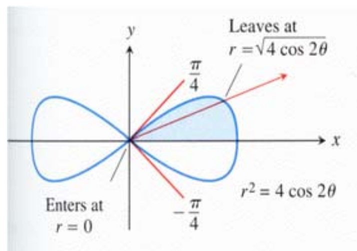
$$A = 3\iint_R dA = 3\int_0^{\pi/3}\int_0^{\sin 3\theta} r\,dr\,d\theta = 3\int_0^{\pi/3} \frac{r^2}{2}\Big|_0^{\sin 3\theta}\,dr\,d\theta$$

$$= \frac{3}{2}\int_0^{\pi/3} \sin^2 3\theta\,d\theta \quad = \frac{3}{4}\int_0^{\pi/3} (1-\cos 6\theta)\,d\theta$$

$$= \frac{3}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = \left[\frac{3}{4}\theta - \frac{3}{24}\sin 6\theta\right]_0^{\pi/3} = \frac{1}{4}\pi$$

EXAMPLE : Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$. The total area is four times the first-quadrant portion.

Solution:



$$A = 4\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{4Cos2\theta}} r\,dr\,d\theta = 4\int_0^{\frac{\pi}{4}}\left[\frac{r^2}{2}\right]_0^{\sqrt{4Cos2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\,d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$

CHANGING CARTESIAN INTEGRALS INTO POLAR INTEGRALS

The procedure for changing a Cartesian integral $\iint_R f(x, y)\,dx\,dy$ into a polar integral has two steps.

Step 1. Substitute $x = r\cos\theta$ and $y = r\sin\theta$, and replace $dx\,dy$ by $r\,dr\,d\theta$ in the Cartesian integral.

Step 2. Supply polar limits of integration for the boundary of R. The Cartesian integral then becomes

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where G denotes the region of integration in polar coordinates.

Notice that *dx dy* is not replaced by *dr dθ* but by *r dr dθ*.

EXAMPLE: Evaluate the double integral $\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$ by changing to polar coordinates.

Solution: The region of integration is bounded by

$$0 \le y \le \sqrt{1-x^2} \text{ and } 0 \le x \le 1$$

$y = \sqrt{1 - x^2}$ is the circle, which gives $x^2 + y^2 = 1$, $r = 1$

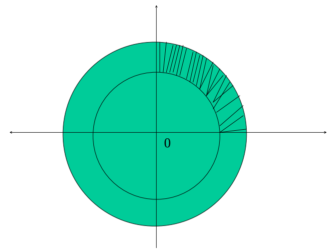
On changing into the polar coordinates, the given integral is

$$\int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta = \int_0^{\pi/2}\left|\frac{r^4}{4}\right|_0^1 d\theta = \int_0^{\pi/2}\frac{1}{4}\,d\theta = \left|\frac{\theta}{4}\right|_0^{\pi/2} = \frac{\pi}{8}$$

EXAMPLE

Evaluate $I = \int_D\int \frac{dx\,dy}{x^2 + y^2}$ by changing to polar coordinates,

where D is the region in the first quadrant between the circles.



Solution:

Two circles are $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$, $0 < a < b$

$$I = \int_0^{\pi/2}\int_a^b \frac{r\,dr\,d\theta}{r^2} = \int_0^{\pi/2}[\ln r]_a^b\,d\theta$$

$$= \int_0^{\pi/2}\ln\left(\frac{b}{a}\right)d\theta = \left[\theta\ln\left(\frac{b}{a}\right)\right]_0^{\pi/2} = \frac{\pi}{2}\ln\left(\frac{b}{a}\right).$$

EXAMPLE

Evaluate the double integral $\int_0^1\int_0^{\sqrt{1-x^2}}(x^2 + y^2)\,dy\,dx$ by changing to polar coordinates.

The region of integration is bounded by $0 < y < \sqrt{1-x^2}$ and $0 \le x \le 1$

$y = \sqrt{1-x^2}$ is the circle $x^2 + y^2 = 1$, $r = 1$

On changing into the polar coordinates, the given integral is

$$\int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta = \int_0^{\pi/2}\left|\frac{r^4}{4}\right|_0^1 d\theta = \int_0^{\pi/2}\frac{1}{4}d\theta = \frac{1}{4}|\theta|_0^{\pi/2} = \frac{1}{4}(\pi/2) = \pi/8$$

LECTURE No. 26

EXAMPLES

Example 1 : Evaluate $I=\int_0^4\int_0^{\sqrt{4y-y^2}} (x^2+y^2)dx\,dy$ by changing into polar coordinates.

Solution: The region of integration is bounded by $0\le x\le\sqrt{4y-y^2}$ and $0 \le y \le 4$

Now $x = \sqrt{4y-y^2}$ is the circle $x^2+y^2-4y=0 \Rightarrow x^2 + y^2 = 4y$.In polar coordinates this takes the form $r^2 = 4r\sin\theta$, $r = 4\sin\theta$

On changing the integral into polar coordinates, we have

$$I = \int_0^{\pi/2}\int_0^{4\sin\theta} r^2 . r\,dr\,d\theta = \int_0^{\pi/2} 64\sin^4\theta\,d\theta = 64.\frac{3.1}{4.2}.\frac{\pi}{2} = 12\pi \quad \text{(using Walli's formula)}$$

Example 2 : Evaluate $\iint_R e^{x^2+y^2}\,dy\,dx$.,where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$

Solution: In Cartesian coordinates, the integral in question is a non-elementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either x or y.

Substituting $x = r\cos\theta$, $y = r\sin\theta$, and replacing dy dx by r dr dθ enables us to evaluate the integral as

$$\iint_R e^{x^2+y^2}dy\,dx = \int_0^{\pi}\int_0^{1} e^{r^2} r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta = \int_0^{\pi}\frac{1}{2}(e-1)d\theta = \frac{\pi}{2}(e-1).$$

Example 3 : Let R_a be the region bounded by the circle $x^2 + y^2 = a^2$. Define

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy = \lim_{a\to\infty}\iint_{R} e^{-(x^2+y^2)}dx\,dy$$

Solution: To evaluate this improper integral,

$$I=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)}\,dx\,dy = \lim_{a\to\infty}\iint_{Da} e^{-\left(x^2+y^2\right)}dx\,dy$$

$$= \lim_{a\to\infty}\int_0^{2\pi}\int_0^{a} e^{-r^2} r\,dr\,d\theta = \lim_{a\to\infty}\int_0^{2\pi}\frac{1}{2}(1-e^{-a^2})d\theta = \lim_{a\to\infty}\frac{1}{2}(1-e^{-a^2})\,\theta\Big|_0^{2\pi}$$

$$=\lim_{a\to\infty}\pi\left(1-e^{-a^2}\right) = =\lim_{a\to\infty}\left(\pi-\frac{\pi}{e^{a^2}}\right) = \pi-\lim_{a\to\infty}\frac{\pi}{e^{a^2}} = \pi-\frac{\pi}{e^{\infty}} = \pi$$

Example 4 : Prove that $\int_0^{\infty} e^{-x^2}\,dx = \dfrac{\sqrt{\pi}}{2}$

Solution : To prove it, we take

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\,dx\,dy = \int_{-\infty}^{\infty} e^{-y^2}\int_{-\infty}^{\infty} e^{-x^2}\,dx\,dy$$

$$= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-x^2}\,dx$$ *y* is a dummy variable, so we can change *y* to *x*.

$$= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \left(2\int_0^{\infty} e^{-x^2}\,dx\right)^2$$

Therefore, $4\left(\int_0^{\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \pi$ by Example 3

$$\left(\int_0^{\infty} e^{-x^2}\,dx\right)^2 = \frac{\pi}{4}$$

By taking square root on both sides, $\int_0^{\infty} e^{-x^2}\,dx = \dfrac{\sqrt{\pi}}{2}$

THEOREM : Let G be the rectangular box defined by the inequalities

$$a \le x \le b, \quad c \le y \le d, \quad k \le z \le \ell$$

If f is continuous on the region G, then $\iiint_G f(x,y,z)\,dv = \int_a^b\int_c^d\int_k^l f(x,y,z)\,dz\,dy\,dx$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

$$= \int_a^b\int_k^l\int_c^d f(x,y,z)\,dy\,dz\,dx = \int_k^l\int_a^b\int_c^d f(x,y,z)\,dy\,dx\,dz = \int_k^l\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$$

$$= \int_c^d\int_k^l\int_a^b f(x,y,z)\,dx\,dz\,dy = \int_c^d\int_a^b\int_k^l f(x,y,z)\,dz\,dx\,dy$$

Example 5: Evaluate the triple integral $\iiint_G 12xy^2z^3\,dV$ over the rectangular box G defined by the inequalities $-1 \le x \le 2$, $0 \le y \le 3$, $0 \le z \le 2$.

Solution: We first integrate with respect to z, holding x and y fixed, then with respect to y holding x fixed, and finally with respect to x.

$$\iiint_G 12xy^2z^3\,dV = \int_{-1}^{2}\int_0^3\int_0^2 12xy^2z^3\,dz\,dy\,dx = \int_{-1}^{2}\int_0^3 \left[3xy^2z^4\right]_{z=0}^{2} dy\,dx = \int_{-1}^{2}\int_0^3 48xy^2\,dy\,dx$$

$$= \int_{-1}^{2}\left[16xy^3\right]_{y=0}^{3} dx = \int_{-1}^{2} 432x\,dx = 216x^2\Big]_{-1}^{2} = 648$$

Example 6 : Evaluate $\iiint_R (x - 2y + z)\, dx\, dy\, dz$ Region R : $0 \le x, 1,\ 0 \le y \le x^2,$ $0 \le z \le x + y$

Solution: $= \int_0^1 \int_0^{x^2} \int_0^{x+y} (x - 2y + z)\, dz\, dy\, dx = \int_0^1 \int_0^{x^2} \left| \frac{(x - 2y + z)^2}{2} \right|_0^{x+y} dy\, dx$

$= \int_0^1 \int_0^{x^2} \left[\frac{(x-2y+x+y)^2}{2} - \frac{(x-2y)^2}{2} \right] dy\, dx = \frac{1}{2} \int_0^1 \int_0^{x^2} (3x^2 - 3y^2)\, dy\, dx = \frac{3}{2} \int_0^1 \left| x^2 y - \frac{y^3}{3} \right|_0^{x^2} dx$

$= \frac{3}{2} \int_0^1 \left(x^4 - \frac{x^6}{3} \right) dx = \frac{3}{2} \left| \frac{x^5}{5} - \frac{x^7}{21} \right|_0^1 = \frac{3}{2} \left[\frac{1}{5} - \frac{1}{21} \right] = \frac{8}{35}$

Example 7 : Evaluate $\iiint_S x\, y\, z\, d\, x\, dy\, dz$ Where S = $\{(x,y,z) : x^2+y^2+z^2 \le 1,\ x \ge 0,\ y \ge 0,$ $z \ge 0\}$

Solution: S is the sphere $x^2 + y^2 + z^2 = 1$ .Since x, y, z are all +ve so we have to consider only the +ve octant of the sphere. Now $x^2 + y^2 + z^2 = 1$. So that $z = \sqrt{1 - x^2 - y^2}$

The Projection of the sphere on *xy* plan is the circle $x^2 + y^2 = 1$.

This circle is covered as y-varies from 0 to $\sqrt{1 - x^2}$ and x varies from 0 to 1.

$$\iiint_R x\, y\, z\, dx\, dy\, dz = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} x\, y\, z\, dz\, dy\, dx = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left| \frac{z^2}{2} \right|_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} x\, y \left(\frac{1 - x^2 - y^2}{2} \right) dy\, dx = \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} x\, (y - x^2 y - y^3)\, dy\, dx$$

$$= \frac{1}{2} \int_0^1 x \left(\frac{y^2}{2} - \frac{x^2 y^2}{2} - \frac{y^4}{4} \right) \Bigg|_0^{\sqrt{1-x^2}} dx = \frac{1}{4} \int_0^1 x \left[1 - x^2 - x^2 (1 - x^2) - \frac{(1 - x^2)^2}{2} \right] dx$$

$$= \frac{1}{8} \int_0^1 (x - 2x^3 + x^5)\, dx = \frac{1}{8} \left| \frac{x^2}{2} - \frac{x^4}{2} + \frac{x^6}{6} \right|_0^1 = \frac{1}{8} \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right) = \frac{1}{48}$$

LECTURE No. 27

VECTOR VALUED FUNCTIONS

Recall that a function is a rule that assigns to each element in its domain one and only one element in its range. Thus far, we have considered only functions for which the domain and range are sets of real numbers; such functions are called real-valued functions of a real variable or sometimes simply real-valued functions. In this section we shall consider functions for which the domain consists of real numbers and the range consists of vectors in 2-space or 3-space; such functions are called vector-valued functions of a real variable or more simply vector-valued functions. In 2-space such functions can be expressed in the form.

$$\mathbf{r}(t) = (\mathbf{x}(t), \mathbf{y}(t)) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$$

and in 3-space in the form

$$\mathbf{r}(t) = (\mathbf{x}(t), \mathbf{y}(t), \mathbf{z}(t)) = x(t)\,\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

where x(t), y(t), and z(t) are real-valued functions of the real variable t. These real-valued functions are called the component functions or components of **r**. As a matter of notation, we shall denote vector-valued functions with boldface type **f**(t), **g**(t) and **r**(t) and real-valued functions, as usual, with lightface italic type *f(t), g(t)* and *r(t)*.

<u>EXAMPLE</u>: $\mathbf{r}(t) = (\ln t)\,\mathbf{i} + \sqrt{t^2+2}\,\mathbf{j} + (\cos t\pi)\mathbf{k}$

Then the component functions are $x(t) = \ln t$, $y(t) = \sqrt{t^2+2}$, and $z(t) = \cos t\pi$

The vector that **r**(t) associates with t = 1 is $\mathbf{r}(1) = (\ln 1)\,\mathbf{i} + \sqrt{3}\mathbf{j} + (\cos\pi)\,\mathbf{k} = \sqrt{3}\mathbf{j} - \mathbf{k}$

The function **r** is undefined if $t \le 0$ because ln *t* is undefined for such *t*.

If the domain of a vector-valued function is not stated explicitly, then it is understood to consist of all real numbers for which every component is defined and yields a real value. This is called the natural domain of the function. Thus the natural domain of a vector-valued function is the intersection of the natural domains of its components.

<u>PARAMETRIC EQUATIONS IN VECTOR FORM</u>

Vector-valued functions can be used to express parametric equations in 2-space or 3-space in a compact form.

For example, consider the parametric equations x = x(t), y = y (t)

Because two vectors are equivalent if and only if their corresponding components are equal, this pair of equations can be replaced by the single vector equation.

$$x = x(t),\ \ y = y(t) \qquad \text{---------(1)}$$

$$x\,\mathbf{i} + y\,\mathbf{j} = x(t)\mathbf{i} + y(t)\,\mathbf{j} \qquad \text{---------(2)}$$

Similarly, in 3-space the three parametric equations

$$x = x(t),\ \ y = y(t),\ \ z = z(t) \qquad \text{---------(3)}$$

can be replaced by the single vector equation

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad \text{---------(4)}$$

if we let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ and $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ in 2-sapce and let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ in 3-space, then both (2) and (4) can be written as $\mathbf{r} = \mathbf{r}(t)$ ---------(5) which is the vector form of the parametric equations in (1) and (3). Conversely, every vector equation of form (5) can be rewritten as parametric equations by equating components on the two sides.

EXAMPLE: Express the given parametric equations as a single vector equation.

(a) $x = t^2, \quad y = 3t$

(b) $x = \cos t, \quad y = \sin t, \quad z = t$

Solution: (a) Using the two sides of the equations as components of a vector yields.

$$\mathbf{x}\,\mathbf{i} + y\,\mathbf{j} = t^2\,\mathbf{i} + 3t\,\mathbf{j}$$

(b) Proceeding as in part (a) yields

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

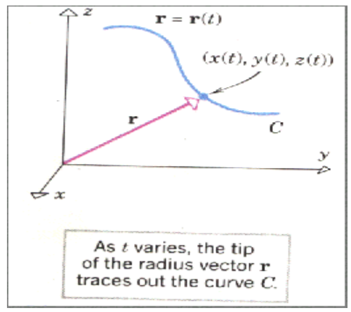
EXAMPLE: Find parametric equations that correspond to the vector equation

$$x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = (t^3 + l)\,\mathbf{i} + 3\,\mathbf{j} + e^t\,\mathbf{k}$$

Equating corresponding components yields.

$$x = t^3 + 1, \quad y = 3, \quad z = e^t$$

GRAPHS OF VECOR-VALUED FUNCTOINS



One method for interpreting a vector-valued function r(t) in 2-space or 3-space geometrically is to position the vector $\mathbf{r} = \mathbf{r}(t)$ with its initial point at the origin, and let C be the curve generated by the tip of the vector $\mathbf{r}$ as the parameter t varies

The vector $\mathbf{r}$, when positioned in this way, is called the radius vector or position vector of C, and C is called the graph of the function $\mathbf{r}(t)$ or, equivalently, the graph of the equation $\mathbf{r} = \mathbf{r}(t)$. The vector equation $\mathbf{r} = \mathbf{r}(t)$ is equivalent to a set of parametric equations, so C is also called the graph of these parametric equations.

EXAMPLE: Sketch the graph of the vector-valued function $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$

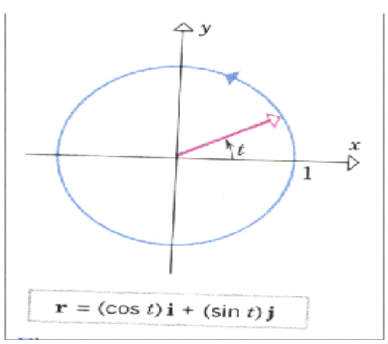
The graph of $\mathbf{r}(t)$ is the graph of the vector equation

$$x\mathbf{i}+y\mathbf{j} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi$$

or equivalently, it is the graph of the parametric equations

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

This is a circle of radius 1 that is centered at the origin with the direction of increasing t counterclockwise. The graph and a radius vector are shown in Fig.



EXAMPLE: Sketch the graph of the vector-valued function r(t) = (cos t)i+(sin t)j+2k, $0 \le t \le 2\pi$

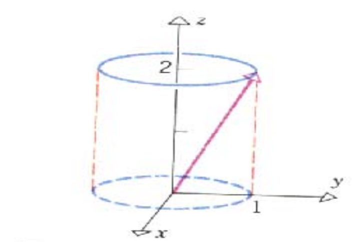
The graph of **r**(t) is the graph of the vector equation

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 2\mathbf{k}, \quad 0 \le t \le 2\pi$$

or, equivalently, it is the graph o the parametric equations

$$x=\cos t, \quad y = \sin t, \quad z=2 \qquad (0 \le t \le 2\pi)$$

From the last equation, the tip o the radius vecor traces a curve in the plane z = 2, and from the first two equations and the preceding example, the curve is a circle of radius 1 centered on the z-axis and traced counterclockwise looking down the z-axis. The graph and a radius vector are shown in Fig.



$\mathbf{r} = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + 2\mathbf{k}$

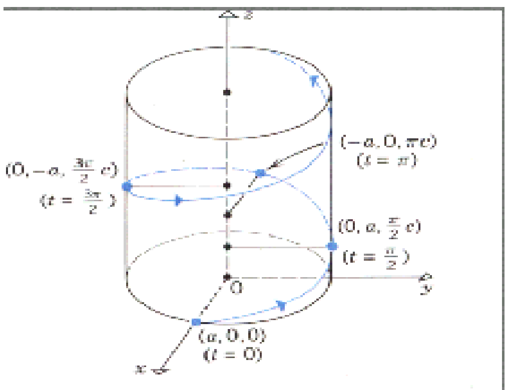
EXAMPLE: Sketch the graph of the vector-valued function r(t) **=** (a cos t)**i +** (a sin t)**j +** (ct)**k ,where a and c are positive constant.**

The graph of **r**(t) is the graph of the parametric equations.

$$x = a\cos t, \quad y = a \sin t, \quad z = ct$$

As the parameter t increases, the value of z = ct also increases, so the point (x, y, z) moves upward. However, as t increases, the point (x, y, z) also moves in a path directly over the circle. x = a cos t, y = a sin t in the xy-plane. The combination of these upward and circular motions produces a corkscrew-shaped curve that wraps around a right-circular cylinder of radius a centered on the z-axis.

This curve is called a circular helix.



$\mathbf{r} = (a\cos t)\,\mathbf{i} + (a \sin t)\,\mathbf{j} + (ct)\,\mathbf{k}$

EXAMPLE: Describe the graph of the vector equation r = (– 2 + t) **i +** 3t**j +** (5 – 4t)**k**

The corresponding parametric equations are $x = -2 + t, \; y = 3t, \;\; z = 5 - 4t$

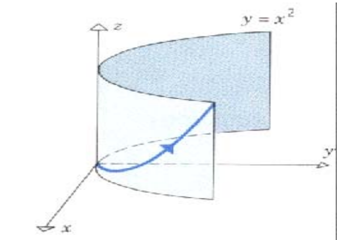
The graph is the line in 3-space that passes through the point (– 2, 0, 5) and is parallel to the vector **i** + 3**j** – 4**k**.

EXAMPLE

The graph of the vector-valued function **r** (t) = t **i** + t^2 **j** + t^3 **k** is called a twisted cubic. Show that this curve lies o the parabolic cylinder $y = x^2$, and sketch the graph for $t \geq 0$

The corresponding parametric equations are $x = t, \quad y = t^2, \quad z = t^3$

Eliminating the parameter t in the equations for x and y yields $y = x^2$, so the curve lies on the parabolic cylinder with this equation. The curve starts at the origin for t = 0; as t increases, so do x, y, and z, so the curve is traced in the upward direction, moving away from the origin along the cylinder.



$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

GRAPHS OF CONSTANT VECOR-VALUED FUNCTIONS

If **c** is a constant vector in the sense that it does not depend on a parameter, then the graph of **r** = **c** is a single point since the radius vector remains fixed with its tip at **c**.

If **c** = x_0**i** + y_0**j** (in 2-space), then the graph is the point (x_0, y_0), and if **c** = x_0**i** + y_0**j** + z_0**k** (in 3-space), then the graph is the point (x_0, y_0, z_0).

EXAMPLE: The graph of the equation **r** = 2**i** + 3**j** – **k** is the point (2, 3, – 1) in 3-space.

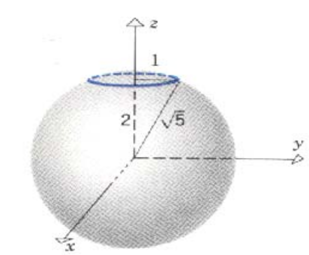
Remark: If **r**(t) is a vector-valued function, then for each value of the parameter t, the expression ||**r**(t)|| is a real-valued function of t because the norm (or length of r(t) is a real number.

For example, if **r**(t) = t **i** + (t – 1) **j** Then $\|\mathbf{r}(t)\| = \sqrt{t^2 + (t-1)^2}$ which is a real-valued function of t.

EXAMPLE: The graph of **r** (t) = (cos t)**i** + (sin t)**j** + 2**k**, $0 \leq t \leq 2\pi$ is a circle of radius 1 centered on the z-axis and lying in the plane z = 2. This circle lies on the surface of a sphere of radius $\sqrt{5}$ because for each value of t

$$\|\mathbf{r}(t)\| = \sqrt{\cos^2 t + \sin^2 t + 2^2} = \sqrt{1+4} = \sqrt{5}$$

which shows that each point on the circle is a distance of $\sqrt{5}$ units from the origin.



$\mathbf{r} = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + 2\mathbf{k}$

LECTURE No. 28

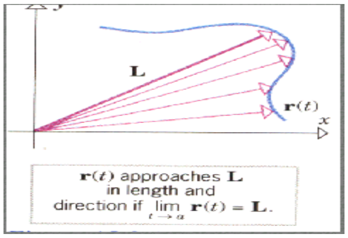
LIMITS OF VECTOR VALUED FUNCTIONS

The limit of a vector-valued function is defined to be the vector that results by taking the limit of each component. Thus, for a function **r**(t) = x (t)**i** + y (t)**j** in 2-space we define.

$$\lim_{t\to\alpha} \mathbf{r}(t) = (\lim_{t\to\alpha} x(t))\mathbf{i} + (\lim_{t\to\alpha} y(t))\mathbf{j}$$

and for a function **r**(t) = x(t)**i** + y(t)**j** + z(t)**k**
in 3-space we define.

$$\lim_{t\to\alpha} \mathbf{r}(t) = (\lim_{t\to\alpha} x(t))\mathbf{i} + (\lim_{t\to\alpha} y(t))\mathbf{j} + (\lim_{t\to\alpha} z(t))\mathbf{k}$$



If the limit of any component does not exist,
then we shall agree that the limit of **r** (t) does not exist.
These definitions are also applicable to the one-sided limits $\lim_{t\to\alpha^+}$, $\lim_{t\to\alpha^-}$ and infinite limits, $\lim_{t\to+\infty}$, and $\lim_{t\to-\infty}$. It follows from (1) and (2) that

$$\lim_{t\to\alpha} r(t) = L$$

if and only if the components of **r**(t) approach the components of L as t → α. Geometrically, this is equivalent to stating that the length and direction of **r** (t) approach the length and direction of L as t → α

CONTINUITY OF VECTOR-VALUED FUNCTIONS

The definition of continuity for vector-valued functions is similar to that for real-valued functions. We shall say that **r** is continuous at t_0 if

1. **r** (t_0) is defined;
2. $\lim_{t\to t_0} \mathbf{r}(t)$ exists;
3. $\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$.

It can be shown that **r** is continuous at t_0 if and only if each component of **r** is continuous. As with real-valued functions, we shall call **r** continuous everywhere or simply continuous if r is continuous at all real values of t. geometrically, the graph of a continuous vector-valued function is an unbroken curve.

DERIVATIVES OF VECOR-VALUED FUNCTIONS

The definition of a derivative for vector-valued functions is analogous to the definition for real-valued functions.

DEFINITION

The derivative **r**$^{/}$(t) of a vector-valued function **r**(t) is defined by

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

Provided this limit exists.

For computational purposes the following theorem is extremely useful; it states that the derivative of a vector-valued function can be computed by differentiating each components.

THEOREM

(a) If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is a vector-valued function in 2-space, and if $x(t)$ and $y(t)$ are differentiable, then $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$

(b) If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a vector-valued function in 3-space, and if $x(t)$, $y(t)$, and $z(t)$ are differentiable, then

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

We shall prove part (a). The proof of (b) is similar.

Proof (a):

$$\mathbf{r}'(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \lim_{h\to 0}\frac{[x(t+h)-x(t)]}{h}\mathbf{i} + \lim_{h\to 0}\frac{[y(t+h)-y(t)]}{h}\mathbf{j}$$

$$= x'(t)\mathbf{i} + y'(t)\mathbf{j}$$

As with real-valued functions, there are various notations for the derivative of a vector-valued function. If $\mathbf{r} = \mathbf{r}(t)$, then some possibilities are $\frac{d}{dt}[\mathbf{r}(t)]$, $\frac{dr}{dt}$, $\mathbf{r}'(t)$, and $\mathbf{r}'$

EXAMPLE

Let $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$. Find $\mathbf{r}'(t)$ and $\mathbf{r}'(1)$

$$\mathbf{r}'(t) = \frac{d}{dt}[t^2]\,\mathbf{i} + \frac{d}{dt}[t^3]\,\mathbf{j}$$

$$= 2t\,\mathbf{i} + 3t^2\,\mathbf{j}$$

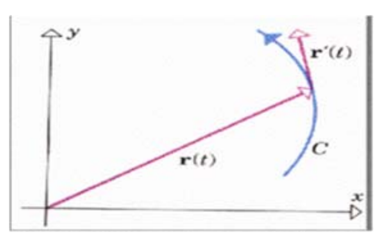
Substituting t=1 yields

$$\mathbf{r}'(1) = 2\mathbf{i}+3\mathbf{j}.$$

TAGENT VECTORS AND TANGENT LINES

GEOMETRIC INTERPRETATIONS OF THE DERIVATIVE.

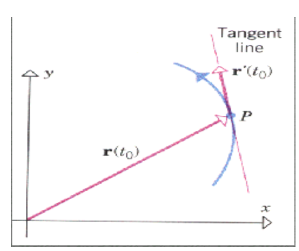
Suppose that C is the graph of a vector-valued function $\mathbf{r}(t)$ and that $\mathbf{r}'(t)$ exists and is nonzero for a given value of t. If the vector $\mathbf{r}'(t)$ is positioned with its initial point at the terminal point of the radius vector



DEFINITION

Let P be a point on the graph of a vector-valued function $\mathbf{r}(t)$, and let $\mathbf{r}(t_0)$ be the radius vector from the origin to P

If $\mathbf{r}'(t_0)$ exists and $\mathbf{r}'(t_0) \neq \mathbf{0}$, then we call $\mathbf{r}'(t_0)$ the tangent vector to the graph of $\mathbf{r}$ at $\mathbf{r}(t_0)$



REMARKS

Observe that the graph of a vector-valued function can fail to have a tangent vector at a point either because the derivative in (4) does not exist or because the derivative is zero at the point. If a vector-valued function **r**(t) has a tangent vector $\mathbf{r}'(t_0)$ at a point on its graph, then the line that is parallel to $\mathbf{r}'(t_0)$ and passes through the tip of the radius vector $\mathbf{r}(t_0)$ is called the tangent line of the graph of **r**(t) at $\mathbf{r}(t_0)$. Vector equation of the tangent line is

$$\mathbf{r} = \mathbf{r}(t_0) + t\,\mathbf{r}'(t_0)$$

EXAMPLE

Find parametric equation of the tangent line to the circular helix

x = cost, y = sint, z = 1 **at the point where t = π/6**

Solution:

To find a vector equation of the tangent line, then we shall equate components to obtain the parametric equations. A vector equation **r = r**(t) of the helix is

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

Thus, $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$

$\Rightarrow \mathbf{r}'(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}$

At the point where t = π/6, these vectors are

$$\mathbf{r}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\pi}{6}\mathbf{k} \quad \text{and}$$

$$\mathbf{r}'\left(\frac{\pi}{6}\right) = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} + \mathbf{k}$$

so from (5) with $t_0 = \pi/6$ a vector equation of the tangent line is

$$\mathbf{r}\left(\frac{\pi}{6}\right) + t\,\mathbf{r}'\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\pi}{6}\mathbf{k}\right) + t\left(-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} + \mathbf{k}\right)$$

Simplifying, then equating the resulting components with the corresponding components of $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ yields the parametric equation.

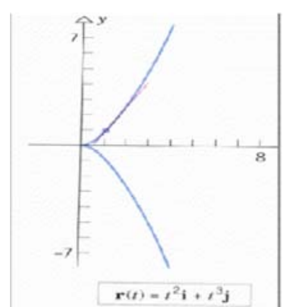
$$x = \frac{\sqrt{3}}{2} - \frac{1}{2}t\,, \quad y = \frac{1}{2} + \frac{\sqrt{3}}{2}t\,, z = \frac{\pi}{6} + t$$

EXAMPLE

The graph of $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ is called a semi-cubical parabola

Find a vector equation of the tangent line to the graph of **r**(t) at

(a) the point (0,0) (b) the point (1,1)



The derivative of **r**(t) is

$$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$$

(a) The point (0, 0) on the graph of **r** corresponds to t = 0. As this point we have $\mathbf{r}'(0) = 0$, so there is no tangent vector at the point and consequently a tangent line does not exist at this point.

(b) The point (1, 1) on the graph of **r** corresponds to t = 1, so from (5) a vector equation of the tangent line at this point is $\mathbf{r} = \mathbf{r}(1) + t\,\mathbf{r}'(1)$

From the formulas for r (t) and $\mathbf{r}'(t)$ with t = 1, this equation becomes

$$\mathbf{r} = (\mathbf{i} + \mathbf{j}) + t\,(2\mathbf{i} + 3\mathbf{j})$$

If **r** is a vector-valued function in 2-space or 3-space, then we say that **r**(t) is smoothly parameterized or that **r** is a smooth function of t if the components of **r** have continuous derivatives with respect to t and $\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t. Thus, in 3-space $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a smooth function of t if $x'(t)$, $y'(t)$, and $z'(t)$ are continuous and there is no value of t at which al three derivatives are zero. A parametric curve C in 2-space or 3-space will be called smooth if it is the graph of some smooth vector-valued function.

It can be shown that a smooth vector-valued function has a tangent line at every point on its graph.

PROPERTIES OF DERIVATIVES

(Rules of Differentiation).

In either 2-space or 3-space let **r**(t), $\mathbf{r}_1(t)$, and $\mathbf{r}_2(t)$ be vector-valued functions, f(t) a real-valued function, k a scalar, and c a fixed (constant) vector. Then the following rules of differentiation hold:

1. $\frac{d}{dt}[\mathbf{c}] = 0$
2. $\frac{d}{dt}[k\,\mathbf{r}(t)] = k\frac{d}{dt}[\mathbf{r}(t)]$
3. $\frac{d}{dt}[\mathbf{r}_1(t) + \mathbf{r}_2(t)] = \frac{d}{dt}[\mathbf{r}_1(t)] + \frac{d}{dt}[\mathbf{r}_2(t)]$
4. $\frac{d}{dt}[\mathbf{r}_1(t) - \mathbf{r}_2(t)] = \frac{d}{dt}[\mathbf{r}_1(t)] - \frac{d}{dt}[\mathbf{r}_2(t)]$
5. $\frac{d}{dt}[f(t)\mathbf{r}(t)] = f(t)\frac{d}{dt}[\mathbf{r}(t)] + \mathbf{r}(t)\frac{d}{dt}[f(t)]$

In addition to the rules listed in the foregoing theorem, we have the following rules for differentiating dot products in 2-space or 3-space and cross products in 3-space:

6. $$\frac{d}{dt}\left[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)\right] = \mathbf{r}_1\cdot\frac{d\,\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt}\cdot\mathbf{r}_2$$
7. $$\frac{d}{dt}\left[\mathbf{r}_1(t)\times\mathbf{r}_2(t)\right] = \mathbf{r}_1\times\frac{d\,\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt}\times\mathbf{r}_2$$

REMARKS:

In (6), the order of the factors in each term on the right does not matter, but in (7) it does.

In plane geometry one learns that a tangent line to a circle is perpendicular to the radius at the point of tangency. Consequently, if a point moves along a circular arc in 2-space, one would expect the radius vector and the tangent vector at any point on the arc to be perpendicular. This is the motivation for the following useful theorem, which is applicable in both 2-space and 3-space.

THEOREM:

If **r (t)** is a vector-valued function in 2-space or 3-space and $\|\mathbf{r}(t)\|$ is constant for all t, then $\mathbf{r(t)} \cdot \mathbf{r}'(\mathbf{t}) = \mathbf{0}$

Proof: That is, **r**(t) and $\mathbf{r}'(t)$ are orthogonal vectors for all t. It follows from (6) with $\mathbf{r}_1(t) = \mathbf{r}_2(t) = \mathbf{r}(t)$ that

$$\frac{d}{dt}[\mathbf{r}(t).\mathbf{r}(t)] = \mathbf{r}(t).\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt}.\mathbf{r}(t)$$

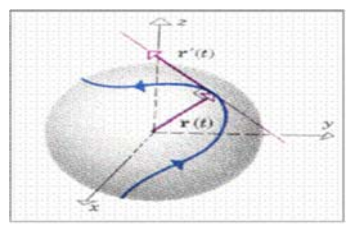
or, equivalently, $\frac{d}{dt}[\|\mathbf{r}(t)\|]^2 = 2\mathbf{r}(t)\cdot\frac{d\mathbf{r}}{dt}$

But $\|\mathbf{r}(t)\|^2$ is constant, so its derivative is zero. Thus $2r(t).\frac{d\mathbf{r}}{dt} = 0$ that is $\mathbf{r}(t)\cdot\frac{d\mathbf{r}}{dt} = 0$

That is the **r**(t) is perpendicular $\frac{d\mathbf{r}}{dt}$

EXAMPLE

Just as a tangent line to a circular arc in 2-space is perpendicular to the radius at the point of tangency, so a tangent line to a curve on the surface of a sphere in 3-space is perpendicular to the radius at the point of tangency.



To see that this is so, suppose that the graph of **r**(t) lies on the surface of the sphere of radius k > 0 centered at the origin. For each value of *t* we have $\|\mathbf{r}(t)\|=k$,

$$\mathbf{r}(t).\ \mathbf{r}'(t) = 0$$

which implies that the radius vector **r**(t) and the tangent vector $\mathbf{r}'(t)$ are perpendicular. This completes the argument because the tangent line, where it exists, is parallel to the tangent vector.

INTEGRALS OF VECTOR VALUED FUNCTION

(a) If **r**(t)=x(t) **i** +y(t) **j** is a vector-valued function in 2-space ,then we define.

$$\int \boldsymbol{r}(t)dt = (\int x(t)dt)\boldsymbol{i} + (\int y(t)dt)\boldsymbol{j} \qquad (1a)$$

$$\int_a^b \boldsymbol{r}(t)dt = (\int_a^b x(t)dt)\boldsymbol{i} + (\int_a^b y(t)dt)\boldsymbol{j} \qquad (1b)$$

(a) If **r**(t)=x(t) **i** +y(t) **j** + z(t) **k** is a vector-valued function in 3-space ,then we define.

$$\int \boldsymbol{r}(t)dt = (\int x(t)dt)\boldsymbol{i} + (\int y(t)dt)\boldsymbol{j} + (\int z(t)dt)\boldsymbol{k} \qquad (2a)$$

$$\int_a^b \boldsymbol{r}(t)dt = (\int_a^b x(t)dt)\boldsymbol{i} + (\int_a^b y(t)dt)\boldsymbol{j} + (\int_a^b z(t)dt)\boldsymbol{k} \qquad (2b)$$

Example:

Let $\boldsymbol{r}(t) = 2t\,\boldsymbol{i} + 3t^2\boldsymbol{j}$, find

$(a)\int \boldsymbol{r}(t)dt \qquad (b)\int_0^2 \boldsymbol{r}(t)dt$

$$\int \boldsymbol{r}(t)dt = \int (2t\,\boldsymbol{i} + 3t^2\boldsymbol{j})dt = (\int 2tdt)\boldsymbol{i} + (\int 3t^2dt)\boldsymbol{j}$$

$$(t^2 + C_1)\boldsymbol{i} + (t^3 + C_2)\boldsymbol{j} = t^2\boldsymbol{i} + C_1\boldsymbol{i} + t^3\boldsymbol{j} + C_2\boldsymbol{j}$$

$$= t^2\boldsymbol{i} + t^3\boldsymbol{j} + C_1\boldsymbol{i} + C_2\boldsymbol{j}$$

$$= t^2\boldsymbol{i} + t^3\boldsymbol{j} + C$$

Where $C = C_1\boldsymbol{i} + C_2\boldsymbol{j}$ is an arbitrary vector constant of integration

$$(b)\int_0^2 \boldsymbol{r}(t)dt = \int_0^2 (2t\boldsymbol{i} + 3t^2\boldsymbol{j})dt = (\int_0^2 2tdt)\boldsymbol{i} + (\int_0^2 3t^2dt)\boldsymbol{j} = \left[t^2\right]_0^2\boldsymbol{i} + \left[t^3\right]_0^2\boldsymbol{j} = (2^2 - 0)\boldsymbol{i} + (2^3 - 0)\boldsymbol{j} = 4\boldsymbol{i} + 8\boldsymbol{j}$$

PROPERTEIS OF INTEGRALS

$\int c\mathbf{r}(t)\,dt = c\int \mathbf{r}(t)\,dt \qquad (3)$

$\int [\mathbf{r}_1(t)+\mathbf{r}_2(t)]\,dt = \int \mathbf{r}1(t)dt + \int \mathbf{r}_2(t)\,dt \qquad (4)$

$\int [\mathbf{r}_1(t)-\mathbf{r}_2(t)]dt = \int \mathbf{r}_1(t)\,dt - \int \mathbf{r}_2(t)\,dt \qquad (5)$

These properties also hold for definite integrals of vector-valued functions. In addition, we leave it for the reader to show that if r is a vector-valued function in 2-space or 3-space, then $\frac{d}{dt}\left[\int \mathbf{r}(t)\,dt\right] = \mathbf{r}(t) \qquad (6)$

This shows that an indefinite integrals of **r**(t) is, in fact, the set of anti-derivatives of **r**(t), just as for real-valued functions.

If **r**(t) is any anti-derivative or **r**(t) in the sense that $\mathbf{R}'(t) = \mathbf{r}(t)$, then

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C} \qquad (7)$$

where C is an arbitrary vector constant of integration.

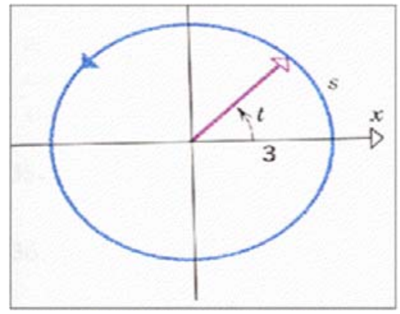
Moreover,

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

LECTURE No.29

CHANGE OF PARAMETER

It is possible for different vector-valued functions to have the same graph.
For example, the graph of the function
r = (3 cos t)**i** + (3 sin t)**j**, $0 \le t \le 2\pi$ -----(1)
is the circle of radius 3 centered at the origin with counterclockwise orientation. The parameter t can be interpreted geometrically as the positive angle in radians from the x-axis to the radius vector. For each value of t, let s be the length of the arc subtended by this angle on the circle



The parameters s and t are related by

$$t = \frac{s}{3}, \quad 0 < s < 6\pi$$

If we substitute this in equation (1), we obtain a vector-valued function of the parameter s, namely

$$\mathbf{r} = 3\cos(s/3)\mathbf{i} + 3\sin(s/3)\mathbf{j}, \quad 0 \le s \le 6\pi$$

whose graph is also the circle of radius 3 centered at the origin with counterclockwise orientation .In various problems it is helpful to change the parameter in a vector-valued function by making an appropriate substitution. For example, we changed the parameter above from t to s by substituting $t = \frac{s}{3}$ in equation(1).

In general, if g is a real-valued function, then substituting t = g(u) in r(t) changes the parameter from t to u.

SMOOTH FUNCTION

When making such a change of parameter, it is important to ensure that the new vector-valued function of u is smooth if the original vector-valued function of t is smooth. It can be proved that this will be so if g satisfies the following conditions:

1. g is differentiable.
2. g' is continuous.
3. $g'(u) \neq 0$ for any u in the domain of g.
4. The range of g is the domain of **r**.

If g satisfies these conditions, then we call t = g(u) a smooth change of parameter. Henceforth, we shall assume that all changes of parameter are smooth, even if it is not stated explicitly.

ARC LENGTH

Because derivatives of vector-valued functions are calculated by differentiating components, it is natural to define integrals of vector-functions in terms of components.

EXAMPLE: If $x'(t)$ and $y'(t)$ are continuous for $a \le t \le b$, then the curve given by the parametric equations

$$x = x(t),\ \ y = y(t) \quad (a \le t \le b) \qquad (9)$$

has arc length

$$\mathbf{L} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (10)$$

This result generalizes to curves in 3-spaces exactly as one would expect:

If $x'(t)$, $y'(t)$, and $z'(t)$ are continuous for $a \le t \le b$, then the curve given by the parametric equations

$x = x(t),\ \ y = y(t),\ \ z = z(t)\ \ (a \le t \le b)$

has arc length

$$\mathbf{L} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \qquad (12)$$

EXAMPLE : Find the arc length of that portion of the circular helix

$$\mathbf{x = \cos t, \qquad y = \sin t, \qquad z = t}$$

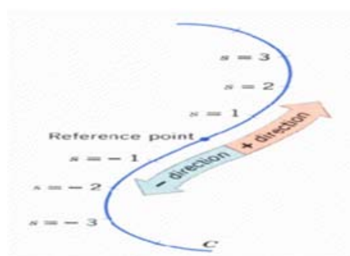
From $t = 0$ to $t = \pi$

The arc length is

$$\mathbf{L} = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \ = \int_0^{\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt$$

$$= \int_0^{\pi} \sqrt{2}\ dt = \sqrt{2}\ \pi$$

<u>ARC LENTH AS A PARAMETER</u>



For many purposes the best parameter to use for representing a curve in 2-space or 3-space parametrically is the length of arc measured along the curve from some fixed reference point. This can be done as follows:

Step 1: Select an arbitrary point on the curve C to serve as a reference point.

Step 2: Starting from the reference point, choose one direction along the curve to be the positive direction and the other to be the negative direction.

Step 3: If P is a point on the curve, let s be the “signed” arc length along C from the reference point to P, where s is positive if P is in the positive direction from the reference point, and s is negative if P is in the negative direction.

By this procedure, a unique point P on the curve is determined when a value for s is given. For example, $s = 2$ determines the point that is 2 units along the curve in the positive direction from the reference point, and $s = -\frac{3}{2}$ determines the point that is $\frac{3}{2}$ units along the curve in the negative direction from the reference point.

Let us now treat s as a variable. As the value of s changes, the corresponding point P moves along C and the coordinates of P become functions of s. Thus, in 2-space the coordinates of P are (x(s), y(s)), and in 3-space they are (x(s), y(s), z(s)). Therefore, in 2-space the curve C is given by the parametric equations x = x(s), y = y (s)
and in 3-space by x = x(s), y = y(s), z = z (s)

REMARKS

When defining the parameter s, the choice of positive and negative directions is arbitrary. However, it may be that the curve C is already specified in terms of some other parameter t, in which case we shall agree always to take the direction of increasing t as the positive direction for the parameter s. By so doing, s will increase as t increases and vice versa.
The following theorem gives a formula for computing an arc-length parameter s when the curve C is expressed in terms of some other parameter t. This result will be used when we want to change the parameterization for C from t to s.

THEOREM

(a) Let C be a curve in 2-space given parametrically by

$$x = x(t), \quad y = y(t)$$

where $x'(t)$ and $y'(t)$ are continuous functions. If an arc-length parameter s is introduced with its reference point at $(x(t_0), y(t_0))$, then the parameters s and t are related by

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du \qquad (13a)$$

(b) Let C be a curve in 3-space given parametrically by

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

where $x'(t)$, $y'(t)$, and $z'(t)$ are continuous functions. If an arc-length parameter s is introduced with its reference point at $(x(t_0), y(t_0), z(t_0))$, then the parameters s and t are related by

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du \qquad (13b)$$

Proof

If $t > t_0$, then from (10) (with *u* as the variable of integration rather than *t*) it follows that

$$\int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du \qquad (14)$$

represents the arc length of that portion of the curve C that lies between $(x(t_0), y(t_0))$ and (x (t), y(t)). If $t < t_0$, then (14) is the negative of this arc length. In either case, integral (14) represents the "signed" arc length *s* between these points, which proves (13a).
It follows from Formulas (13a) and (13b) and the Second Fundamental Theorem of Calculus (Theorem 5.9.3) that in 2-space.

$$\frac{ds}{dt} = \frac{d}{dt}\textbf{Error!} \quad = \textbf{Error!}$$

and in 3-space

$$\frac{ds}{dt} = \frac{d}{dt}\textbf{Error!} = \textbf{Error!}$$

Thus, in 2-space and 3-space, respectively,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad (15a)$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \qquad (15b)$$

REMARKS:

Formulas (15a) and (15b) reveal two facts worth noting. First, $\frac{ds}{dt}$ does not depend on t_0; that is, the value of $\frac{ds}{dt}$ is independent of where the reference point for the parameter s is located. This is to be expected since changing the position of the reference point shifts each value of s by a constant (the arc length between the reference points), and this constant drops out when we differentiate. The second fact to be noted from (15a) and (15b) is that $\frac{ds}{dt} \geq 0$ for all t. This is also to be expected since s increases with t by the remark preceding Theorem 15.3.2. If the curve C is smooth, then it follows from (15a) and (15b) that $\frac{ds}{dt} \geq 0$ for all t .

EXAMPLE

$$x = 2t + 1, \ y = 3t - 2 \qquad (16)$$

using arc length s as a parameter, where the reference point for s is the point (1, – 2).
In formula (13a) we used u as the variable of integration because t was needed as a limit of integration. To apply (13a), we first rewrite the given parametric equations with u in place of t; this gives
from which we obtain

$$x = 2u + 1, \qquad y = 3u - 2$$

$$\frac{dx}{du} = 2, \qquad \frac{dy}{du} = 3$$

we see that the reference point (1,–2) corresponds to $t = t_0 = 0$

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du = \int_{t_0}^{t} \sqrt{13}\, du = \sqrt{13}u \Big]_{u=0}^{u=t} = \sqrt{13}t$$

Therefore, $t = \frac{1}{\sqrt{13}} s$

Substituting this expression in the given parametric equations yields.

$$x = 2\left(\frac{1}{\sqrt{13}}s\right) + 1 = \frac{2}{\sqrt{13}} s + 1$$

$$y = 3\left(\frac{1}{\sqrt{13}}s\right) - 2 = \frac{3}{\sqrt{13}} s - 2$$

EXAMPLE: Find parametric equations for the circle x = a cos t, y = a sin t $(0 \le t \le 2\pi)$ using arc length s as a parameter, with the reference point for s being (a, 0), where a > 0.

Solution: We first replace t by u in the given equations so that x = a cos u, y = a sin u

$$\text{And} \quad \frac{dx}{du} = -a \sin u, \qquad \frac{dy}{du} = a \cos u$$

Since the reference point (a, 0) corresponds to t = 0, we obtain

$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du = \int_{t_0}^{t} \sqrt{(-a \sin u)^2 + (a \cos u)^2}\, du = \int_0^t a\, du = au \Big]_{u=0}^{u=t} = at$$

Solving for t in terms of s yields $t = \frac{s}{a}$

Substituting this in the given parametric equations and using the fact that s = at ranges from 0 to $2\pi a$ as t ranges from 0 to 2π, we obtain

$$x = a\cos\left(\frac{s}{a}\right), \quad y = a\sin\left(\frac{s}{a}\right) \quad (0 \le s \le 2\pi a)$$

EXAMPLE

Find Arc length of the curve $\mathbf{r}(t) = t^3\mathbf{i} + t\mathbf{j} + \frac{1}{2}\sqrt{6}\, t^2\, \mathbf{k}$, $1 \le t \le 3$

Here $x = t^3$, $y = t$, $z = \frac{1}{2}\sqrt{6}\, t^2$

$$\frac{dx}{dt} = 3t^2, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = \sqrt{6}\, t$$

$$\text{Arc length} = \int_1^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_1^3 \sqrt{9t^4 + 1 + 6t^2}\, dt = \int_1^3 \sqrt{(3t^2 + 1)^2}\, dt$$

$$= \left| t^3 + t \right|_1^3 = (3)^3 + 3 - (1)^3 - 1 = 27 + 3 - 1 - 1 = 28$$

EXAMPLE: Calculate $\frac{d\mathbf{r}}{d\mathbf{u}}$ by chain Rule, where r $= e^t\mathbf{i} + 4e^{-t}\mathbf{j}$ **and t = u²**

Solution:

$$\frac{dr}{dt} = e^t\mathbf{i} - 4e^{-t}\mathbf{j}$$

$$\frac{dt}{du} = 2u$$

$$\frac{d\mathbf{r}}{du} = \frac{d\mathbf{r}}{dt} \cdot \frac{dt}{du} = (e^t\mathbf{i} - 4e^t\mathbf{j}).(2u) = 2u\, e^{u^2}\mathbf{i} - 8ue^{-u^2}\mathbf{j}$$

By expressing **r** in terms of u

$$\mathbf{r} = e^u\mathbf{i} + 4e^{-u^2}\mathbf{j}$$

$$\frac{d\mathbf{r}}{du} = 2u\, e^{u^2}\mathbf{i} - 8ue^{-u^2}\mathbf{j}$$

LECTURE No. 30

EXACT DIFFERENTIAL

If $z = f(x, y)$, then $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

The result can be extended to functions of more than two independent variables.

If $z = f(x, y, w)$, $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy + \frac{\partial z}{\partial w} dw$

Make a note of these results in differential form as shown.

Exercise

Determine the differential dz for each of the following functions.

1. $z = x^2 + y^2$
2. $z = x^3 \sin 2y$
3. $z = (2x - 1) e^{3y}$
4. $z = x^2 + 2y^2 + 3w^2$
5. $z = x^3y^2 w$.

Finish all five and then check the result.

---

1. $dz = 2 (x\,dx + y\,dy)$
2. $dz = x^2 (3 \sin 2y\,dx + 2x \cos 2y\,dy)$
3. $dz = e^{3y} \{2dx + (6x - 3)\,dy\}$
4. $dz = 2 (xdx + 2ydy + 3wdw)$
5. $dz = x^2y (3ywdx + 2xwdy + xydw)$

<u>Exact Differential</u>

We have just established that if $z = f(x, y)$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

We now work in reverse.

Any expression $dz = Pdx + Qdy$, where P and Q are functions of x and y, is an exact differential if it can be integrated to determine z.

$\therefore\ P = \frac{\partial z}{\partial x}$ and $Q = \frac{\partial z}{\partial y}$

Now $\frac{\partial P}{\partial y} = \frac{\partial^2 z}{\partial y \partial x}$ and $\frac{\partial Q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y}$ and we know that $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

Therefore, for dz to be an exact differential $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and this is the test we apply.

EXAMPLE

$$dz = (3x^2 + 4y^2)\,dx + 8xy\,dy.$$

If we compare the right-hand side with Pdx + Qdy, then

$$P = 3x^2+4y^2 \ \therefore\ \frac{\partial P}{\partial y} = 8y$$

$$Q = 8xy \qquad \therefore \; \frac{\partial Q}{\partial x} = 8y$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \therefore \text{ dz is an exact differential}$$

Similarly, we can test this one.

EXAMPLE

$$dz = (1 + 8xy)\, dx + 5x^2\, dy.$$

From this we find dz is not an exact differential
for $dz = (1 + 8xy)\, dx + 5x^2\, dy$

$$\therefore \; P = 1 + 8xy \quad \therefore \; \frac{\partial P}{\partial y} = 8x$$

$$Q = 5x^2 \quad \therefore \; \frac{\partial Q}{\partial x} = 10x$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \; \therefore \text{ dz is not an exact differential}$$

EXERCISE

Determine whether each of the following is an exact differential.

1. $dz = 4x^3y^3dx + 3x^4y^2\, dy$
2. $dz = (4x^3y+2xy^3)\, dx+(x^4+3x_2y^2)\, dy$
3. $dz=(15y^2e^{3x}+2xy^2)dx+(10ye^{3x}+x^2y)dy$
4. $dz=(3x^2e^{2y}-2y^2e^{3x})dx+(2x^3e^{2y}-2ye^{3x})dy$
5. $dz=(4y^3\cos 4x+3x^2\cos 2y)dx+(3y^2\sin 4x-2x^3 \sin 2y)\, dy.$

1. Yes 2. Yes 3. No 4. No 5. Yes

We have just tested whether certain expressions are, in fact, exact differentials–and we said previously that, by definition, an exact differential can be integrated. But how exactly do we go about it? The following examples will show.

<u>Integration Of Exact Differentials</u>

$dz = Pdx+Qdy$ where $P=\dfrac{\partial z}{\partial x}$ and $Q=\dfrac{\partial z}{\partial y}$

$\therefore \; z = \int Pdx$ and also $z = \int Qdy$

Example

$$dz = (2xy + 6x)\, dx + (x^2 + 2y3)\, dy.$$

$$P = \frac{\partial z}{\partial x} = 2xy + 6x \quad \therefore \; z=\int (2xy+6x)dx$$

$\therefore \; z = x^2y + 3x^2 + f(y)$ where f(y) is an arbitrary function of y only, and is akin to the constant of integration in a normal integral.

Also $Q = \frac{\partial z}{\partial y} = x^2 + 2y^3 \quad \therefore \quad z = \int (x^2+2y^3)\, dy$

$\therefore \quad z = x^2y + \frac{y^4}{2} + F(x)$ where F(x) is an arbitrary function of x only.

$$z = x^2y + 3x^2 + f(y) \qquad \text{(i)}$$

and $$z = x^2y + \frac{y^4}{2} + F(x) \quad \text{(ii)}$$

For these two expressions to represent the same function, then

f(y) in (i) must be $\frac{y^4}{2}$ already in (i)

and F(x) in (ii) must be $3x^2$ already in (i)

$$\therefore \quad z = x^2y + 3x^2 + \frac{y^4}{2}$$

EXAMPLE

Integrate dz = $(8e^{4x} + 2xy^2)\, dx + (4 \cos 4y + 2x^2y)\, dy$.

$$dz = (8e^{4x} + 2xy^2)\, dx + (4 \cos 4y + 2x^2y)\, dy$$

$$P = \frac{\partial z}{\partial x} = 8e^{4x} + 2xy^2$$

$$\therefore \quad z = \int (8e^{4x} + 2xy^2)\, dx$$

$$\therefore \quad z = 2e^{4x} + x^2y^2 + f(y) \quad \text{(i)}$$

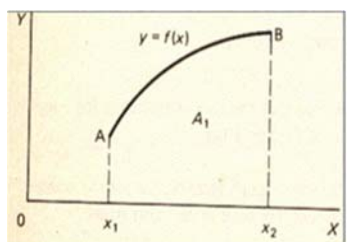
$$Q = \frac{\partial z}{\partial y} = 4 \cos 4y + 2x^2y$$

$$\therefore \quad z = \int (4\cos 4y + 2x^2y)\, dy$$

$$\therefore \quad z = \sin 4y + x^2y^2 + F(x) \qquad \text{(ii)}$$

For (i) and (ii) to agree, $f(y) = \sin 4y$ and $F(x) = 2e^{4x}$

$$\therefore \quad z = 2e^{4x} + x^2y^2 + \sin 4y$$



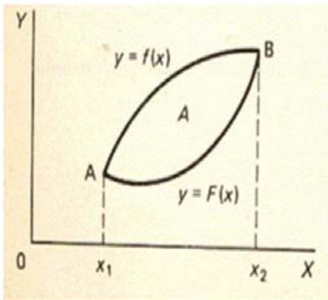
Area enclosed by the closed curve

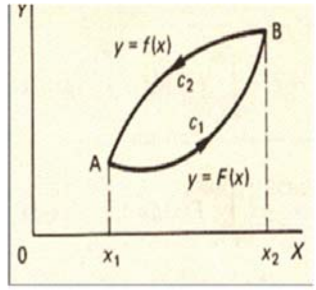
One of the earliest applications of integration is finding the area of a plane figure bounded by the x-axis, the curve $y = f(x)$ and ordinates at $x=x_1$ and $x=x_2$.

$$A_1 = \int_{x_1}^{x_2} y\,dx = \int_{x_1}^{x_2} f(x)\,dx$$

If points A and B are joined by another curve, $y = F(x)$

$$A_2 = \int_{x_1}^{x_2} f(x)\,dx$$





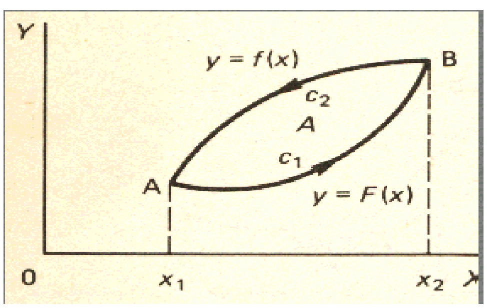
Combining the two figures, we have

$$A = A_1 - A_2 \qquad \therefore\ A = \int_{x_1}^{x_2} F(x)dx - \int_{x_1}^{x_2} f(x)dx$$

The final result above can be written in the form

$$A = -\oint y\,dx$$

Where the symbol $\oint$ indicates that the integral is to be evaluated round the closed boundary in the positive



EXAMPLE

Determine the area enclosed by the graph of $y = x^3$ and $y = 4x$ for $x \geq 0$.

First we need to know the points of intersection. These are $x = 0$ and $x = 2$

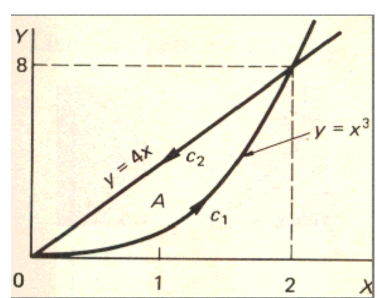
We integrate in a an anticlockwise manner

c_1: $y = x^3$, limits $x = 0$ to $x = 2$

c_2: $y = 4x$, limits $x = 2$ to $x = 0$.

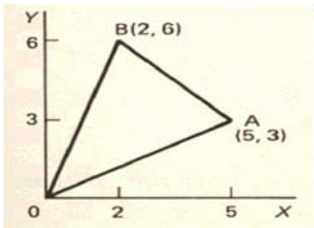
$A = -\oint y\,dx = A = 4$ square units

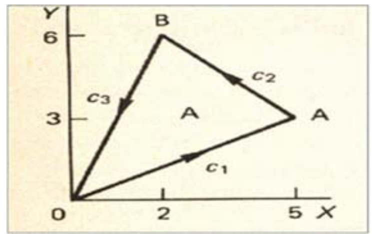
For $A = -\oint ydx = -$**Error!** $= -$**Error!**$= 4$



EXAMPLE

Find the area of the triangle with vertices (0, 0), (5, 3) and (2, 6).





The equation of OA is $y = \frac{3}{5}x$, BA is $y = 8 - x$, OB is $y = 3x$

Then $A = -\oint y\,dx$

Write down the component integrals with appropriate limits.

$A = -\oint y\,dx = -$ **Error!**

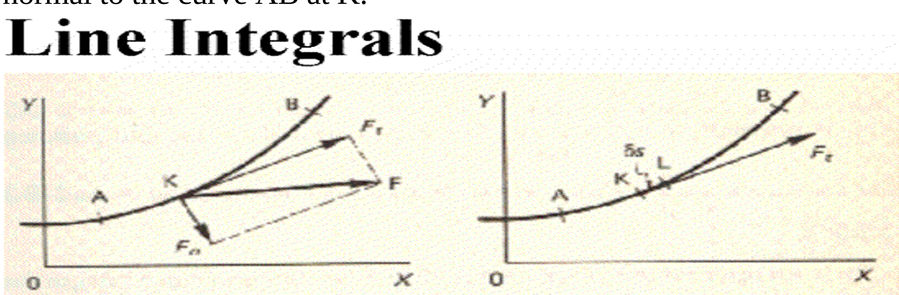
The limits chosen must progress the integration round the boundary of the figure in an anticlockwise manner. Finishing off the integration, we have

A = 12 square units

The actual integration is easy enough. The work we have just done leads us on to consider line integrals, so let us make a fresh start in the next frame.

<u>Line Integrals</u>

If a field exists in the xy-plane, producing a force F on a particle at K, then F can be resolved into two components.F_1 along the tangent to the curve AB at K. F_2 along the normal to the curve AB at K.



The work done in moving the particle through a small distance δs from K to L along the curve is then approximately $F_1\,\delta s$. So the total work done in moving a particle along the curve from A to B is given by

$\lim_{\delta \to 0} \sum F_t\,\delta s = \int F_t\,ds$ from A to B

This is normally written $\int_{AB} F_t\,ds$ where A and B are the end points of the curve, or as $\int_C F_t\, ds$ where the curve c connecting A and B is defined.

Such an integral thus formed, is called a line integral since integration is carried out along the path of the particular curve c joining A and B.

$$\therefore \quad I = \int_{AB} F_t\, dx = \int_C F_t\, ds$$

where c is the curve y = f(x) between A(x_1, y_2) and B (x_2, y_2).

There is in fact an alternative form of the integral which is often useful, so let us also consider that.

Alternative form of a line integral

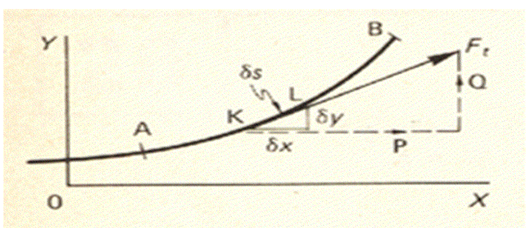
It is often more convenient to integrate with respect to x or y than to take arc length as the variable.

If F_t has a component

P in the x-direction

Q in the y-direction

then the work done from K to L can be stated as $P\delta x + Q\delta y$



$$\therefore \quad \int_{AB} F_t\, ds = \int_{AB} (P\,dx + Q dy)$$

where P and Q are functions of x and y.

In general then, the line integral can be expressed as

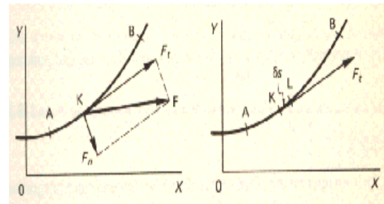
$$I = \int_C F_t\, ds = \int_C (P\,dx + Q dy)$$

where c is the prescribed curve and F, or P and Q, are functions of x and y.

Make a note of these results –then we will apply them to one or two examples.

LECTURE No.31

LINE INTEGRAL



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$$\lim_{\delta \to 0} \sum F_t\, \delta s = \int F_t\, ds \text{ from A to B}$$

This is normally written $\int_{AB} F_t\, ds$ where A and B are the end points of the curve, or as $\int_C F_t\, ds$ where the curve c connecting A and B is defined.Such an integral thus formed, is called a line integral since integration is carried out along the path of the particular curve c joining A and B.

$$\therefore\ I = \int_{AB} F_t\, dx = \int_C F_t\, ds$$

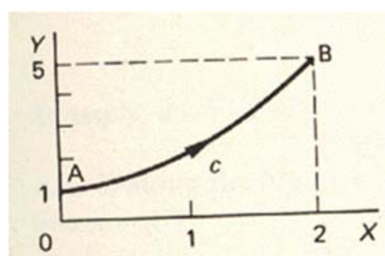
where c is the curve $y = f(x)$ between $A(x_1, y_1)$ and $B(x_2, y_2)$.
There is in fact an alternative form of the integral which is often useful, so let us also consider that.

<u>Alternative form of a line integral</u>

It is often more convenient to integrate with respect to x or y than to take arc length as the variable.
If F_t has a component ,P in the x-direction ,Q in the y-direction then the work done from K to L can be stated as $P\delta x + Q\delta y$

Example 1:

Evaluate $\int_C (x + 3y)\, dx$ from A (0, 1) to B (2, 5) along the curve $y = 1 + x^2$.

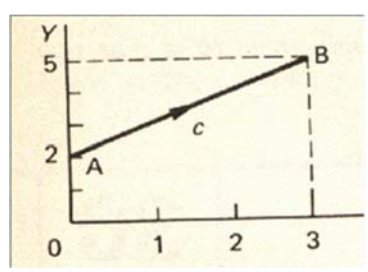


Solution: The line integral is of the form $\int_C (P\, dx + Q dy)$ where, in this case, Q = 0 and c is the curve $y = 1 + x^2$.

It can be converted at once into an ordinary integral by substituting for y and applying the appropriate limits of x.

$$I = \int_C (Pdx+Qdy) = \int_C (x+3y)dx = \int_0^2 (x+3+3x^2)dx$$

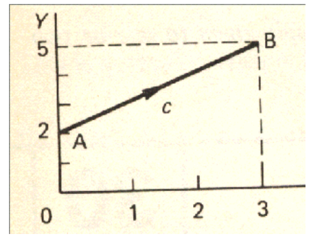
$$= \left[\frac{x^2}{2}+3x+x^3\right]_0^2 = 16$$



Example 2

Evaluate $I = \int_C (x^2 + y)\,dx + (x - y^2)dy$ from A (0, 2) to B (3, 5) along the curve y = 2 + x.

Solution: $I = \int_C (Pdx + Qdy)$



$P = x^2 + y = x^2 + 2 + x = x^2 + x + 2$

$Q = x - y^2 = x - (4+4x+x^2) = -(x^2+3x+4)$

Also y = 2 + x

$\therefore$ dy = dx and the limits are x=0 to x=3

$$I = \int_0^3 \{(x^2+x+2)\,dx - (x^2+3x+4)\,dx\} = \int_0^3 -(2x+2)\,dx = \left|-x^2 - 2x\right|_0^3 = -9 - 6 = -15$$

Example 3

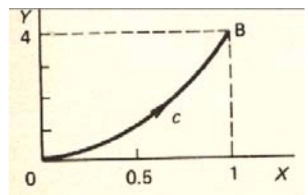
Evaluate $I = \int_C \{(x^2+2y)dx + xydy\}$ from O(0, 0) to B(1, 4) along the curve $y=4x^2$.

Solution: In this case, c is the curve $y = 4x^2$.

$\therefore$ dy = 8x dx

Substitute for y in the integral and apply the limits.

$$I = \int_C \{(x^2+2y)\,dx + xydy\}$$



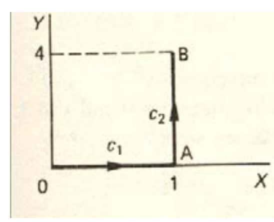
also $x^2 + 2y = x^2 + 8x^2 = 9x^2$; $xy = 4x^3$

$$\therefore\ I = \int_0^1 \left(9x^2\,dx + x\left(4x^2\right)\left(8x\,dx\right)\right) = \int_0^1 \{9x^2dx + 32x^4dx\} = \frac{47}{5} = 9.4$$

They are all done in very much the same way.

Example 4

Evaluate $I = \int_C \{(x^2 + 2y)\,dx + xydy\}$ from O(0, 0) to A (1, 0) along the line y = 0 and then from A (1, 0) to B (1, 4) along the line x = 1.



Solution: (i) OA : c_1 is the line y = 0 $\therefore$ dy = 0.

Substituting y = 0 and dy = 0 in the given integral gives.

$$I_{OA} = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

(ii) AB: Here c_2 is the line x = 1 $\therefore$ dx=0

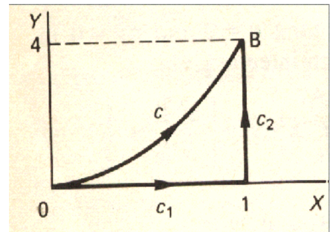
$\therefore$ $I_{AB} = 8$

$$\text{For } I_{AB} = \int_0^4 \{(1 + 2y)(0) + ydy\} = \int_0^4 ydy = \left[\frac{y^2}{2}\right]_0^4 = 8$$

$$\text{Then } I = I_{OA} + I_{AB} = \frac{1}{3} + 8 = 8\frac{1}{3} \qquad \therefore\ I = \frac{25}{3} = 8\frac{1}{3}$$

If we now look back to Example 3 and 4 just completed, we find that we have evaluated the same integral between the same two end points, but along different paths of integration. If we combine the two diagrams, we have where c is the curve $y = 4x^2$ and $c_1 + c_2$ are the lines $y = 0$ and $x = 1$. The result obtained were

$I_c = 9\frac{2}{3}$ and $I_{c_1+c_2} = 8\frac{1}{3}$



Remark: The integration along two distinct paths joining the same two end points does not necessarily give the same results.

Properties of line integrals

1. $\int_C F\,ds = \int_C \{P\,dx + Q\,dy\}$

2. $\int_{AB} F\,ds = -\int_{BA} F\,ds$ and $\int_{AB} \{P\,dx+Q\,dy\} = \int_{BA} \{P\,dx+Q\,dy\}$

i.e. the sign of a line integral is reversed when the direction of the integration along the path is reversed.

3. (a) For a path of integration parallel to the y-axis, i.e. $x = k$, $dx = 0$

$$\therefore \int_C P\,dx = 0 \quad \therefore\ I_C = \int_C Q\,dy.$$

(b) For a path of integration parallel to the x-axis, i.e. $y = k$, $dy = 0$.

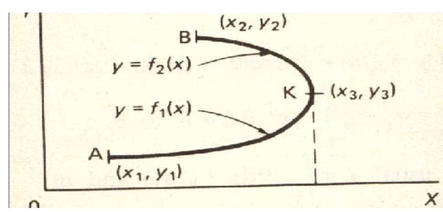
$$\therefore \int_C Q\,dy = 0 \ \therefore\ I_C = \int_C P\,dx.$$

4. If the path of integration c joining A to B is divided into two parts AK and KB, then

$$I_c = I_{AB} = I_{AK} + I_{KB}.$$

5 .If the path of integration c is not single valued for part of its extent, the path is divided into two sections.

$y = f_1(x)$ from A to K, $y = f_2(x)$ from K to B.



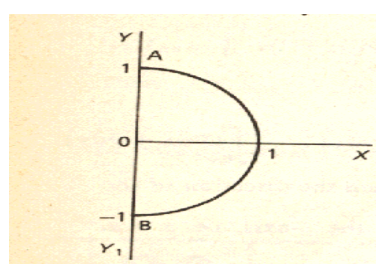
6. In all cases, the actual path of integration involved must be continuous and single-valued.

Example 5

Evaluate $I = \int_C (x + y)\,dx$ from A(0, 1) to B (0, – 1) along the semi-circle $x^2+y^2=1$ for $x \geq 0$.

Solution: The first thing we notice is that the path of integration c is not single-valued

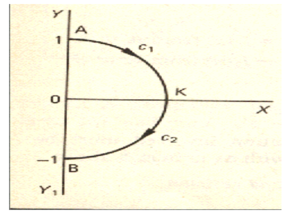
For any value of x, $y = \pm\sqrt{1 - x^2}$. Therefore, we divided c into two parts



(i) $y = \sqrt{1 - x^2}$ from A to K (x=0 to x=1)

(ii) $y = -\sqrt{1 - x^2}$ from K to B (x=1 to x=0)

As usual, $I = \int_C (Pdx + Qdy)$ and in this particular case, Q = 0



$$\therefore \; I = \int_C Pdx = \int_0^1 (x+\sqrt{1-x^2})\,dx + \int_1^0 (x - \sqrt{1-x^2}\,)\,dx$$

$$= \int_0^1 (x + \sqrt{1-x^2} - x + \sqrt{1-x^2}\,dx \;=\; 2\int_0^1 \sqrt{1-x^2}\,dx$$

Now by trigonometric substitution, put $x = \sin\theta$

$$\therefore \quad dx = \cos\theta\, d\theta \quad \text{and} \; \sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$$

Limits : $x = 0, \;\; \theta = 0; \;\; x = 1, \;\; \theta = \dfrac{\pi}{2}$

$$I = 2\int_0^1 \sqrt{1-x^2}\,dx$$

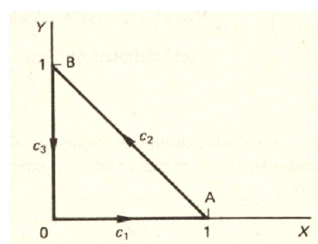
$$= 2\int_0^{\pi/2} \cos\theta\cos\theta\, d\theta = 2\int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} (1+\cos 2\theta)\,d\theta = \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{2}$$

Now let us extend this line of development a stage further.

Example 6

Evaluate the line integral

$I = \oint (x^2dx - 2xy\,dy)$ where c comprises the three sides of the triangle joining O(0, 0), A (1, 0) and B (0, 1).



Solution: First draw the diagram and mark in c_1, c_2 and c_3, the proposed directions of integration. Do just that. The three se of the path of integration must be arranged in an anticlockwise manner round the figure.

Now we deal with each pat separately.

(a) OA : c_1 is the line $y = 0$

Therefore, $dy = 0$.

Then $I = \oint (x^2dx - 2xy\,dy)$ for this part becomes

$$I_1 = \int_0^1 x^2dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

(b) AB : for c_2 is the line $y = 1 - x$

$\therefore$ $dy = -\,dx$.

$$I_2 = \int_1^0 \{x^2dx + 2x(1-x)dx\} = \int_1^0 (x^2+2x-2x^2)dx = \int_1^0 (2x-x^2)\,dx = \left| x^2 - \frac{x^3}{3} \right|_1^0 = -\frac{2}{3}$$

$$\therefore \qquad I_2 = -\frac{2}{3}$$

Note that anticlockwise progression is obtained by arranging the limits in the appropriate order. Now we have to determine I_3 for BO.

(c) BO: c_3 is the line $x = 0$

$\therefore\ dx = 0 \quad \therefore\ I_3 = \int 0dy = 0 \quad \therefore\ I_3 = 0$

Finally, $I = I_1+I_2+I_3 = \frac{1}{3} - \frac{2}{3} + 0 = -\frac{1}{3} \quad \therefore\ I = -\frac{1}{3}$

Example 7

Evaluate $\oint_c y\,dx$ when c is the circle $x^2+y^2 = 4$.

Solution: $x^2 + y^2 = 4 \quad \therefore \quad y = \pm\sqrt{4 - x^2}$

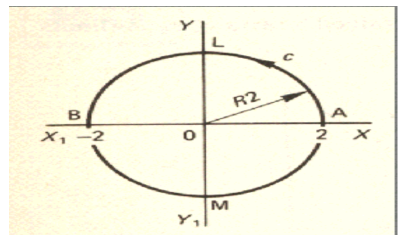
y is thus not single-valued. Therefore use

$y = \sqrt{4 - x^2}$ for ALB between

$x = 2$ and $x = -2$ and

$y = -\sqrt{4 - x^2}$ for BMA between

$x = -2$ and $x = 2$.



$$\therefore\ I = \int_2^{-2} \sqrt{4 - x^2}\,dx + \int_{-2}^{2} \{-\sqrt{4 - x^2}\}\,dx$$

$$= -2\int_{-2}^{2} \sqrt{4 - x^2}\,dx \quad = -4\int_0^2 \sqrt{4 - x^2}\,dx.$$

$$I = -4\int_0^2 \sqrt{4-x^2}\,dx$$

Put $x = 2\sin\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = 2\cos\theta \quad \Rightarrow dx = 2\cos\theta\,d\theta$

When $x=0,\ \ 0= 2\sin\theta \Rightarrow 0= \sin\theta \Rightarrow \theta=0$

When $x=2,\ \ 2= 2\sin\theta \Rightarrow 1= \sin\theta \Rightarrow \theta=\frac{\pi}{2}$

$$I = -4\int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta}\ \ 2\cos\theta\,d\theta = -16\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\ \ \cos\theta\,d\theta$$

$$= -16\int_0^{\frac{\pi}{2}} \cos\theta\ \ \cos\theta\,d\theta = -16\int_0^{\frac{\pi}{2}} \cos^2\theta\ \ d\theta = -8\int_0^{\frac{\pi}{2}} \frac{1 + Cos\,2\theta}{2}\ \ d\theta$$

$$= -8\int_0^{\frac{\pi}{2}} \left(1 + Cos\,2\theta\right)\ d\theta = -8\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{2}} = -8\left(\left(\frac{\pi}{2}-0\right) + \frac{1}{2}\left(0-0\right)\right)$$

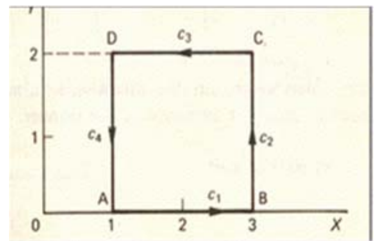
$$= -4\pi$$

LECTURE No. 32

EXAMPLES

Example 1: Evaluate $I = \oint \{xydx+(1+y^2)dy\}$ where c is the boundary of the rectangle joining A(1,0), B (3, 0), C(3, 2), D (1, 2).

Solution: First draw the diagram and insert c_1, c_2, c_3, c_4.
That give
Now evaluate I_1 for AB; I_2 for BC; I_3 for CD;
I_4 for DA; and finally I.



$$I = \oint \{xydx + (1 + y^2)\, dy\}$$

(a) AB: c_1 is $y = 0 \ \therefore\ dy = 0$ $\quad \therefore I_1 = 0$

(b) BC: c_2 is $x = 3 \ \therefore\ dx = 0$

$$\therefore I_2 = \int_0^2 (1+y^2)dy = \left[y + \frac{y^3}{3}\right]_0^2 = 4\frac{2}{3} \qquad \therefore I_2 = 4\frac{2}{3}$$

(c) CD : c_3 is $y = 2 \quad \therefore\ dy = 0$

$$\therefore I_3 = \int_3^1 2xdx = \left[x^2\right]_3^1 = -8 \qquad \therefore I_3 = -8$$

(d) DA : c_4 is $x = 1 \quad \therefore\ dx = 0$

$$\therefore\ I_4 = \int_2^0 (1+y^2)\, dy = \left[y + \frac{y^3}{3}\right]_2^0 = -4\frac{2}{3}$$

Finally $I = I_1 + I_2 + I_3 + I_4 = 0 + 4\frac{2}{3} - 8 - 4\frac{2}{3} = -8 \qquad \therefore I = -8$

Remember that, unless we are directed otherwise, we always proceed round the closed boundary in an anticlockwise manner.

Line integral with respect to arc length

We have already established that

$$I = \int_{AB} F_t ds = \int_{AB} \{Pdx+Qdy\}$$

where F_t denoted the tangential force along the curve c at the sample point K(x,y). The same kind of integral can, of course, relate to any function f(x,y) which is a function of the position of a point on the stated curve, so that

$$I = \int_C f(x, y)\, ds.$$

This can readily be converted into an integral in terms of x:

$$I = \int_C f(x,y)dx = \int_C f(x,y) \frac{ds}{dx} dx$$

$$\text{where } \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\therefore \int_C f(x,y)\,dx = \int_{x_1}^{x_2} f(x,y)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \text{------------------} \quad (1)$$

Example 2

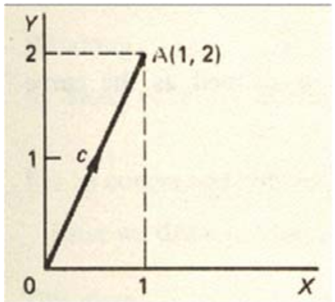
Evaluate I = $\int_C (4x+3xy)ds$ where c is the straight line joining O(0,0) to A (1,2).

Solution: c is the line y = 2x $\therefore \frac{dy}{dx} = 2$

$$\therefore \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{5}$$

$$\therefore I = \int_{x=0}^{x=1}(4x+3xy)ds = \int_0^1 (4x+3xy)(\sqrt{5})\,dx. \quad \text{But } y = 2x$$

for $$I = \int_0^1 (4x+6x^2)(\sqrt{5})\,dx = 2\sqrt{5}\int_0^1 (2x+3x^2)\,dx = 4\sqrt{5}$$



Parametric Equations

When x and y are expressed in parametric form, e.g. x = y (t), y = g(t), then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} \qquad \therefore\ ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$I = \int_C f(x,y)\,ds = \int_{t_1}^{t_2} f(x,y)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \text{----------------}(2)$$

Example 3 : Evaluate I = $\oint$ 4xyds where c is defined as the curve x = sin t, y = cos t between t=0 and t=$\frac{\pi}{4}$.

Solution: We have x = sin t $\therefore \frac{dx}{dt} = \cos t$,

$$y = \cos t \ \therefore \frac{dy}{dt} = -\sin t$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{\cos^2 t+\sin^2 t} = 1 \qquad \therefore\ ds = dt$$

$$\therefore I = \int_C f(x,y)ds = \int_{t_1}^{t_2} f(x,y)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{\pi/4} 4\sin t\cos t\,dt = 2\int_0^{\pi/4}\sin 2t\,dt = -2\left[\frac{\cos 2t}{2}\right]_0^{\pi/4} = 1$$

Dependence of the line integral on the path of integration

We know that integration along two separate paths joining the same two end points does not necessarily give identical results. With this in mind, let us investigate the following problem.

Example 4 : **Evaluate $I = \oint_C \{3x^2 y^2\, dx + 2x^3 y\, dy\}$ between O (0, 0) and A (2, 4)**

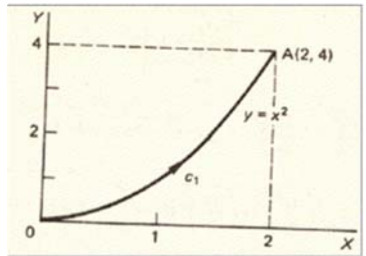
(a) along c_1 i.e. $y = x^2$

(b) along c_2 i.e. $y = 2x$

(c) along c_3 i.e. $x = 0$ from (0,0) to (0,4) and $y = 4$ from (0,4) to (2,4).

Solution:

(a).First we draw the figure and insert relevant information.



$$I = \int_C \{3x^2y^2dx + 2x^3ydy\}$$

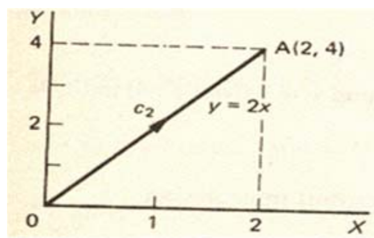
The path c_1 is $y = x^2 \quad \therefore\ dy = 2x\, dx$

$$\therefore\quad I_1 = \int_0^2 \{3x^2x^4dx+2x^3x^22xdx\}= \int_0^2 (3x^6 + 4x^6)\, dx$$

$$\therefore\quad = \left[x^7\right]_0^2 = 128 \quad \therefore\quad I_1 = 128$$

(b) Here the path of integration is c_2, i.e. $y = 2x$

So, in this case, for with c_2, $y = 2x \quad \therefore\ dy = 2dx$



$$\therefore\ I_2 = \int_0^2 (3x^2\, 4x^2\, dx + 2x^3\, 2x\, 2dx\}$$

$$= \int_0^2 (12x^4\, dx + 8\, x^4\, dx\}$$

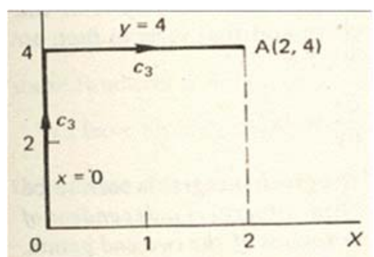
$$= \int_0^2 20\, x^4\, dx = 4\left[x^5\right]_0^2 = 128 \qquad \therefore\quad I_2 = 128$$

(c) In the third case, the path c_3 is split

$x = 0$ from (0,0) to (0, 4),

$y = 4$ from (0, 4) to (2, 4)

Sketch the diagram and determine I_3.



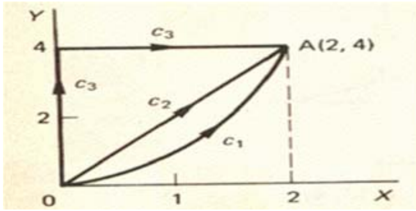
from (0,0) to (0,4) $x=0 \quad \therefore\ dx=0 \ \therefore\ I_{3a}=0$

from (0,4) to (2,4) $y=4 \quad \therefore\ dy=0$

$$\therefore\ I_{3b}=\int_0^2 48x^2\, dx = 128$$

$$\therefore\quad I_3 = 0+128 = 128$$

In the example we have just worked through, we took three different paths and in each case, the line integral produced the same result. It appears, therefore, that in this case, the value of the integral is independent of the path of integration taken.



We have been dealing with $I = \int_C \{3x^2y^2dx+2x^3ydy\}$

On reflection, we see that the integrand $3x^2 y^2 dx + 2x^3y dy$ is of the form $P dx+Q dy$ which we have met before and that it is, in fact, an exact differential of the function

$z = x^3y^2$, for $\frac{\partial z}{\partial x} = 3x^2 y^2$ and $\frac{\partial z}{\partial y} = 2x^3 y$

This always happens. If the integrand of the given integral is seen to be an exact differential, then the value of the line integral is independent of the path taken and depends only on the coordinates of the two end points.

LECTURE No.33

EXAMPLES

Example 1: Evaluate I = $\int_C$ {3ydx + (3x+2y)dy} from A(1, 2) to B (3, 5).

Solution: No path is given, so the integrand is doubtless an exact differential of some function z = f (x,y). In fact $\frac{\partial P}{\partial y} = 3 = \frac{\partial Q}{\partial x}$. We have already dealt with the integration of exact differentials, so there is no difficulty. Compare with I = $\int_C \{P\,dx + Q\,dy\}$.

$$P = \frac{\partial z}{\partial x} = 3y \qquad \therefore\ z = \int 3ydz = 3xy + f(y) \qquad \text{-------- (i)}$$

$$Q = \frac{\partial z}{\partial y} = 3x + 2y \qquad \therefore\ z = \int (3x+2y)\,dy = 3xy + y^2 + F(x) \qquad \text{----(ii)}$$

For (i) and (ii) to agree $f(y) = y^2$; $F(x) = 0$

Hence $z = 3xy + y^2$

$$\therefore I = \int_C \{3ydx + (3x+2y)dy\} = \int_{(1,2)}^{(3,5)} d(3xy+y^2) = \left[3xy+y^2\right]_{(1,2)}^{(3,5)} = (45+25) - (6+4) = 60$$

Example2: Evaluate I = $\int_C$ {(x^2+ye^x)dx+(e^x+y)dy} between A (0, 1) and B (1, 2).

Solution: As before, compare with $\int_C$ {Pdx+Q dy}.

$$P = \frac{\partial z}{\partial x} = x^2 + ye^x \qquad \therefore\ z = \frac{x^3}{3} + ye^x + f(y)$$

$$Q = \frac{\partial z}{\partial y} = e^x + y \qquad \therefore\ z = ye^x + \frac{y^2}{2} + F(x)$$

For these expressions to agree,

$$f(y) = \frac{y^2}{2};\ F(x) = \frac{x^3}{3} \qquad \text{Then } I = \left[\frac{x^3}{3} + ye^x + \frac{y^2}{2}\right]_{(0,1)}^{(1,2)} = \frac{5}{6} + 2e$$

REMARKS: The main points are that, if (Pdx+Qdy) is an exact differential

(a) $I = \int_C (Pdx + Qdy)$ is independent of the path of integration

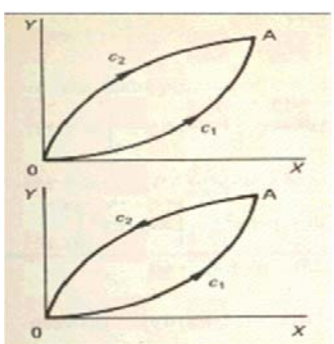
(b) $I = \oint_C (P\,dx + Q\,dy)$ is zero.

If $I = \int_C \{P\,dx + Q\,dy\}$ and (Pdx + Qdy) is an exact differential,

Then $I_{c_1} = - I_{c_2}$

$I_{c_1} + I_{c_2} = 0$

Hence, the integration taken round a closed curve is zero, provided (Pdx+Q dy) is an exact differential.



$\therefore$ If (P dx + Q dy) is an exact differential, $\oint$ (P dx + Q dy) = 0

Exact differentials in three independent variables

A line integral in space naturally involves three independent variables, but the method is very much like that for two independent variables.

$$dz = Pdx + Q\,dy + R\,dw \text{ is an exact differential of } z = f(x, y, w)$$

$$\text{if } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x};\ \frac{\partial P}{\partial w} = \frac{\partial R}{\partial x};\ \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial w}$$

If the test is successful, then

(a) $\int_C$ (P dx + Q dy + R dw) is independent of the path of integration.

(b) $\oint_C$ (P dx + Q dy + R dw) is zero.

Example 3: **Verify that dz = ($3x^2yw$ + 6x) dx + (x^3w – 8y) dy + (x^3y + 1) dw is an exact differential and hence evaluate $\int_C$ dz from A (1, 2, 4) to B (2, 1 3).**

Solution: First check that dz is an exact differential by finding the partial derivatives above, when

$$P = 3x^2yw + 6x; \quad Q = x^3w - 8y; \text{ and } \quad R = x^3y + 1$$

$$\frac{\partial P}{\partial y} = 3x^2w;\ \frac{\partial Q}{\partial x} = 3x^2w \quad \therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial P}{\partial w} = 3x^2y;\ \frac{\partial R}{\partial x} = 3x^2y \quad \therefore \frac{\partial P}{\partial w} = \frac{\partial R}{\partial x}$$

$$\frac{\partial R}{\partial y} = x^3;\ \frac{\partial Q}{\partial w} = x^3 \quad \therefore \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial w}$$

$\therefore$ dz is an exact differential

Now to find z. $P = \frac{\partial z}{\partial x}$; $Q = \frac{\partial z}{\partial y}$; $R = \frac{\partial z}{\partial w}$

$$\therefore \frac{\partial z}{\partial x} = 3x^2yw+6x \ \therefore\ z=\int(3x^2yw+6x)dx \ = x^3yw+3x^2+f(y)+F(w)$$

$$\therefore \frac{\partial z}{\partial y} = x^3w-8x \ \therefore\ z = \int(x^3w-8y)dy \ = x^3yw-4y^2+g(x)+F(w)$$

$$\frac{\partial z}{\partial w} = x^3y+1 \quad \therefore\ z = \int(x^3y+1)dw \ = y^3yw+w+f(y)+g(x)$$

For these three expressions for z to agree

$$f(y) = -4y^2; \quad F(w) = w; \quad g(x) = 3x^2$$

$$\therefore \quad z = x^3yw + 3x^2 - 4y^2 + w$$

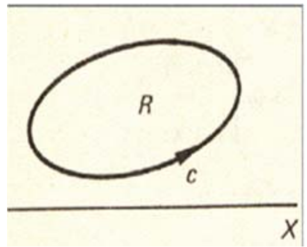
$$\therefore \quad I = \Big[x^3yw + 3x^2 - 4y^2 + w\Big]_{(1,2,4)}^{(2,1,3)}$$

for $\quad I = \Big[x^3yw + 3x^2 - 4y^2 + w\Big]_{(1,2,4)}^{(2,1,3)} = (24+12-4+3)-(8+3-16+4)=36$

The extension to line integrals in space is thus quite straightforward.

Finally, we have a theorem that can be very helpful on occasions and which links up with the work we have been doing. It is important, so let us start a new section.

<u>Green's Thorem</u>



Let P and Q be two function of x and y that are finite and continuous inside and the boundary c of a region R in the xy-plane.If the first partial derivatives are continuous within the region and on the boundary, then Green's theorem states that.

$$\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy = -\oint_C (P\,dx + Q\,dy)$$

That is, a double integral over the plane region R can be transformed into a line integral over the boundary c of the region and the action is reversible.

Let us see how it works.

EXAMPLE 4

Evaluate $I = \oint_C \{(2x - y)dx + (2y+x)dy\}$ around the boundary c of the ellipse $x^2 + 9y^2 = 16$.

Solution: The integral is of the form

$I = \oint_C \{P\,dx + Q\,dy)$ where $P = 2x - y \;\therefore\; \dfrac{\partial P}{\partial y} = -1$ and $Q = 2y + x \;\therefore\; \dfrac{\partial Q}{\partial x} = 1.$

$$\therefore\; I = -\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dxdy = -\iint_R (-1-1)dx\,dy = 2\iint_R dx\,dy = 2A$$

But $\iint_R dx\,dy$ over any closed region give the area of the figure.

In this case, then, I = 24 where A is the area of the ellipse $(A = \pi ab)$

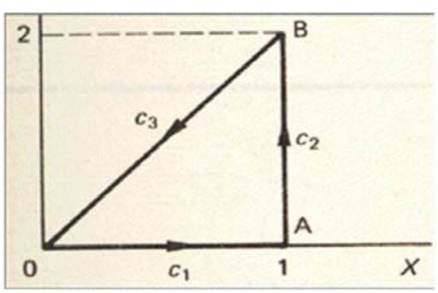
$x^2+9y^2 = 16$ i.e. $\dfrac{x^2}{16} + \dfrac{9y^2}{16} = 1$

$\therefore\; a = 4;\; b = \dfrac{4}{3} \qquad \therefore\; A = \pi ab = \dfrac{16\pi}{3} \qquad \therefore\; I = 2A = \dfrac{32\pi}{3}$

To demonstrate the advantage of Green's theorem, let us work through the next example (a) by the previous method, and (b) by applying Green's theorem.

Example 5: Evaluate $I = \oint_C \{(2x+y)\,dx + (3x-2y)\,dy\}$ taken in anticlockwise manner round the triangle with vertices at O (0,0) A (1, 0) B (1, 2).

Solution: $I = \oint_C \{(2x + y)\,dx + (3x - 2y)\,dy\}$



(a) By the previous method

There are clearly three stages with c_1, c_2, c_3. Work through the complete evaluation to determine the value of I. It will be good revision. When you have finished, check the result with the solution in the next frame. I = 2

(a) (i) c_1 is $y = 0 \quad \therefore \; dy = 0$

$$\therefore \; I_1 = \int_0^1 2x\,dx = \left[x^2\right]_0^1 = 1 \quad \therefore \; I_1 = 1$$

(ii) c_2 is $x = 1 \;\; \therefore \; dx = 0$

$$\therefore \; I_2 = \int_0^2 (3-2y)\,dy = \left[3y - y^2\right]_1^0 = 2 \;\; \therefore \; I_2 = 2$$

(iii) c_3 is $y = 2x \quad \therefore \; dy = 2\,dx$

$$\therefore \; I_3 = \int_1^0 \{4x\,dx + (3x - 4x)\,2\,dx\}$$

$$= \int_1^0 2x\,dx = \left[x^2\right]_1^0 = -1 \quad \therefore \; I_3 = -1$$

$$I = I_1 + I_2 + I_3 = 1 + 2 + (-1) = 2 \quad \therefore \; I = 2$$

Now we will do the same problem by applying Green's theorem, so more

(b) By Green's theorem

$$I = \oint_C \{(2x + y)\,dx + (3x - 2y)\,dy\}$$

$$P = 2x + y \quad \therefore \quad \frac{\partial P}{\partial y} = 1;$$

$$Q = 3x - 2y \; \therefore \quad \frac{\partial Q}{\partial x} = 3$$

$$I = -\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$$

$$= -\iint_R (1-3)\,dx\,dy = 2\iint_R dx\,dy = 2A$$

$$= 2 \times \text{the area of the triangle} = 2 \times \left(\frac{1}{2} \times 1 \times 2\right) = 2$$

$$\therefore \quad I = 2$$

Remark: Application of Green's theorem is not always the quickest method. It is useful, however, to have both methods available.

If you have not already done so, make a note of Green's theorem.

$$\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy = -\oint_C (P\,dx + Q\,dy)$$

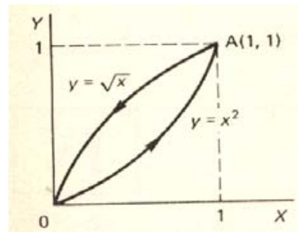
Note: Green's theorem can, in fact, be applied to a region that is not simply connected by arranging a link between outer and inner boundaries, provided the path of integration is such that the region is kept on the left-hand side.

LECTURE No.34

EXAMPLES

Example 1: Evaluate the line integral $I = \oint_C \{xy\,dx + (2x - y)\,dy\}$ round the region bounded by the curves $y = x^2$ and $x = y^2$ by Green's theorem

Solution: Points of intersection are O(0, 0) and A(1, 1).

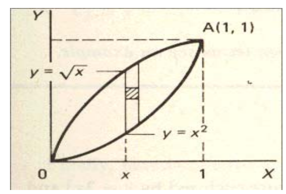


$$I = \oint_C \{xy\,dx + (2x - y)\,dy\}$$

$$= \oint_C \{Pdx + Qdy\} = -\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$$

$$P = xy \quad \therefore \frac{\partial P}{\partial y} = x; \qquad Q = 2x - y \quad \therefore \frac{\partial Q}{\partial x} = 2$$

$$I = -\iint_R (x - 2)\,dx\,dy = -\int_0^1 \int_{y=x^2}^{y=\sqrt{x}} (x - 2)\,dy\,dx$$



$$= -\int_0^1 (x - 2)\left[\,y\,\right]_{x^2}^{\sqrt{x}} dx$$

$$\therefore\ I = -\int_0^1 (x - 2)\left(\sqrt{x} - x^2\right) dx = -\int_0^1 (x^{3/2} - x^3 - 2x^{1/2} + 2x^2)\,dx$$

$$= -\left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4 - \frac{4}{3}x^{3/2} + \frac{2}{3}x^3\right]_0^1 = \frac{31}{60}$$

In this special case when P=y and Q= – x so $\frac{\partial P}{\partial y} = 1$ and $\frac{\partial Q}{\partial x} = -1$

Green's theorem then states $\iint_R \{1 - (-1)\}\,dx\,dy = -\oint_C (P\,dx + Q\,dy)$

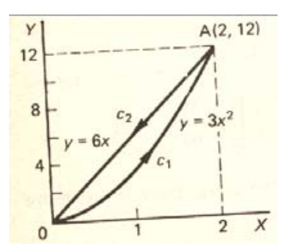
i.e. $2\iint_R dx\,dy = -\oint_C (y\,dx - x\,dy) = \oint_C (x\,dy - y\,dx)$

Therefore, the area of the closed region $A = \iint_R dx\,dy = \frac{1}{2}\oint_C (x\,dy - y\,dx)$

Example 2: Determine the area of the figure enclosed by $y = 3x^2$ and $y = 6x$.

Solution: Points of intersection: $3x^2 = 6x \quad \therefore\ x = 0$ or 2

Area $A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$



We evaluate the integral in two parts, i.e.
OA along c_1 and AO along c_2

$$2A=\int_{c_1 \text{ (along OA)}} (xdy-ydx) + \int_{c_2 \text{ (along OA)}} (xdy-ydx) = I_1+I_2$$

I_1: c_1 is $y = 3x^2$ $\therefore$ $dy = 6x\,dx$

$$\therefore \quad I_1 = \int_0^2 (6x^2dx - 3x^2dx) = \int_0^2 3x^2\,dx = \left[\, x^3 \,\right]_0^2 = 8 \;\therefore\; I_1 = 8$$

Similarly, for c_2 is $y = 6x$ $\therefore$ $dy = 6\,dx$

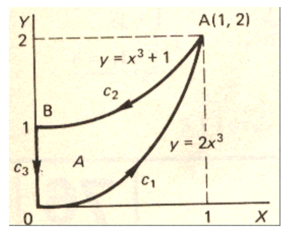
$$\therefore \quad I_2 = \int_2^0 (6x\,dx - 6x\,dx) = 0$$

$\therefore$ $I_2 = 0$

$\therefore$ $I = I_1 + I_2 = 8 + 0 = 8$

$\therefore$ $A = 4$ square units

Example 3: Determine the area bounded by the curves $y = 2x^3$, $y = x^3 + 1$ and the axis $x = 0$ for $x \geq 0$.



Solution: Here it is $y = 2x^3$; $y = x^3 + 1$; $x = 0$

Point of intersection $2x^3 = x^3 + 1$ $\therefore$ $x^3 = 1$ $\therefore$ $x = 1$

Area $A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$ $\therefore$ $2A = \oint_C (x\,dy - y\,dx)$

(a) OA : c_1 is $y = 2x^3$ $\therefore$ $dy = 6x^2\,dx$

$$\therefore I_1=\int_{c_1} (xdy - ydx)=\int_0^1 (6x^3dx-2x^3\,dx) \qquad = \int_0^1 4x3\,dx = \left[\, x^4 \,\right]_0^1 = 1$$

$\therefore$ $I_1 = 1$

(b) AB: c_2 is $y = x^3 + 1$ $\therefore$ $dy = 3x^2\,dx$

$$\therefore \quad I_2 = \int_1^0 \{3x^3\,dx - (x^3 + 1)\,dx\} = \int_1^0 (2x^3 - 1)\,dx = \left[\frac{x^4}{2} - x\right]_1^0 = -\left(\frac{1}{2} - 1\right) = \frac{1}{2}$$

$\therefore$ $I_2 = \frac{1}{2}$

(c) BO: c_3 is $x = 0$ $\therefore$ $dx = 0$

$$I_3 = \int_{y=1}^{y=0} (xdy - ydx) = 0 \quad \therefore\; I_3 = 0$$

$$\therefore \quad 2A = I = I_1 + I_2 + I_3 = 1 + \frac{1}{2} + 0 = 1\frac{1}{2} \qquad \therefore\; A = \frac{3}{4} \text{ square units}$$

Revision Summary

Properties of line integrals

- Sign of line integral is reversed when the direction of integration along the path is reversed.
- Path of integration parallel to y-axis, $dx = 0$ $\therefore$ $I_c = \int_c Q\,dy$.

- Path of integration parallel to x-axis, dy = 0 $\therefore$ $I_c = \int_c P\,dx$.
- Path of integration must be continuous and single-valued.
- Dependence of line integral on path of integration.
- In general, the value of the line integral depends on the particular path of integration.

- **Exact differential**
  If P dx + Q dy is an exact differential, then
  (a) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$
  (b) $I = \int_c (P\,dx + Q\,dy)$ is independent of the path of integration
  (c) $I = \oint_C (P\,dx + Q\,dy)$ is zero.
- **Exact differential in three variables.**
  If P dx + Q dy + R dw is an exact differential
  (a) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$; $\dfrac{\partial P}{\partial w} = \dfrac{\partial R}{\partial x}$; $\dfrac{\partial R}{\partial y} = \dfrac{\partial Q}{\partial w}$
  (b) $\int_c (P\,dx + Q\,dy + R\,dw)$ is independent of the path of integration.
  (c) $\oint_C (P\,dx + Q\,dy + R\,dw)$ is zero.
- **Green's theorem**
  $\oint_C (P\,dx+Q\,dy) = -\iint_R \left\{\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right\} dx\,dy$ and, for a simple closed curve,
  $\oint_C (x\,dy - y\,dx) = 2\iint_R dx\,dy = 2A$
  where A is the area of the enclosed figure.

<u>Gradient of a scalar function</u>

Del operator is given by $\nabla = \left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}\right)$

$$\nabla\phi = \text{grad }\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$

$$\text{grad }\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

<u>Div (Divergence of a vector function)</u>

If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ **,** then

$$\text{div A} = \nabla.\text{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).\left(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\right)$$

$$\therefore \text{ div }\mathbf{A} = \nabla.\mathbf{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

Note that

(a) the grad operator ∇ acts on a scalar and gives a vector

(b) the div operator ∇ acts on a vector and gives a scalar.

Example 4: If A = $x^2y\mathbf{i} - xyz\mathbf{j} + yz^2\mathbf{k}$, then find Div **A.**

Solution:

$$\text{Div } \mathbf{A} = \nabla.\mathbf{A} = \frac{\partial}{\partial x}(x^2 y) - \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(yz^4) = 2xy - xz + 2yz$$

Example 5: If A = $2x^2y\mathbf{i} - 2(xy^2 + y^3\mathbf{z})\mathbf{j} + 3y^2z^2\mathbf{k}$, determine **∇.A** i.e. div **A**.

Solution: **A**= $2x^2y\mathbf{i} - 2(xy^2 + y^3z)\mathbf{j} + 3y^2z^2\mathbf{k}$

$$\nabla.\mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 4xy - 2(2xy+3y^2\mathbf{z}) + 6y^2z = 4xy - 4xy - 6y^2z + 6y^2z = 0$$

Such a vector A for which **∇.A** = 0 at all points, i.e. for all values of x, y, z, is called a solenoid vector. It is rather a special case.

<u>Curl (Curl of a Vector Function)</u>

The curl operator denoted by ∇ × **A**, acts on a vector and gives another vector as a result.

If **A**= $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ then curl **A=∇×A**.

$$\text{i.e. curl } \mathbf{A} = \nabla\times\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\times\left(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$\therefore\ \nabla\times\mathbf{A} = \mathbf{i}\left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) + \mathbf{j}\left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right) + \mathbf{k}\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)$$

Curl **A** is thus a vector function.

Example 6: If **A**=$(y^4 - x^2z^2)\mathbf{i} + (x^2+y^2)\mathbf{j} - x^2yzk$, determine curl **A** at the point (1,3, −2).

Solution: Curl **A** = ∇ × **A** =
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y4 - x^2z^2 & x^2+y^2 & -x^2yz \end{vmatrix}$$

Now we expand the determinant

$$\nabla\times\mathbf{A} = \mathbf{i}\left\{\frac{\partial}{\partial y}(-x^2yz) - \frac{\partial}{\partial z}(x^2+y^2)\right\} - \mathbf{j}\left\{\frac{\partial}{\partial x}(-x^2yz) - \frac{\partial}{\partial z}(y^4 - x^2z^2)\right\}$$

$$+ \mathbf{k}\left\{\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(y^4 - x^2z^2)\right\}$$

$\nabla\times\mathbf{A} = \mathbf{i}\{-x^2z\} - \mathbf{j}\{-2xyz + 2x^2z\} + \mathbf{k}(2x - 4y^3\}$. ∴ At (1, 3, − 2),

$\nabla\times\mathbf{A} = \mathbf{i}\,(2) - \mathbf{j}\,(12 - 4) + \mathbf{k}\,(2 - 108) = 2\mathbf{i} - 8\mathbf{j} - 106\mathbf{k}$

Example 7:

Determine curl F at the point (2,0,3) given that $\mathbf{F} = z\,e^{2xy}\mathbf{i} + 2\,x\,\mathbf{z}\cos y\mathbf{j} + (x+2y)\mathbf{k}$.

Solution: In determinant form, curl $\mathbf{F} = \nabla \times \mathbf{F}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^{2xy} & 2xz\cos y & x+2y \end{vmatrix}$$

Now expand the determinant and substitute the values for x, y and z, finally obtaining curl

$$\nabla \times \mathbf{F} = \mathbf{i}\{2 - 2x\cos y\} - \mathbf{j}\{1 - e^{2xy}\} + \mathbf{k}\,(\{2z\cos y - 2xze^{2xy}\}$$

$$\therefore \text{At } (2,0,3) \quad \nabla \times \mathbf{F} = \mathbf{i}(2-4) - \mathbf{j}(1-1) + \mathbf{k}(6-12) = -2\mathbf{i} - 6\mathbf{k} = -2\,(\mathbf{i} + 3\mathbf{k})$$

Summary of grad, div and curl

(a) Grad operator acts on a scalar field to give a vector field.

(b) Div operator acts o a vector field to give a scalar field.

(c) Curl operator acts on a vector field to give a vector field.

(d) With a scalar function ϕ (x,y,z)

$$\text{Grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

(e) With a vector function $\mathbf{A} = a_x i + a_y \mathbf{j} + a_z \mathbf{k}$

(i) $\text{div } \mathbf{A} = \nabla.\,\mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$

(ii) $\text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$

Multiple Operations

We can combine the operators grad, div and curl in multiple operations, as in the examples that follow.

EXAMPLE 8: If $A = x^2y\mathbf{i} + yz^3\mathbf{j} - zx^3\mathbf{k}$, then find grad div A.

Solution: $\text{div } \mathbf{A} = \nabla.\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).(x^2y\mathbf{i} + yz^3\mathbf{j} - zx^3\mathbf{k})$

$$= 2xy + z^3 - x^3 = \phi \quad \text{(say)}$$

Now $\text{grad (div } \mathbf{A}) = \nabla(\nabla.\mathbf{A}) = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = (2y - 3x^2)\mathbf{i} + (2x)\mathbf{j} + (3z^2)\mathbf{k}$

i.e., $\text{grad (div } \mathbf{A}) = \nabla(\nabla.\,\mathbf{A}) = (2y - 3x^2)\mathbf{i} + 2x\mathbf{j} + 3z^2\mathbf{k}$

Example 9: If $\phi = xyz - 2y^2z + x^2z^2$ determine **div grad** ϕ at the point (2, 4, 1).

Solution: First find grad ϕ and then the div of the result.

$$\text{div grad } \phi = \nabla.(\nabla\phi)$$

We have $\phi = xyz - 2y^2z + x^2z^2$

$$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = (yz+2xz^2)\mathbf{i} + (xz-4yz)\mathbf{j} + (xy-2y^2+2x^2z)\mathbf{k}$$

$$\therefore \text{ div grad } \phi = \nabla.(\nabla\phi) = 2z^2 - 4z + 2x^2$$

$$\therefore \text{ At } (2,4,1), \quad \text{div grad } \phi = \nabla.(\nabla\phi) = 2 - 4 + 8 = 6$$

REMARK: Let $\text{grad } \phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$

Then $\text{div grad } \phi = \nabla.(\nabla\phi) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial z^2}$

$$\therefore \text{div grad } \phi = \nabla.(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Example 10: If $F = x^2yz\mathbf{i} + xyz^2\mathbf{j} + y^2z\mathbf{k}$ determine curl **F** at the point (2, 1, 1). Determine an expression for curl F in the usual way, which will be a vector, and then the curl of the result. Finally substitute values.

Solution: Curl curl $\mathbf{F} = \nabla\times(\nabla\times\mathbf{F}) = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xyz^2 & y^2z \end{vmatrix} = (2yz-2xyz)\mathbf{i} + x^2y\mathbf{j} + (yz^2 - x^2z)\mathbf{k}$$

$$\text{Curl Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz-2xyz & x^2y & yz^2-x^2z \end{vmatrix} = z^2\mathrm{i} - (-2xz - 2y + 2xy)\mathbf{j} + (2xy - 2z + 2xz)\mathbf{k}$$

$$\therefore \text{ At } (2, 1, 1), \quad \text{curl cul } \mathbf{F} = \nabla\times(\nabla \times \mathbf{F}) = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

Two interesting general results

(a) Curl grad ϕ where ϕ is a scalar

$$\text{grad } \phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$\therefore \text{ curl grad } \phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \mathbf{i}\left\{\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right\} - \mathbf{j}\left\{\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right\} + \mathbf{k}\left\{\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right\}$$

$$= \mathbf{i}\,0 - \mathbf{j}\,0 + \mathbf{k}\,0 = \mathbf{0}$$

$\therefore$ curl grad $\phi = \nabla \times (\nabla\phi) = \mathbf{0}$

(b) Div curl A where A is a vector.

$$\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \mathbf{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) - \mathbf{j}\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) + \mathbf{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

Then div curl $\mathbf{A} = \nabla.(\nabla\times\mathbf{A}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \mathbf{.}\, (\nabla \times \mathbf{A})$

$$= \frac{\partial^2 a_z}{\partial x\partial y} - \frac{\partial^2 a_y}{\partial z\partial x} - \frac{\partial^2 a_z}{\partial x\partial y} + \frac{\partial^2 a_x}{\partial y\partial z} + \frac{\partial^2 a_y}{\partial z\partial x} - \frac{\partial^2 a_x}{\partial y\partial z} = 0$$

$\therefore$ div curl $\mathbf{A} = \nabla \mathbf{.} (\nabla \times \mathbf{A}) = 0$

(c) Div grad ϕ where ϕ is a scalar.

$$\text{grad } \phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Then div grad $\phi = \nabla.(\nabla\phi) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\mathbf{.}\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial z^2}$

$$\therefore \text{div grad } \phi = \nabla.(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

This result is sometimes denoted by $\nabla^2\phi$.

So these general results are

(a) curl grad $\phi = \nabla\times(\nabla\phi) = 0$
(b) div curl $\mathbf{A} = \nabla.(\nabla\times\mathbf{A}) = 0$
(c) div grad $\phi = \nabla.(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

LECTURE No. 35

DEFINITE INTEGRALS

Definite Integral for $\sin^n x$ and $\cos^n x$, $0 \le x \le \pi/2$

(1) $$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x)dx = \frac{1}{2}\left| x - \frac{\sin 2x}{2} \right|_0^{\frac{\pi}{2}} = \frac{1}{2}\left| \frac{\pi}{2} - \frac{\sin \pi}{2} \right| = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{1}{2}\frac{\pi}{2}$$

(2) $$\int_0^{\frac{\pi}{2}} \cos^2 x dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2x)dx = \frac{1}{2}\left| x + \frac{\sin 2x}{2} \right|_0^{\frac{\pi}{2}} = \frac{1}{2}\left| \frac{\pi}{2} + \frac{\sin \pi}{2} \right| = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx = \frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^3 x dx = \int_0^{\frac{\pi}{2}} \sin^2 x \sin x dx = \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\sin x dx = \int_0^{\frac{\pi}{2}} \sin x dx + \int_0^{\frac{\pi}{2}} \cos^2 x(-\sin x)dx$$

$$= \left| -\cos x \right|_0^{\frac{\pi}{2}} + \left| \frac{\cos^3 x}{3} \right|_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} + \cos 0 + \frac{1}{3}\left[\cos^3 \frac{\pi}{2} - \cos^3 0 \right] = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \cos^3 x dx = \int_0^{\frac{\pi}{2}} \cos^2 x \cos x dx = \int_0^{\frac{\pi}{2}} (1-\sin^2 x)\cos x dx = \int_0^{\frac{\pi}{2}} \cos x dx - \int_0^{\frac{\pi}{2}} \sin^2 x(\cos x)dx$$

$$= \left| \sin x \right|_0^{\frac{\pi}{2}} - \left| \frac{\sin^3 x}{3} \right|_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 - \frac{1}{3}\left[\sin^3 \frac{\pi}{2} - \sin^3 0 \right] = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^4 x dx = \int_0^{\frac{\pi}{2}} (\sin^2 x)^2 dx = \int_0^{\frac{\pi}{2}} \left[\frac{1-\cos 2x}{2} \right]^2 dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} (1 - 2\cos 2x + \cos^2 2x)dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (1 - 2\cos 2x + \frac{1+\cos 4x}{2})dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} (\frac{3}{2} - 2\cos 2x + \frac{c\,os 4x}{2})dx$$

$$= \frac{1}{4}\left| \frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8} \right|_0^{\frac{\pi}{2}} = \frac{1}{4}\left[\frac{3}{2}\frac{\pi}{2} - \sin\pi + \frac{\sin 2\pi}{8} \right]$$

$$\int_0^{\frac{\pi}{2}} \sin^4 x dx = \frac{1}{4}\left[\frac{3}{2}\frac{\pi}{2} \right] \qquad so \qquad \int_0^{\frac{\pi}{2}} \sin^4 x dx = \frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos^4 x dx = \int_0^{\frac{\pi}{2}} (\cos^2 x)^2 dx = \int_0^{\frac{\pi}{2}} \left[\frac{1+\cos 2x}{2}\right]^2 dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} (1+2\cos 2x+\cos^2 2x)dx$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (1+2\cos 2x+\frac{1+\cos 4x}{2})dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} (\frac{3}{2}+2\cos 2x+\frac{c\,os\,4x}{2})dx$$

$$= \frac{1}{4}\left|\frac{3}{2}x+\sin 2x+\frac{\sin 4x}{8}\right|_0^{\frac{\pi}{2}} = \frac{1}{4}\left[\frac{3}{2}\frac{\pi}{2}+\sin\pi+\frac{\sin 2\pi}{8}\right]$$

$$\int_0^{\frac{\pi}{2}} \cos^4 x dx = \frac{1}{4}\left[\frac{3}{2}\frac{\pi}{2}\right] \qquad \text{So} \qquad \int_0^{\frac{\pi}{2}} \cos^4 x dx = \frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^5 x dx = \frac{4}{5}\frac{2}{3} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^5 x dx = \frac{4}{5}\frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^6 x dx = \frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^6 x dx = \frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^7 x dx = \frac{6}{7}\frac{4}{5}\frac{2}{3} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^7 x dx = \frac{6}{7}\frac{4}{5}\frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^8 x dx = \frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^8 x dx = \frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^9 x dx = \frac{8}{9}\frac{6}{7}\frac{4}{5}\frac{2}{3} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^9 x dx = \frac{8}{9}\frac{6}{7}\frac{4}{5}\frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^{10} x dx = \frac{9}{10}\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^8 x dx = \frac{9}{10}\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

Wallis Sine Formula

When n is even

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.\frac{n-7}{n-6}.---------\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

When n is odd

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.\frac{n-7}{n-6}.---------\frac{6}{7}.\frac{4}{5}.\frac{2}{3}.$$

$$\int_0^{\frac{\pi}{2}} \sin^{11} x dx = \frac{10.8.6.4.2}{11.9.7.5.3} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^{11} x dx = \frac{10.8.6.4.2}{11.9.7.5.3}$$

$$\int_0^{\frac{\pi}{2}} \sin^{12} x dx = \frac{11.9.7.5.3.1}{10.8.6.4.2}\frac{\pi}{2} \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \cos^{12} x dx = \frac{11.9.7.5.3.1}{10.8.6.4.2}\frac{\pi}{2}$$

Integration By Parts

$$\int UVdx = U\int V\,dx - \int\left[\int Vdx.\frac{dU}{dx}\right]dx$$

Example: *Evaluate* $\int$ x lnx dx

$\int$ x lnx dx = lnx $\int$ xdx – $\int$ [$\int$ x dx.$\frac{d}{dx}$(ln x)] dx (Using integrating by parts)

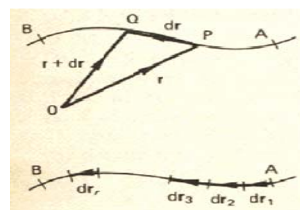
$= \ln x(\frac{x^2}{2}) - \int (\frac{x^2}{2})(\frac{1}{x})dx = (\frac{x^2}{2})\ln x - \int (\frac{x}{2})d\,x = (\frac{x^2}{2})\ln x - \frac{1}{2}(\frac{x^2}{2})$

Example: *Evaluate* $\int$ x sinx dx

$\int$ **x** sinx dx =x$\int$ sin xdx – $\int$ [$\int$ sinx dx.$\frac{d}{dx}$(x)] dx (We are integrating by parts)

$= x(-\cos x) - \int (-\cos x)(1)dx = -x(\cos x) + \int \cos x\,d\,x = -x(\cos x)+\sin x$

Line Integrals

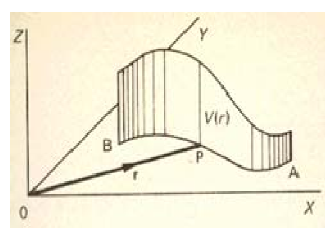


Let a point p on the curve c joining A and B be denoted by the position vector **r** with respect to origin O. If q is a neighboring point on the curve with position vector **r + dr**, then $\overline{PQ}$ = **r**

The curve c can be divided up into many n such small arcs, approximating to d**r**$_1$, d**r**$_2$, d**r**$_3$,………. d**r**$_p$,……

so that $\overline{AB}\sum_{p=1}^{n} dr_p$ where d**r**$_p$ is a vector representing the element of the arc in both magnitude and direction. If dr $\to$ 0, then the length of the curve AB=$\int_c dr$.

Scalar Field



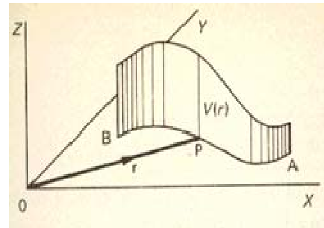
If a scalar field V(r) exists for all points on the curve , the $\sum_{p=1}^{n} V(r)dr_p$ with dr $\to$ 0 , defines the line integral of V i.e line integral = $\int_c V(r)dr$.

We can illustrate this integral by erecting a continuous Ordinate to V(r) at each point of the curve $\int_c V(r)dr$ is then represented by the area of the curved surface between the ends A and B the curve c. To evaluate a line integral, the integrand is expressed in terms of x, y, z with d**r** = dx **i** + dy **j** + dz **k**

In practice, x, y and z are often expressed in terms of parametric equation of a fourth variable (say u), i.e. x = x(u) ; y = y(u) ; z = z(u) . From these, dx, dy and dz can be written in terms of u and the integral evaluate in terms of this parameter u.

LECTURE No.36
SCALAR FIELD

Scalar Field



If a scalar field V(r) exists for all points on the curve, then $\sum_{p=1}^{n} V(r)dr_p$ with dr $\rightarrow$ 0 defines the line integral of V i.e. line integral = $\int_c V(r)dr.$

We can illustrate this integral by erecting a continuous Ordinate to V(r) at each point of the curve $\int_c V(r)dr$ is then represented by the area of the curved surface between the ends A and B the curve c. To evaluate a line integral, the integrand is expressed in terms of x, y, z with d**r** =dx i + dy j + dz k

In practice, x, y and z are often expressed in terms of parametric equation of a fourth variable (say u), i.e. x = x(u) ; y = y(u) ; z = z(u) . From these, dx, dy and dz can be written in terms of u and the integral evaluate in terms of this parameter u.

Example: If $V=xy^2z$, evaluate $\int_c V(r)dr$ along the curve c having parametric equations $x = 3u; y=2u^2 ;z=u^3$ between A(0,0,0) and B(3,2,1)

Solution: $V= xy^2z = (3u)(4u^4)(u^3)=12u^8$

$d\mathbf{r}= dx\mathbf{i}+ dy\,\mathbf{j}+ dz\,\mathbf{k} \Rightarrow d\mathbf{r}= 3du\,\mathbf{i} +4udu\,\mathbf{j} +3u^2du\,\mathbf{k}$

for $x = 3u$; $\therefore$ $dx = 3du$; $y = 2u^2$ $\therefore$ $dy = 4u\,du$; $z = u^3$ $\therefore$ $dz =3u^2dz$

Limiting: A(0,0,0) corresponds to B(3,2,1) corresponds to u

A(0,0,0) $\equiv$ u=0 ; B(3,2,1) $\equiv$ u = 1

$$\int_c V(r)dr = \int_0^1 12u^8(3\,\boldsymbol{i} +4u\,\boldsymbol{j} +3u^2\,\boldsymbol{k})du=\left|36\frac{u^9}{9}\boldsymbol{i}+48\frac{u^{10}}{10}\boldsymbol{j}+36\frac{u^{11}}{11}\boldsymbol{k}\right|_0^1 = 4\boldsymbol{i}+\frac{24}{5}\boldsymbol{j}+\frac{36}{11}\boldsymbol{k}$$

Example : If $V = xy + y^2z$ Evaluate $\int_c V(r)dr$ along the curve c defined by $x= t^2$; $y = 2t$; $z= t+5$ between A(0,0,5) and B(4,4,7) . As before, expressing V and d**r** in term of the parameter t .

Solution:

since $V=xy+y^2z$

$= (t^2)(2t)+(4t^2)(t+5)$

$= 6t^3 + 20t^2.$

$$\left.\begin{array}{l} x = t^2 \quad dx = 2t\,dt \\ y = 2t \quad dy = 2\,dt \\ z = t+5 \quad dz = dt \end{array}\right\}$$

$\therefore$ $dr = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$

$= 2t\,dt\,\mathbf{i} + 2\,dt\,\mathbf{j} + dt\mathbf{k}$

$\therefore \int_C Vdr = \int_C (6t^3+20t^2)(2t\,\mathbf{i} +2\mathbf{j}+\mathbf{k})\,dt$

Limits: A (0, 0, 5) $\equiv t = 0$;
B (4, 4, 7) $\equiv t = 2$

$$\therefore \int_C V dr = \int_0^2 (6t^3+20t^2)(2t\,\mathbf{i} +2\,\mathbf{j} +\mathbf{k})\,dt$$

$$\int_C V dr = 2\int_0^2 \{(6t^4+20t^3)\mathbf{i}+(6t^3+20t^2)\mathbf{j} +(3t^3+10t^2)\mathbf{k}\}dt.$$

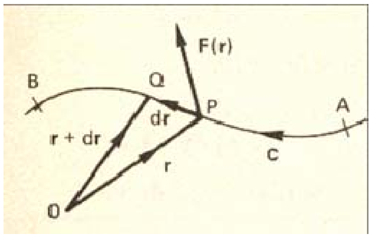
$$= \frac{8}{15}(444\mathbf{i} + 290\mathbf{j} + 145\mathbf{k})$$

Vector Field

If a vector field **F**(r) exists for all points of the curve c, then for each element of arc we can form the scalar product $\mathbf{F}\cdot d\mathbf{r}$. Summing these products for all elements of arc, we have $\sum_{p=1}^{n} F.dr_p$

The line integral of F(r) from A to B along the stated curve = $\int_C$ F.dr.

In this case, since **F**.d**r** is a scalar product, then the line integral is a scalar. To evaluate the line integral, F and d**r** are expressed in terms of x,y,z, and the curve in parametric form. We have



$\mathbf{F}= F_1\,\mathbf{i} + F_2\,\mathbf{j} +F_3\,\mathbf{k}$

And $d\mathbf{r} = dx\,\mathbf{i} +dy\,\mathbf{j} +dz\,\mathbf{k}$

Then $\mathbf{F}.d\mathbf{r} = (F_1\,\mathbf{i} + F_2\,\mathbf{j} +F_3\,\mathbf{k}).(dx\,\mathbf{i} +dy\,\mathbf{j} +dz\,\mathbf{k}) = \int_c (F_1dx + F_2dy +F_3dz)$

Now for an example to show it in operation.

Example

If F(r) = $x^2y\,\mathbf{i} + xz\,\mathbf{j} + 2yz\,\mathbf{k}$ **,** Evaluate $\int_c F.dr$ between A(0,0,0) and B(4,2,1) along the curve c having parametric equations $x=4t$; $y =2t^2$; $z = t^3$

Solution: Expressing everything in terms of the parameter t, we have

$dx = 4\,dt$; $dy = 4t\,dt$; $dz = 3t^2\,dt$

$x^2y = (16t^2)(2t^2) = 32\,t^4$

$x = 4t \qquad \therefore \quad dx = 4\,dt$

$xz = (4t)(t^3) = 4\,t^4$

$y = 2\,t^2 \qquad \therefore \quad dy = 4t\,dt$

$2yz = (4\,t^2)(t^3) = 4\,t^5$

$z = t^3 \qquad \therefore \quad dz = 3t^2\,dt$

$\mathbf{F} = 32\,t^4\,\mathbf{i} + 4\,t^4\,\mathbf{j} - 4\,t^5\,\mathbf{k}$

$d\mathbf{r} = 4dt\,\mathbf{i} + 4t\,dt\,\mathbf{j} + 3t^2\,\mathbf{k}$

Then $\int \mathbf{F}.d\mathbf{r} = \int (32t^4\mathbf{i}+4t^4\mathbf{j}-4t^5\mathbf{k}).$

$(4dt\,\mathbf{i} + 4t\,dt\,\mathbf{j} + 3t^2\,dt\,\mathbf{k})$

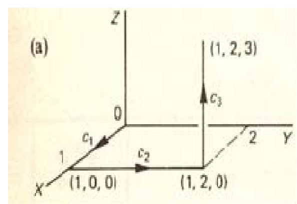
$= \int (128t^4+16t^5+12t^7)\,dt$

Limits: $A(0,0,0) \equiv t = 0;$

$B\,(4, 2, 1) \equiv t = 1$

$\int_C \mathbf{F}.d\mathbf{r} = (128t^4 + 16t^5 + 12t^7)dt = \frac{128}{5}\,t^5 + \frac{16}{6}\,t^6 + \frac{12}{8}\,t^8 = \frac{128}{5} + \frac{8}{3} + \frac{3}{2} = 29.76$

Example



If $\mathbf{F}(\mathbf{r}) = x^2y\mathbf{i} + 2yz\mathbf{j} + 3z^2x\mathbf{k}$

Evaluate $\int_C \mathbf{F}.d\mathbf{r}$ between A(0,0,0) and B(1,2,3)

B (1, 2, 3)

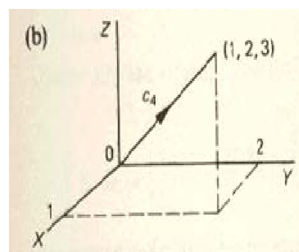
(a) along the straight line

c_1 from (0, 0, 0) to (1, 0, 0)

then c_2 from (1, 0, 0) to (1, 2, 0)

and c_3 from (1, 2, 0) to (1, 2, 3)

(b) along the straight line c_4 joining (0, 0, 0) to (1, 2, 3).



We first obtain an expression for $\mathbf{F}.d\mathbf{r}$ which is

$\mathbf{F}.d\mathbf{r} = (x^2y\mathbf{i} + 2yz\mathbf{j} + 3z^2x\mathbf{k}).$

$(dx\,\mathbf{i}+ dy\mathbf{j} + dz\,\mathbf{k})$

$\mathbf{F}.d\mathbf{r} = x^2y\,dx + 2yz\,dy + 3z^2x\,dz$

$\int\mathbf{F}.d\mathbf{r} = \int x^2ydx + \int 2yzdy+\int 3z^2xdz$

Here the integration is made in three sections, along c_1, c_2 and c_3.

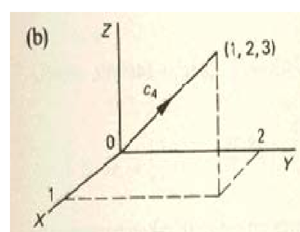
(i) c_1: $y = 0, z = 0, dy = 0, dz = 0$

$\therefore \int_{C_1} \mathbf{F}.d\mathbf{r}=0+0+0 = 0$

(ii) c_2: The conditions along c_2 are

c_2: $x = 1, z = 0, dx = 0, dz = 0$

$\therefore \int_{C_2} F.dr = 0 + 0 + 0 = 0$



(iii) c_3: $x = 1, y = 2, dx = 0, dy = 0$

$\int_{C_3} F.dr = 0 + 0 + \int_0^3 3z^2dz = 27$

Summing the three partial results

$\int_{(0,0,0)}^{(1,2,3)} \mathbf{F}.d\mathbf{r} = 0 + 0 + 27 = 27$

$\therefore \int_{c_1+c_2+c_3} \mathbf{F}.d\mathbf{r} = 27$

If t taken as the parameter, the parametric equation of c are $x = t;\ y = 2t;\ z = 3t$

$(0, 0, 0) \Rightarrow t = 0,\ (1, 2, 3) \Rightarrow t = 1$ and the limits of t are $t = 0$ and $t = 1$

$\mathbf{F} = 2t^3\mathbf{i} + 12t^2\mathbf{j} + 27t^3\mathbf{k}$

$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + kdz = dt\,\mathbf{i} + 2dt\,\mathbf{j} + 3dt\,\mathbf{k}$

$$\int_{C_4} \mathbf{F}.d\mathbf{r} = \int_0^1 (2t^3\mathbf{i}+12t^3\mathbf{j}+27t^3\mathbf{k}).(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\,dt = \int_0^1 (2t^3+24t^2+81t^3)dt$$

$$= \int_0^1 (83t^3 + 24t^2)\,dt = \left[83\frac{t^4}{4} + 8t^3\right]_0^1 = \frac{115}{4} = 28.75$$

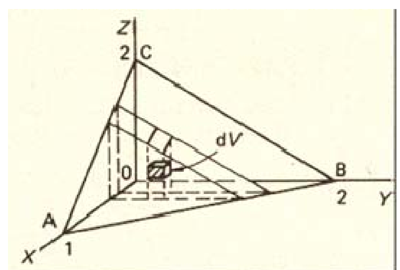
So the value of the line integral depends on the path taken between the two end points A and B

(a) $\int \mathbf{F}.d\mathbf{r}$ via c_1, c_2 and $c_3 = 27$

(b) $\int \mathbf{F}.d\mathbf{r}$ via $c_4 = 28.75$

Example

Evaluate $\int_v F\,dv$ where V is the region bounded by the planes x = 0, y = 0, z = 0 and 2x + y = 2 , and F = 2z **i** +y **k.** To sketch the surface 2x + y + z = 2, note that



when z = 0, 2x+y=2 i.e. y = 2 − 2x
when y = 0, 2x+z=2 i.e. z = 2 − 2x
when x = 0, y+z=2 i.e. z = 2 − y
Inserting these in the planes
x = 0, y = 0, z = 0 will help.
The diagram is therefore.

So 2x + y + z = 2 cuts the axes at
A(1,0,0); B (0, 2, 0); C (0, 0, 2).
Also F = 2z**i** + y**k**;
z = 2 − 2x − y = 2 (1 − x) − y

$$\therefore \int_V F dV = \int_0^1 \int_0^{2(1-x)} \int_0^{2(1-x)-y} (2x\mathbf{i}+y\mathbf{k})dzdydx$$

$$= \int_0^1 \int_0^{2(1-x)} \left[z^2\mathbf{i}+yz\mathbf{k}\right]_{z=0}^{z=2(1-x)-y} dydx$$

$$= \int_0^1 \int_0^{2(1-x)} \{[4(1-x)^2-4(1-x)y+y^2]\mathbf{i} + [2(1-x)y-y^2]k\}dydx$$

$$\int_V FdV = \frac{1}{3}(2\mathbf{i} + \mathbf{k})$$

Lecture No. 37

Higher order derivative and Leibniz theorem

Derivative of a function

The concept of **Derivative** is at the core of Calculus and modern mathematics. The definition of the derivative can be approached in two different ways. One is geometrical (as a slope of a curve) and the other one is physical (as a rate of change).

We know that if y=f(x) is a single valued function of a continuous variable, and if the ratio $\frac{1}{h}\{f(x+h)-f(x)\}$ tends to a definite limit as the value of h tends to zero through positive or negative directions, then we say that the function has a derivative at the point 'x' . If the ratio has no limiting value then the function has no derivative at the point x. Symbolically it is represented as

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

If the derivative of a function $y = f(x)$ is itself a continuous function $y' = f'(x)$, we can take the derivative of $f'(x)$, which is generally referred to as the *second derivative of f(x)* and written $f''(x)$. Similarly, the third derivative is obtained by differentiating second derivative as given below.

$$f'''(x)=(f''(x))'$$

This can continue as long as the resulting derivative is itself differentiable, with the fourth derivative, the fifth derivative, and so on.

Any derivative beyond the first derivative can be referred to as a higher order derivative.

Interpretation:

A first derivative tells how fast a function is changing i.e., how fast it's going up or down which is graphically the slope of the curve. A second derivative tells how fast the first derivative is changing or, in other words, how fast the slope is changing. A third derivative informs about how fast the second derivative is changing, i.e., how fast the rate of change of the slope is changing.

Notation

Let $f(x)$ be a function of x. The following are notations for higher order derivatives.

2nd derivative	**3rd derivative**	**4th derivative**	**nth derivative**	**remarks**
$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$f^{(n)}(x)$	Probably the most common notation
$\frac{d^2 f}{dx^2}$	$\frac{d^3 f}{dx^3}$	$\frac{d^4 f}{dx^4}$	$\frac{d^n f}{dx^n}$	Leibniz notation.
$\frac{d^2}{dx^2}[f(x)]$	$\frac{d^3}{dx^3}[f(x)]$	$\frac{d^4}{dx^4}[f(x)]$	$\frac{d^n}{dx^n}[f(x)]$	Another form of Leibniz notation.
$D^2 f$	$D^3 f$	$D^4 f$	$D^n f$	Euler's notation.

Because the “prime” notation for derivatives would eventually become somewhat messy, it is preferable to use the numerical notation $f^{(n)}(x) = y^{(n)}(x)$ to denote the *n*th derivative of *f*(x).

Example:

$$f(x) = 15x^3 - 3x^2 + 20x - 5$$

Its **first derivative** is given as
$f'(x) = 45x^2 - 6x + 20$
Now, this is again a continuous function and therefore can be differentiated. Its derivative which will be the **second derivative** of given function will become
$f''(x) = (f'(x))' = 90x - 6$
As, this is a continuous function so we can differentiate it again. This will be called the **third derivative which is**

$$f'''(x) = (f''(x))' = 90$$

Continuing, **fourth derivative** will be

$$f^{(4)}(x) = (f'''(x))' = 0$$

(We have changed the notation at this point. We can keep adding on primes, but that will get cumbersome as we calculate the derivatives higher than third.)

This process can continue but notice that we will get zero for all derivatives after this point.
This above example leads us to the following fact about the differentiation of polynomials.

Note:

1) If *p(x)* is a polynomial of degree *n* (*i.e.* the largest exponent in the polynomial) then,

$$p^{(k)}(x) = 0 \qquad \text{for } k \geq n+1$$

2) We will need to be careful with the "non-prime" notation for derivatives. Consider each of the following

$$f^{(2)}(x) = f''(x)$$

$$f^{2}(x) = [f(x)]^2$$

The presence of parenthesis in the exponent denotes differentiation while the absence of parenthesis denotes exponentiation.

Example:

If

$f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2,$ then

$f'(x) = 12x^3 - 6x^2 + 2x - 4$

$f''(x) = 36x^2 - 12x + 2$

$f'''(x) = 72x - 12$

$f^{(4)}(x) = 72$

$f^{(5)}(x) = 0$

.

.

$f^{(n)}(x) = 0 \qquad (n \geq 5)$

In the above two examples, we have seen that all polynomial functions eventually go to zero when you differentiate repeatedly. On the other hand, rational functions like

$$f(x) = \frac{x^2 - 8}{x + 5}$$

get messier and messier as you take higher and higher derivatives.

Cyclical derivatives:

The higher derivatives of some functions may start repeating themselves. For example, the derivatives of sine and cosine functions behave cyclically.

$y = \sin x$

$y' = \cos x$

$y'' = -\sin x$

$y''' = -\cos x$

$y^{(iv)} = \sin x$

The cycle repeats indefinitely with every multiple of four.

Example: Find the third derivative of $f(x) = 4\sin x - \dfrac{1}{x+3} + 5x$ with respect to x.

Solution:

$$f(x) = 4\sin x - \frac{1}{x+3} + 5x$$
$$f'(x) = 4\cos x + \frac{1}{(x+3)^2} + 5$$
$$f''(x) = -4\sin x - \frac{2}{(x+3)^3} + 0$$
$$f'''(x) = -4\cos x + \frac{6}{(x+3)^4}$$

Some standard nth derivatives

1)

Let

$y = (ax+b)^m$ Then

$$y' = ma(ax+b)^{m-1}$$
$$y'' = m(m-1)a^2(ax+b)^{m-2}$$

. . . .

. . . .

. . . .

$$y^{(n)} = (m-1)(m-2)......(m-n+1)a^n(ax+b)^{m-n}$$

If m is positive integer and n ≤ m, we can write

$$y^{(n)} = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$$

if m=n, then $y^{(n)} = n!a^n$, a constant, so that $y^{(n+1)}$ and subsequent derivatives of y are zero.

Corollary 1:

If $m = -1, y = \dfrac{1}{ax+b}$

Therefore, $y^{(n)} = (-1)(-2)(-3).....(-n)a^n(ax+b)^{-1-n}$

$$= \frac{(-1)^n n!a^n}{(ax+b)^{n+1}} = \frac{d^n}{dx^n}\left[\frac{1}{ax+b}\right]$$

Corollary 2:

Let $y = \ln(ax+b)$ so that

$$y' = \frac{a}{ax+b} = a.\frac{1}{ax+b}$$

Taking its $(n\text{-}1)th$ derivative, we have

$$y^{(n)} = \frac{d^n}{dx^n}\left[\ln(ax+b)\right] = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{a}{ax+b}\right]$$

$$= a.\frac{(-1)^{n-1}(n-1)!a^{n-1}}{(ax+b)^n} = .\frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

2)

$y = e^{ax}$

$y' = ae^{ax}$

$y'' = a^2e^{ax}$

.

.

.

$y^{(n)} = a^{(n)}e^{ax}$

3)

$y = \sin(ax+b)$

$$y' = a\cos(ax+b) = a\sin(ax+b+\frac{\pi}{2})$$

$$y'' = a^2\cos(ax+b+\frac{\pi}{2}) = a^2\sin(ax+b+\frac{\pi}{2}+\frac{\pi}{2}) = a^2\sin(ax+b+2.\frac{\pi}{2})$$

$$y''' = a^3\cos(ax+b+2.\frac{\pi}{2})$$

$$y''' = a^3\sin(ax+b+2.\frac{\pi}{2}+\frac{\pi}{2}) = a^3\sin(ax+b+3.\frac{\pi}{2})$$

.

.

.

$$y^{(n)} = a^n\sin(ax+b+n.\frac{\pi}{2})$$

Similarly

$$\frac{d^n}{dx^n}[\cos(ax+b)] = a^n \cos(ax+b+n.\frac{\pi}{2})$$

Example:

If $y = \dfrac{x}{2x^2+3x+1}$, find $y^{(n)}$.

Solution:

$$y = \frac{x}{2x^2+3x+1} = \frac{x}{2x^2+2x+x+1} = \frac{x}{2x(x+1)+1(x+1)} = \frac{x}{(2x+1)(x+1)}$$

Applying partial fraction

$$\frac{x}{(2x+1)(x+1)} = \frac{A}{(2x+1)} + \frac{B}{(x+1)} \qquad ....(1)$$

$$\frac{x}{(2x+1)(x+1)} = \frac{A(x+1)+B(2x+1)}{(2x+1)(x+1)}$$

$$x = A(x+1)+B(2x+1)$$

put $x+1=0 \Rightarrow x=-1$

$$-1 = B(-2+1)$$

$$-1 = -B$$

$$1 = B$$

put $2x+1=0 \Rightarrow x = -\frac{1}{2}$

$$-\frac{1}{2} = A(-\frac{1}{2}+1) = \frac{1}{2}A$$

$$-1 = A$$

put values of A and B in (1)

$$\frac{x}{(2x+1)(x+1)} = \frac{-1}{(2x+1)} + \frac{1}{(x+1)}$$

$$\frac{x}{(2x+1)(x+1)} = \frac{1}{(x+1)} - \frac{1}{(2x+1)}$$

$$y^n = \frac{d^n}{dx^n}\left[\frac{1}{x+1}-\frac{1}{2x+1}\right]$$
$$= \frac{d^n}{dx^n}\left[\frac{1}{x+1}\right]-\frac{d^n}{dx^n}\left[\frac{1}{2x+1}\right]$$
$$= \frac{(-1)^n n!}{(x+1)^{n+1}}-\frac{(-1)^n n!2^n}{(2x+1)^{n+1}}$$
$$= (-1)^n n!\left[\frac{1}{(x+1)^{n+1}}-\frac{2^n}{(2x+1)^{n+1}}\right]$$

Leibniz theorem

In calculus, the **general Leibniz rule**, named after Gottfried Leibniz, generalizes the product rule (which is also known as "Leibniz's rule".) It states that if *u* and *v* are *n*-times differentiable functions, then the *n*th derivative of the product uv is given by

$$(u.v)^n = \sum_{k=0}^{n}\binom{n}{k}u^{(n-k)}\, v^{(k)}$$

Where $\binom{n}{k}$ is the binomial coefficient.

Proof:

The proof of this theorem will be given through mathematical induction.

We know that

$$(uv)' = u'v + uv'$$
$$(uv)'' = D^1\left[(uv)'\right]$$
$$= D\,(u'v + uv')$$
$$= D\,(u'v) + D(uv')$$
$$= u''v + u'v' + u'v' + uv''$$
$$= u''v + 2u'v' + uv''$$

Which can be written as

$$= {}^2C_0 u''v + {}^2C_1 u'v' + {}^2C_2 uv''.$$

Thus the theorem is true for n=1, 2. Suppose that the theorem is true for a particular value of n, say n=r. then

$$y^{(r)}=(uv)^{(r)}={}^{r}C_0\,u^{(r)}v+{}^{r}C_1\,u^{(r-1)}v'+\ldots.+{}^{r}C_{r-1}\,u'v^{(r-1)}+{}^{r}C_r\,u\,v^{(r)}$$

Differentiating both sides of the above equation, we have

$$y^{(r+1)}=(uv)^{(r+1)}={}^{r}C_0\,[u^{(r+1)}v+u^{(r)}v']+{}^{r}C_1[u^{(r)}v'+u^{(r-1)}v'']+\ldots.+{}^{r}C_{r-1}[u''v^{(r-1)}+u'v^{(r)}]+{}^{r}C_r[u'v^{(r)}+uv^{(r+1)}]$$

$$={}^{r}C_0\,u^{(r+1)}v+u^{(r)}v'[{}^{r}C_0+{}^{r}C_1]+u^{(r-1)}v''[{}^{r}C_2+{}^{r}C_3]+\ldots.+u'v^{(r)}[{}^{r}C_{r-1}+{}^{r}C_r]+{}^{r}C_r uv^{(r+1)}$$

But ${}^{n}C_r+{}^{n}C_{r+1}={}^{n+1}C_{r+1}$ for all n, so that

$$y^{(r+1)}={}^{r+1}C_0\,u^{(r+1)}v+{}^{r+1}C_1u^{(r)}v'+{}^{r+1}C_2u^{(r-1)}v''+\ldots.+{}^{r+1}C_ru'v^{(r)}+{}^{r+1}C_{r+1}uv^{(r+1)}$$

Thus the theorem is true for n=r+1 .By the principal of mathematical induction, the result is true for all positive integer n. Hence the theorem is proved.

Example:

Find the n^{th} derivative of

$y=e^x lnx$ By using Leibniz theorem

Solution:

Leibniz theorem states that

$$(u.v)^n=\sum_{k=0}^{n}\binom{n}{k}u^{(n-k)}\,v^{(k)}$$

It will be expanded like

$$(u.v)^n={}^{n}C_0\,u^{(n)}v+{}^{n}C_1\,u^{(n-1)}v'+{}^{n}C_2\,u^{(n-2)}v''+\ldots\ldots\ldots\ldots+{}^{n}C_{n-1}\,u'v^{(n-1)}+{}^{n}C_n\,u\,v^{(n)}$$

$$(u.v)^n=u^{(n)}v+nu^{(n-1)}v'+\frac{n(n-1)}{2!}u^{(n-2)}v''+\ldots\ldots\ldots\ldots\ldots.+n\,u'v^{(n-1)}+u\,v^{(n)} \qquad (1)$$

Here $u=e^x$ and $v=\ln x$

$u'=e^x$ $\qquad v'=\dfrac{1}{x}=\dfrac{(-1)^0\,0!}{x}$

$u''=e^x$ $\qquad v''=-\dfrac{1}{x^2}=\dfrac{(-1)^1 1!}{x^2}$

. .

. .

. .

$u^{n-1}=e^x$ $\qquad v^{n-1}=\dfrac{(-1)^{n-2}(n-2)!}{x^{n-1}}$

$u^n=e^x$ $\qquad v^n=\dfrac{(-1)^{n-1}(n-1)!}{x^n}$

Now inserting all values in (1)

$$(e^x.\ln x)^n = e^x \ln x + ne^x.\frac{1}{x} + \frac{n(n-1)}{2!}e^x(-\frac{1}{x^2}) + ... + ne^x\,\frac{(-1)^{n-2}(n-2)!}{x^{n-1}} + e^x\,\frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$= e^x\left[\ln x + \frac{n}{x} + \frac{(-1)n(n-1)}{2x^2} + ... + n\frac{(-1)^{n-2}(n-2)!}{x^{n-1}} + \frac{(-1)^{n-1}(n-1)!}{x^n}\right]$$

Example:

If $y = a\cos(\ln x) + b\sin(\ln x)$, then prove that

$x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$

Solution:

$y = a\cos(\ln x) + b\sin(\ln x)$

$$y' = -a\sin(\ln x)\frac{1}{x} + b\cos(\ln x)\frac{1}{x}$$

$$y' = \frac{1}{x}(-a\sin(\ln x) + b\cos(\ln x))$$

$xy' = -a\sin(\ln x) + b\cos(\ln x)$

Differentiating it again, we get

$$xy'' + y' = -a\cos(\ln x)\frac{1}{x} - b\sin(\ln x)\frac{1}{x}$$

$$xy'' + y' = -\frac{1}{x}(a\cos(\ln x) + b\sin(\ln x))$$

$x^2y'' + xy' = -(a\cos(\ln x) + b\sin(\ln x)) = -y$

$x^2y'' + xy' + y = 0$

Differentiating 'n' times by using leibniz theorem

$({}^n\mathrm{C}_0\mathrm{y}^{(n+2)}x^2 + {}^n\mathrm{C}_1\mathrm{y}^{(n+1)}.2x + {}^n\mathrm{C}_2\mathrm{y}^{(n)}.2) + ({}^n\mathrm{C}_0\mathrm{y}^{(n+1)}x + {}^n\mathrm{C}_1\mathrm{y}^{(n)}) + y^{(n)} = 0$

$$\mathrm{y}^{(n+2)}x^2 + 2xn\mathrm{y}^{(n+1)} + \frac{n(n-1)}{2!}.2.\mathrm{y}^{(n)} + \mathrm{y}^{(n+1)}x + n\mathrm{y}^{(n)} + \mathrm{y}^{(n)} = 0$$

$\mathrm{y}^{(n+2)}x^2 + (2n+1)x\mathrm{y}^{(n+1)} + (n^2 - n + n + 1)\mathrm{y}^{(n)} = 0$

$\mathrm{y}^{(n+2)}x^2 + (2n+1)x\mathrm{y}^{(n+1)} + (n^2+1)\mathrm{y}^{(n)} = 0$

$x^2\mathrm{y}^{(n+2)} + (2n+1)x\mathrm{y}^{(n+1)} + (n^2+1)\mathrm{y}^{(n)} = 0$

hence proved.

Example:

Find the nth order derivative of $e^{ax}\sin x$.

Solution:

We know that by using Leibniz theorem

$$(u.v)^n = u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{2!}u^{(n-2)}v'' + ... + n\,u'\,v^{(n-1)} + u\,v^{(n)}$$

Here

$u = e^{ax}$ $\qquad v = \sin x$

$u' = ae^{ax}$ $\qquad v' = \cos x = \sin(x + \frac{\pi}{2})$

$u'' = a^2 e^{ax}$ $\qquad v'' = \cos(x + \frac{\pi}{2}) = \sin(x + \frac{\pi}{2} + \frac{\pi}{2}) = \sin(x + 2.\frac{\pi}{2})$

$u''' = a^3 e^{ax}$ $\qquad v''' = \cos(x + 2.\frac{\pi}{2}) = \sin(x + 2.\frac{\pi}{2} + \frac{\pi}{2}) = \sin(x + 3.\frac{\pi}{2})$

. .

. .

. .

$u^{(n-1)} = a^{(n-1)} e^{ax}$ $\qquad v^{(n-1)} = \sin(x + (n-1).\frac{\pi}{2})$

$u^{(n)} = a^n e^{ax}$ $\qquad v^{(n)} = \sin(x + n.\frac{\pi}{2})$

$$(e^{ax}.\sin x)^{(n)} = a^n e^{ax} \sin x + na^{(n-1)} e^{ax} \sin(x + \frac{\pi}{2}) + \frac{n(n-1)}{2!} a^{(n-2)} e^{ax} \sin(x + 3.\frac{\pi}{2})$$

$$+ .... + nae^{ax} \sin(x + (n-1).\frac{\pi}{2}) + e^{ax} \sin(x + n.\frac{\pi}{2})$$

Exercise

1) Find the third derivative of $f(x) = 4x^5 + 6x^3 + 2x + 1$ with respect to x.

2) Find the nth order derivative of

(i) $\dfrac{x}{x^2 - a^2}$

(ii) $\dfrac{x^3}{(x-1)(x-2)}$

3) Prove that

$$\frac{d^n}{dx^n}\left[\frac{\ln x}{x}\right] = \frac{(-1)^n n!}{x^{n+1}}\left[\ln x - 1 - \frac{1}{2} - ..... - \frac{1}{n-1} - \frac{1}{n}\right]$$

4)If $f(x) = \ln(1 + \sqrt{1-x}),$ prove that

$$4x(1-x)f''(x) + 2(2-3x)f'(x) + 1 = 0$$

5) Find the nth order derivative of $e^{ax} \cos x$.

Lecture No. 38
Taylor and Maclaurin Series

Introduction

When we talk about Approximation, the first question comes in to mind is that, why we have developed these expansion formulas, there is not merely a mathematical curiosity, but rather, is an essential means to exploring and computing those functions (transcendental), whose characteristics are not very much familiar.
What we actually do in approximation problem, we chose a function from the well-defined class that closely matches a target function (which we want to approximate at a certain point) in a task specific way. This is typically done with polynomial or rational (ratio of polynomials) approximations, as we are very well aware of characteristics of polynomials and we know how to mathematically manipulate them to get our required results.
It is common practice to approximate a function by using Taylor series. A Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. Any finite number of initial terms of the Taylor series of a function is called a Taylor polynomial. The Taylor series of a function is the limit of that function's Taylor polynomials, provided that the limit exists.

Approximation problem

Suppose we are interested in approximating a function $f(x)$ in the neighborhood of a point $a=0$ by a polynomial

$$P(x)=c_0+c_1x+c_2x^2+\cdots+c_nx^n \qquad (1)$$

Because $P(x)$ has $n+1$ coefficients, so we have to impose $n+1$ condition on the polynomial to achieve good approximation to $f(x)$. As "0" is the point about which we are approximating the function so we will chose the coefficient of $P(x)$, such that the $P(x)$ and the 1st n derivatives are same as the $f(x)$ and the 1st n derivatives of $f(x)$ at the point "0" i.e.

$$P(0)=f(0), P'(0)=f'(0),\ P''(0)=f''(0),\ldots,P^n(0)=f^n(0) \qquad (2)$$

We have

$$
\begin{aligned}
&P(x)=c_0+c_1x+c_2x^2+\cdots+c_nx^n\\
&P'(x)=c_1+2c_2x+\cdots+nc_nx^{n-1}\\
&P''(x)=2c_2+3.2c_3x+\cdots+n(n-1)c_nx^{n-2}\\
&P'''(x)=3.2c_3+\cdots+n(n-1)(n-2)c_nx^{n-3}\\
&.\\
&.\\
&.\\
&P^n(x)=n(n-1)(n-2)...(1)c_n
\end{aligned}
$$

From (2) we get

$$
\begin{aligned}
&P(0)=f(0)=c_0\\
&P'(0)=f'(0)=c_1\\
&P''(0)=f''(0)=2!c_2\\
&P'''(0)=f'''(0)=3.2c_3=3!c_3\\
&.\\
&.\\
&.\\
&P^n(0)=f^n(0)=n(n-1)(n-2)...(1)c_n=n!c_n
\end{aligned}
$$

So we get the following values for the coefficients of

$$P(x)\ c_0=f(0),c_1=f'(0),c_2=\frac{f''(0)}{2!},c_3=\frac{f'''(0)}{3!},...,c_n=\frac{f^n(0)}{n!}$$

Now we have evaluated all the unknowns.

Taylor Polynomial

Let a function f has continuous derivatives of nth order on the interval $\left[a,a+h\right]$. Then

$$f(x)=\sum_{k=0}^{n}\frac{f^k(a)}{k!}(x-a)^k=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^n}{n!}f^n(a)$$

Alternate form

$$f(a+h)=\sum_{k=0}^{n}\frac{h^k}{k!}f^k(a)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^n}{n!}f^n(a)$$

is called the Taylor polynomial of degree n.

Taylor Series

Let a function f has continuous derivatives of every order on the interval $[a,a+h]$. Then

$$f(x)=\sum_{k=0}^{\infty}\frac{f^k(a)}{k!}(x-a)^k=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^n}{n!}f^n(a)...$$

Alternate form

$$f(a+h)=\sum_{k=0}^{\infty}\frac{h^k}{k!}f^k(a)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^n}{n!}f^n(a)+\cdots$$

is called the Taylor Series.

This expression (**Taylor Series)** can be easily converted to **Maclaurin Series** just by putting $a=0$ and $h=x$ the

$$f(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}f^k(0)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^n}{n!}f^n(0)+\cdots$$

The above expression is called **Maclaurin Series.**

Taylor's Theorem

Now we will discuss a result called Taylor's Theorem which relates a function, its derivative and its higher derivatives. It basically deals with approximation of functions by polynomials.

Statement

Suppose f has n+1 continuous derivatives on an open interval $]a,a+h[$. Then there exist a number $\theta, 0<\theta<1$, such that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}f^n(a+\theta h)$$

Proof:
Consider the function F defined by

$$F(x)=f(x)+(a+h-x)f'(x)+\frac{(a+h-x)^2}{2!}f''(x)+\cdots+\frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x)$$

$$+\frac{(a+h-x)^n}{n!}A$$

where A is a constant to be determined such that $F(a)=F(a+h)$
So we have

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}A=f(a+h)$$

The function F clearly satisfied the condition of roll's Theorem. Hence there exist a number

number θ with $0<\theta<1$, such that , $F'(a+\theta h)=0$

Now

$$F'(x)=f'(x)-f'(x)+(a+h-x)f''(x)-(a+h-x)f''(x)$$

$$+\frac{(a+h-x)^2}{2!}f'''(x)-\cdots+\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)-\frac{(a+h-x)^{n-1}}{(n-1)!}A$$

$$=\frac{(a+h-x)^{n-1}}{(n-1)!}[f^n(x)-A]$$

Therefore

$$F'(a+\theta h)=\frac{(h-\theta h)^{n-1}}{(n-1)!}[f^n(a+\theta h)-A]=0$$

$$\frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1}[f^n(a+\theta h)-A]=0$$

Since $h\neq 0, 1-\theta\neq 0$, so we have

$$f^n(a+\theta h)-A=0$$

$$f^n(a+\theta h)=A$$

Substituting the value of A into (1) we get,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a+\theta h)$$

$0 < \theta < 1$

This formula is known as Taylor Development of a function in finite form with $(n+1)th$ term as **Lagrange's Form of Remainder** after n terms.

Taylor's Theorem with Cauchy Form of Remainder

Suppose *f* has *n*+1 continuous derivatives on an open interval $]a, a+h[$. Then there exist a number θ with $0 < \theta < 1$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^n(a+\theta h)$$

This formula is known as Taylor Development of a function in finite form with term as **Cauchy's Form of Remainder** after n terms.

Corollary

If the interval in Taylor's Theorem is taken as $[0, x]$ in place of $[a, a+h]$ then this form is called Maclaurin's Theorem, it again has two forms

I Maclaurin's Theorem with Lagrange's Remainder

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^n(\theta x)$$

$0 < \theta < 1$

II Maclaurin's Theorem with Cauchy's Remainder

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{(n-1)!}(1-\theta)f^n(\theta x)$$

$0 < \theta < 1$

Example

Apply Taylor's Theorem to prove that

$$(a+b)^m = a^m + \frac{m}{1!}a^{m-1}b + \frac{m(m-1)}{2!}a^{m-2}b^2 + \cdots$$

For all the real $m, a > 0, -a < b < a$.

Solution

Here $f(x) = x^m$, $f'(x) = mx^{m-1}$, $f''(x) = m(m-1)x^{m-2}$, and

$f'''(x) = m(m-1)(m-2)x^{m-3}, \cdots, f^n(x) = m(m-1)(m-2)\cdots(m-n+1)x^{m-n}$

Then by Taylor's Theorem

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^n(a+\theta h)$$

By putting values

$$f(x+b) = f(x) + bf'(x) + \frac{b^2}{2!}f''(x) + \frac{b^3}{3!}f'''(x) + \cdots + \frac{b^n}{n!}f^n(x+\theta b)$$

$$= x^m + bmx^{m-1} + \frac{b^2}{2!}m(m-1)x^{m-2} + \frac{b^3}{3!}m(m-1)(m-2)x^{m-3} + \cdots$$

$$+ \frac{b^n}{(n-1)!}m(m-1)(m-2)\cdots(m-n+1)x^{m-n}$$

Then

$$f(a+b) = a^m + bma^{m-1} + \frac{b^2}{2!}m(m-1)a^{m-2} + \frac{b^3}{3!}m(m-1)(m-2)a^{m-3} + \cdots$$

Here

$$R_n = \frac{b^n}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta b), 0<\theta<1.$$

$$f^n(a+\theta b) = m(m-1)(m-2)\cdots(m-n+1)(a+\theta b)^{m-n}$$

$$R_n = \frac{b^n}{(n-1)!}(1-\theta)^{n-1} m(m-1)(m-2)\cdots(m-n+1)(a+\theta b)^{m-n}$$

$$= \frac{b^n(1-\theta)^{n-1} n!}{(m-n)!(n-1)!}(a+\theta b)^{m-n}$$

$R_n \to 0$ as $n \to \infty$ for all real $m, a>0, -a<b<a$

Hence

$$(a+b)^m = a^m + \frac{m}{1!}a^{m-1}b + \frac{m(m-1)}{2!}a^{m-2}b^2 + \cdots$$

Example

Apply Taylor's Theorem to prove that

$$\ln\sin(x+h) = \ln S\text{in}\, x + hC\text{ot}\, x - \frac{1}{2}h^2 C\text{s}c^2 x + \frac{1}{3}h^3 CotxC\text{s}c^2 x + \cdots$$

Solution

Let $f(x) = \ln S\text{in}\, x,$ then

$$f'(x) = \frac{1}{S\text{in}\, x}.Cosx = Cotx,\ f''(x) = -Csc^2 x$$

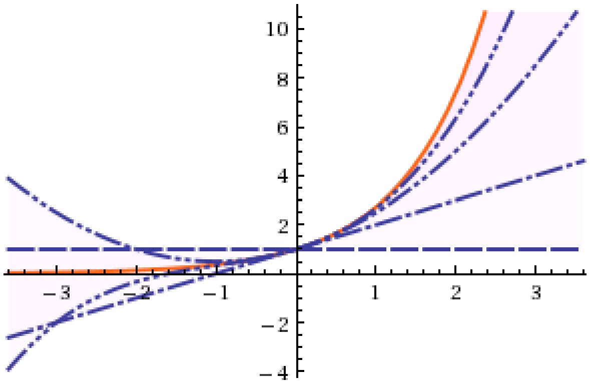
$$f'''(x) = -2Cscx(-Cscx\, Cotx) = 2Csc^2 x\, C\text{ot}\, x$$

By applying Taylor's Theorem

$$\ln\sin(x+h) = \ln S\text{in}\, x + hC\text{ot}\, x - \frac{1}{2}h^2 C\text{s}c^2 x + \frac{1}{3}h^3 CotxC\text{s}c^2 x + \cdots$$

Example

Find the Maclaurin Series $f(x) = e^x$, expanded about $x = 0$



Solution

Here $f'(x) = e^x$, $f''(x) = e^x$ and so on $f^{(n)}(x) = e^x$ for $n = 0, 1, 2, ...$

$$f(0) = f'(0) = f''(0) = ... = f^{(n)}(0) = e^0 = 1$$

The nth Maclaurin polynomial is

$$\sum_{k=0}^{n} \frac{f^k(0)}{k!} x^k = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0)$$

Thus the Maclaurin Series is

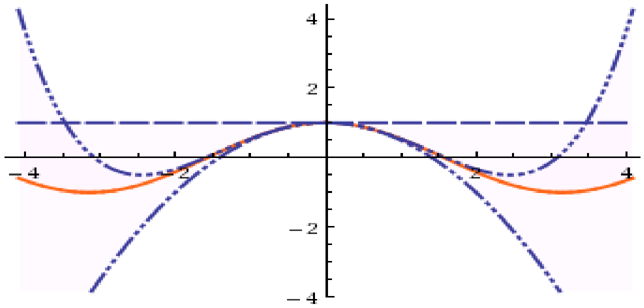
$$\sum_{k=0}^{\infty} \frac{f^k(0)}{k!} x^k = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots$$

Putting the values, we get

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

Example

Find the Maclaurin Series of $f(x) = Cosx$, expanded about $x = 0$.



Solution: Here $f(x) = Cosx$, $f'(x) = -Sinx$, $f''(x) = -Cosx, \ldots$
and $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1, \ldots$

The nth Maclaurin polynomial is

$$\sum_{k=0}^{n} \frac{f^{k}(0)}{k!} x^{k} = f(0) + xf'(0) + \frac{x^{2}}{2!} f''(0) + \cdots + \frac{x^{n}}{n!} f^{(n)}(0)$$

the Maclaurin Series is

$$\sum_{k=0}^{\infty} \frac{f^{k}(0)}{k!} x^{k} = f(0) + xf'(0) + \frac{x^{2}}{2!} f''(0) + \cdots + \frac{x^{n}}{n!} f^{(n)}(0) + \cdots$$

Putting the values, we get

$$\sum_{k=0}^{\infty} \frac{f^{k}(0)}{k!} x^{k} = 1 - \frac{x^{2}}{2!} + \frac{x^{4}}{4!} - \frac{x^{6}}{6!} + \cdots + (-1)^{k} \frac{x^{2k}}{(2k)!} + \cdots$$

Exercise

1. Find the Maclaurin series of given functions

i) $Sin\,x$

ii) $e^{Sin\,x}$

2. Find the Taylor series of given functions

i) $\ln x$ about $x = 1$

ii) a^{x} about $x = 2$

Lecture No. 39
Numerical Integration

To evaluate the definite integral of certain functions whose anti derivatives cannot be found easily or in more practical situations the integrand is expressed in tabular form, numerical techniques provide efficient way to approximate the definite integral.

A definite integral $\int_a^b f(x)dx$ can be interpreted as area under the curve $y = f(x)$ bounded by the x-axis and the line $x = a$ and $x = b$. In numerical integration to approximate the definite integral, we estimate the area under the curve by evaluating the integrand $f(x)$ at a set of distinct points $(x_0, x_1, ..., x_n)$, where $x_i \in [a,b]$ for $0 \le i \le n$. Of course, we assume that the function to be integrated is continuous on $[a,b]$.

Integration Methods

The commonly used integration methods can be classified into two groups: the Newton-Cotes formulae that employ functional values at equally spaced points, and the Gaussian quadrature formulae that employ unequally spaced points.

Closed Newton-Cotes Quadrature Formula

The method of integration will be based on interpolation polynomial $P_n(x)$ of degree n appropriate for a given function. When this polynomial $P_n(x)$ is used to approximate $f(x)$ over $[a,b]$, and then the integral of $f(x)$ is approximated by the integral of $P_n(x)$, the resulting formula is called a Newton-Cotes quadrature formula. When the sample points $x_0 = a$ and $x_n = b$ are used, it is called a closed Newton-Cotes formula. Thus the idea of Newton-Cotes formulas is to replace a complicated function or tabulated data with an approximating function that is easy to integrate.

$$I = \int_a^b f(x)dx \approx \int_a^b P_n(x)dx$$

where $P_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$

The next result gives the formulae when approximating polynomials of degree n=1, 2 are used.

Theorem

Assume that $x_k = x_0 + kh,$ are equally spaced nodes and $f_k = f(x_k)$. The first two closed Newton-Cotes quadrature formulae:

(1) Trapezoidal Rule $\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}\left(f(x_0) + f(x_1)\right)$

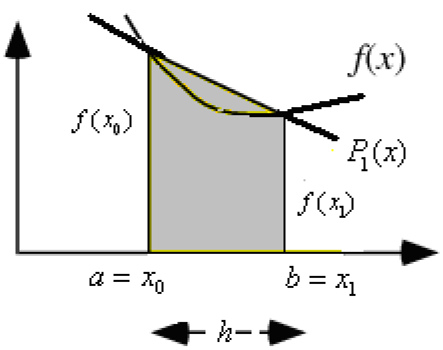
(2) Simpson's Rule $\int_{x_0}^{x_2} f(x)dx \approx \frac{h}{3}\left(f(x_0)+4f(x_1)+f(x_2)\right)$

The Trapezoidal Rule

One of the simplest ways to estimate an integral $I=\int_a^b f(x)dx$ is to employ linear interpolation, i.e., to approximate the curve $y=f(x)$ by a straight line $y=P_1(x)$ also called secant line passing through the points $(a, f(a))$ and $(b, f(b))$ and then to compute the area under the line i.e. area is approximated by the trapezium formed by replacing the curve with its secant line drawn between the end points $(a, f(a))$ and $(b, f(b))$.

Let $a=x_0$, $b=x_1$, and $h=x_1-x_0$. To approximate

$$\int_a^b f(x)dx=\int_{x_0}^{x_1} f(x)dx \approx \int_{x_0}^{x_1} P_1(x)dx$$



Now the area of trapezoid is the product of its altitude and the average length of its parallel sides. The area of trapezoid with altitude x_1-x_0 is

$$(x_1-x_0)\left(\frac{f(x_0)+f(x_1)}{2}\right)$$

$$=\frac{h}{2}\left(f(x_0)+f(x_1)\right)$$

Thus $\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}\left(f(x_0)+f(x_1)\right)$

Note: the error term involved in trapezoidal rule is

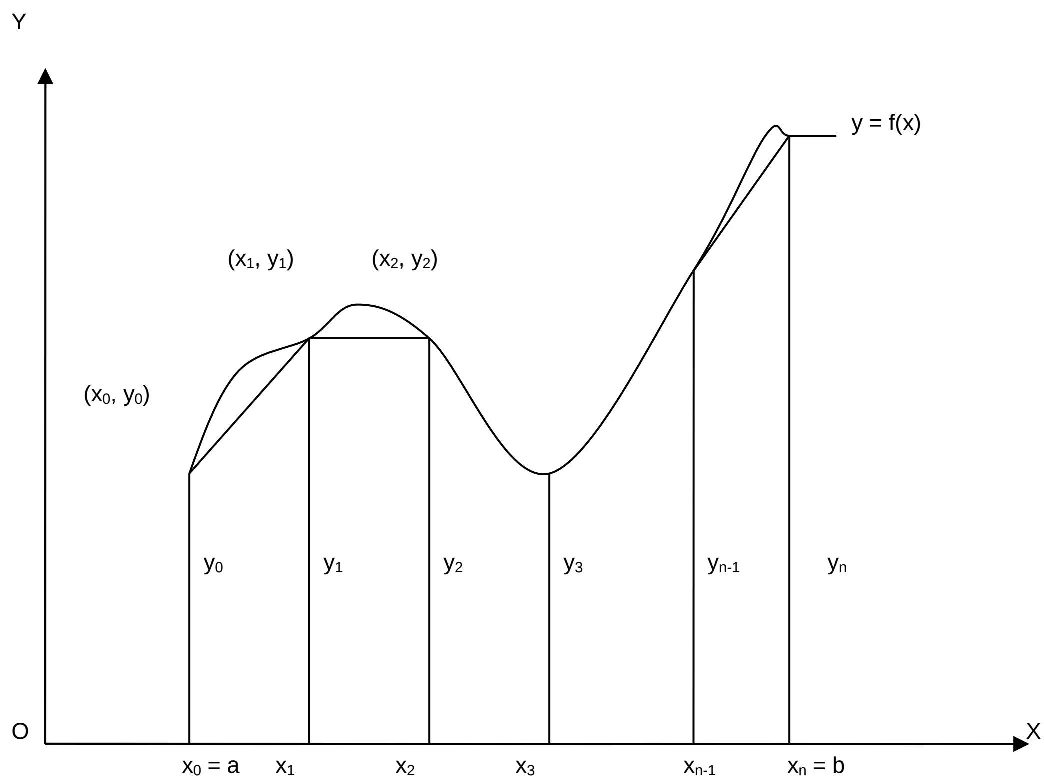
$$=-\frac{h^3}{12}f''(\xi), \quad \xi \in [x_0, x_1]$$

Thus the trapezoidal rule with error term is

$$\int_{x_0}^{x_1} f(x)dx=\frac{h}{2}\left(f(x_0)+f(x_1)\right)-\frac{h^3}{12}f''(\xi)$$

The Trapezoidal Rule (Composite Form)

In order to evaluate the definite integral $I = \int_a^b f(x)dx$ we divide the interval $[a,b]$ into n sub-intervals, each of size $h = \frac{b-a}{n}$ and denote the sub-intervals by $[x_0, x_1],[x_1, x_2], \dots,[x_{n-1}, x_n]$, such that $x_0 = a,\ x_n = b,$ and $x_k = x_0 + kh,\ k = 1,2,\dots,n$ and then use the trapezoidal rule on each subinterval



Thus, we can write the above definite integral as a sum. Therefore,

$$I = \int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

The area under the curve in each sub-interval is approximated by a trapezium. The integral I, which represents an area between the curve $y = f(x)$, the x-axis and the ordinates at $x = x_0,\ x = x_n$ is obtained by adding all the trapezoidal areas in each sub-interval. Now, using the trapezoidal rule into equation:

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(y_0 + y_1) - \frac{h^3}{2} y''(\xi)$$

We get

$$I = \int_{x_0}^{x_n} f(x)dx = \frac{h}{2}(y_0 + y_1) - \frac{h^3}{2} y''(\xi_1) + \frac{h}{2}(y_1 + y_2) - \frac{h^3}{12} y''(\xi_2)$$
$$+ \cdots + \frac{h}{2}(y_{n-1} + y_n) - \frac{h^3}{12} y''(\xi_n)$$

Where $x_{k-1} < \xi < x_k$, for $k = 1, 2, ..., n-1$.
Thus, we arrive at the result

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) + E_n$$

Where the error term En is given by

$$E_n = -\frac{h^3}{12}[y''(\xi_1) + y''(\xi_2) + \cdots + y''(\xi_n)]$$

Equation represents the trapezoidal rule over $[x_0, x_n]$, which is also called the composite form of the trapezoidal rule. The error term given by Equation:

$$E_n = -\frac{h^3}{12}[y''(\xi_1) + y''(\xi_2) + \cdots + y''(\xi_n)]$$
$$= -\frac{h^3}{12}\sum_{k=1}^{n} y''(\xi_k), \quad \xi_k \in (x_{k-1}, x_k)$$

is called the global error.
However, if we assume that $y''(x)$ is continuous over $[x_0, x_n]$ then there exists some ξ in $[x_0, x_n]$ such that and $x_n = x_0 + nh$ and the maximum error incurred in the approximate value obtained by trapezoidal rule is

$$E_n = -\frac{h^3 M}{12n^2} \quad \text{where} \quad M = \max\left|f''(\xi)\right|, \quad \xi \in [x_0, x_n]$$

Example (The Trapezoidal Rule):

Evaluate the integral $I = \int_0^1 \frac{dx}{1+x^2}$, by using Trapezoidal rule, take $h = \frac{1}{4}$.

Solution

At first, we shall tabulate the function as

x	0	1/4	1/2	3/4	1
$y = \frac{1}{1+x^2}$	1	0.9412	0.8000	0.6400	0.5000

Using trapezoidal rule, and taking $h = \frac{1}{4}$

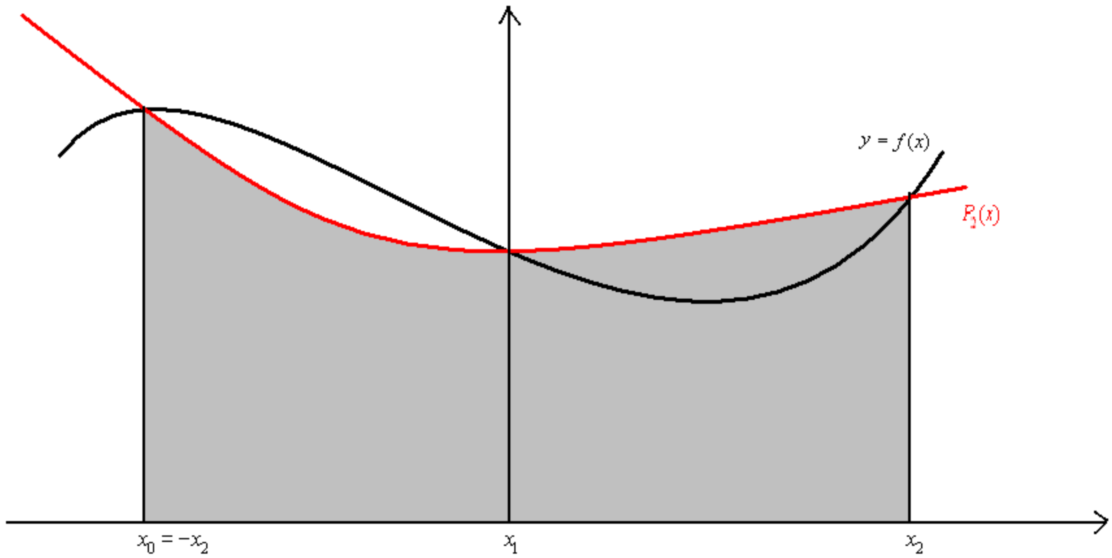
$$
\begin{aligned}
I &= \int_0^1 \frac{dx}{1+x^2}, \\
&= \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right] \\
&= \frac{1}{8}\left[1 + 2(0.9412 + 0.8000 + 0.6400) + 0.5000\right] \\
&= \frac{1}{8}\left[1 + 2(2.3812) + 0.5000\right] \\
&= \frac{1}{8}\left[1 + 2(2.3812) + 0.5000\right] \\
&= \frac{1}{8}\left[6.2624\right] \\
&= 0.7828
\end{aligned}
$$

But the closed form solution to the given integral is

$$
\begin{aligned}
\int_0^1 \frac{dx}{1+x^2} &= \left[\tan^{-1} x\right]_0^1 \\
&= \frac{\pi}{4} = 0.7854
\end{aligned}
$$

Simpson's 1/3 Rule

The trapezoidal rule tries to simplify integration by approximating the function to be integrated by a straight line or a series of straight line segments. In Simpson's rule we try to approximate by a series of parabolic segments, hoping that the parabola will more closely match a given curve $y = f(x)$, than would the straight line in the trapezoidal rule. To estimate $I = \int_a^b f(x)dx$, the curve $y = f(x)$, is approximated by a parabola $y = P_2(x)$ passing through three points $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))$ and then the area under the parabolic segment is computed. We assume that x_1 coincides with the origin so that $x_0 = -x_2$ and parabola is $P_2(x) = ax^2 + bx + c$



$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} P_2(x)dx$$

$$a = x_0,\ x_1 = x_0 + h \ \ and \ \ x_2 = x_0 + 2h = b$$

$$\int_{x_0}^{x_2} P_2(x)dx = \int_{-x_2}^{x_2} P_2(x)dx$$

$$= \int_{-x_2}^{x_2} (ax^2 + bx + c)dx$$

$$= \frac{ax^3}{3}\Big|_{-x_2}^{x_2} + \frac{bx^2}{2}\Big|_{-x_2}^{x_2} + cx\Big|_{-x_2}^{x_2}$$

$$= \frac{ax^3}{3}\Big|_{-x_2}^{x_2} + \frac{bx^2}{2}\Big|_{-x_2}^{x_2} + cx\Big|_{-x_2}^{x_2}$$

$$= \frac{ax_2^{\ 3}}{3} + \frac{ax_2^{\ 3}}{3} + \frac{bx_2^{\ 2}}{2} - \frac{bx_2^{\ 2}}{2} + cx_2 + cx_2$$

$$= \frac{2ax_2^{\ 3}}{3} + 2cx_2$$

$$= \frac{x_2}{3}(2ax_2^{\ 2} + 6c) \qquad ....(1)$$

We also have

$h = x_1 - x_0 = 0 - (-x_2) = x_2$

$f(x_0) = f(-x_2) = ax_2^{\,2} - bx_2 + c = ah^2 - bh + c$

$f(x_1) = f(0) = c$

$f(x_2) = ax_2^{\,2} + bx_2 + c = ah^2 + bh + c$

$f(x_0) + f(x_2) = 2ah^2 + 2c$

and so

$f(x_0) + 4f(x_1) + f(x_2) = 2ah^2 + 6c$

Substituting this in the area formula (1), we have

$$\int_{x_0}^{x_2} P_2(x)dx = \frac{h}{3}(2ah^2 + 6c)$$

$$= \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$$

Thus $\int_{x_0}^{x_2} f(x)dx \approx \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$

Note: the error term involved in Simpson's 1/3 rule is

$$= -\frac{h^5}{90} f^{(iv)}(\xi), \quad \xi \in [x_0, x_1]$$

Thus the Simpson's 1/3 rule with error term is

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) - \frac{h^5}{90} f^{(iv)}(\xi)$$

or

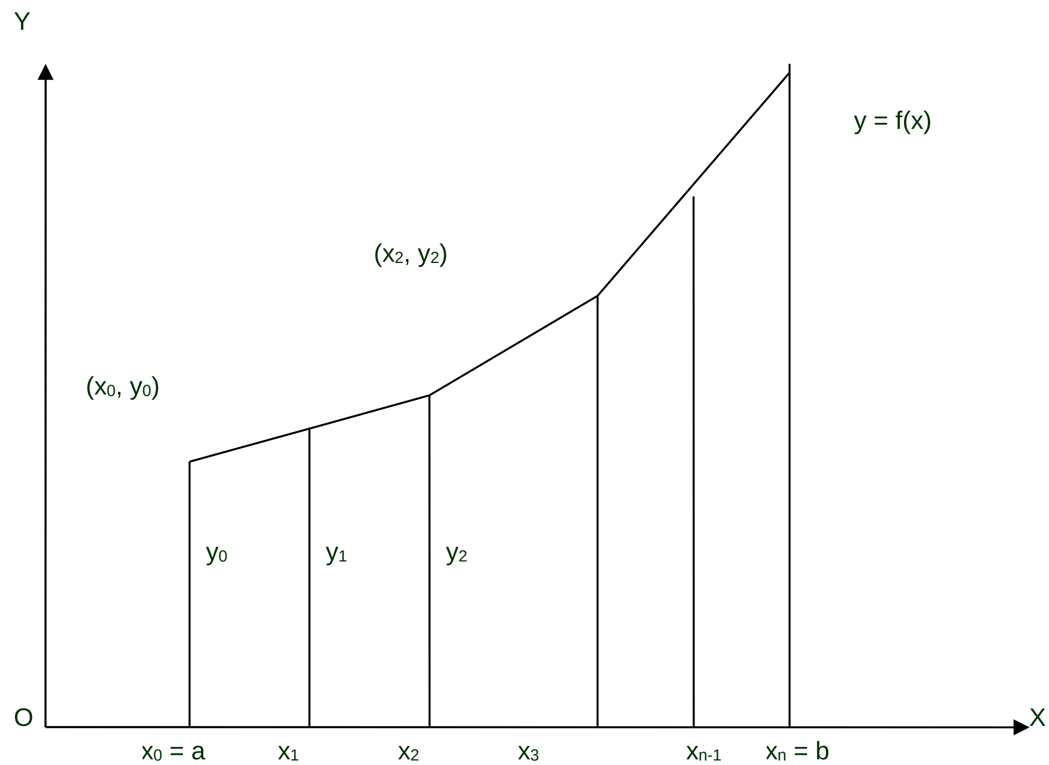
$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}(y_0 + 4y_1 + y_2) - \frac{h^5}{90} y^{(iv)}(\xi)$$

Simpson's 1/3 Rule (Composite Form)

In deriving equation,

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}(y_0 + 4y_1 + y_2) - \frac{h^5}{90} y^{(iv)}(\xi)$$

Geometrically, this equation represents the area between the curve $y = f(x)$, the x-axis and the ordinates at $x = x_0$ and x_2 after replacing the arc of the curve between (x_0, y_0) and (x_2, y_2) by an arc of a quadratic polynomial as in the figure.



In Simpson's 1/3 rule, we have used two sub-intervals of equal width. In order to get a composite formula, we shall divide the interval of integration [a, b] into an even number of sub intervals say 2N, each of width (b – a)/2N, thereby we have

$$x_0 = a, x_1, \ldots, x_{2N} = b \text{ and } x_k = x_0 + kh, \quad k = 1, 2, \ldots, (2N-1)$$

Thus, the definite integral I can be written as

$$I = \int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{2N-2}}^{x_{2N}} f(x)dx$$

Applying Simpson's 1/3 rule as in equation

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}(y_0 + 4y_1 + y_2) - \frac{h^5}{90} y^{(iv)}(\xi)$$

to each of the integrals on the right-hand side of the above equation, we obtain

$$I = \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots$$

$$+ (y_{2N-2} + 4y_{2N-1} + y_{2N})] - \frac{N}{90} h^5 y^{(iv)}(\xi)$$

That is

$$\int_{x_0}^{x_{2N}} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + \cdots + y_{2N-1}) + 2(y_2 + y_4 + \cdots + y_{2N-2}) + y_{2N}] + \text{Error term}$$

This formula is called composite Simpson's 1/3 rule. The error term E, which is also called global error, is given by

$$E = -\frac{N}{90}h^5 y^{(iv)}(\xi) = -\frac{x_{2N} - x_0}{180}h^4 y^{(iv)}(\xi)$$

for some ξ in $[x_0, x_{2N}]$.

Example (Simpson's 1/3 Rule):

Estimate the value of $\int_1^5 \ln x dx$ using Simpson's 1/3 rule. Also, obtain the value of h, so that the value of the integral will be accurate up to five decimal places.

Solution Let for number of sub-intervals $2N = 8,$ and $x_{2N} = 5,\ x_0 = 1$

$$h = \frac{x_{2N} - x_0}{2N}$$

$$= \frac{5-1}{8} = 0.5$$

k	$x_k = x_0 + kh$	$y = f(x) = \ln x$
1	$x_1 = 1 + 1*0.5 = 1.5$	$y_1 = f(x_1) = \ln x_1 = \ln 1.5 = 0.4055$
2	$x_2 = 1 + 2*0.5 = 2$	$y_2 = f(x_2) = \ln x_2 = \ln 2.0 = 0.6931$
3	$x_3 = 1 + 3*0.5 = 2.5$	$f(x_3) = 0.9163$
4	$x_4 = 3.0$	$y_4 = 1.0986$
5	$x_5 = 3.5$	$f(x_5) = 1.2528$
6	$x_6 = 4.0$	1.3863
7	$x_7 = 4.5$	1.5041
8	$x_8 = 5$	1.6094

Now using Simpson's 1/3 rule,

$$\int_1^5 \ln x dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6) + y_8]$$

$$= \frac{0.5}{3}[0 + 4(0.4055 + 0.9163 + 1.2528 + 1.5041)$$
$$+ 2(0.6931 + 1.0986 + 1.3863) + 1.6094]$$

$$= \frac{0.5}{3}[0 + 4(4.0786) + 2(3.178) + 1.6094]$$

$$= \frac{0.5}{3}(24.2798) = 4.0466$$

The error in Simpson's rule is given by

$$E = -\frac{x_{2N} - x_0}{180} h^4 y^{(iv)}(\xi)$$

(ignoring the sign)

Since
$$y = \ln x, y' = \frac{1}{x}, y'' = -\frac{1}{x^2}, y''' = \frac{2}{x^3}, y^{(iv)} = -\frac{6}{x^4}$$

$$\underset{1 \le x \le 5}{Max} \quad y^{(iv)}(x) = 6,$$

$$\underset{1 \le x \le 5}{Min} \quad y^{(iv)}(x) = 0.0096$$

Therefore, the error bounds are given by

$$\frac{(0.0096)(4)h^4}{180} < E < \frac{(6)(4)h^4}{180}$$

If the result is to be accurate up to five decimal places, then

$$\frac{24h^4}{180} < 10^{-5}$$

That is, $h^4 < 0.000075$ or $h < 0.09$. It may be noted that the actual value of integrals is

$$\int_1^5 \ln x dx = [x \ln x - x]_1^5 = 5 \ln 5 - 4 = 4.0472$$

Example (Simpson's 1/3 Rule):

Evaluate the integral $I = \int_0^1 \frac{dx}{1+x^2}$, by using Simpson's 1/3 rule, take $h = \frac{1}{4}$.

Solution

At first, we shall tabulate the function as

X	0	¼	½	¾	1
$y = 1/ 1+x^2$	1	0.9412	0.8000	0.6400	0.5000

Using Simpson's 1/3 rule, and taking $h = \frac{1}{4}$, we have

$$
\begin{aligned}
I &= \int_0^1 \frac{dx}{1+x^2}, \\
&= \frac{h}{3}\left[y_0 + 4(y_1 + y_3) + 2y_2 + y_4\right] \\
&= \frac{1}{12}\left[1 + 4(0.9412 + 0.6400) + 2(0.8000) + 0.5000\right] \\
&= \frac{1}{12}\left[1 + 4(1.5812) + 1.6 + 0.5000\right] \\
&= \frac{1}{12}\left[9.4248\right] \\
&= 0.7854
\end{aligned}
$$

Exercise

Evaluate the following integrals by using
(i) Trapezoidal rule
(ii) Simpson's 1/3 rule

1. $\int_0^4 x^2 dx, \quad h = 1/2$
2. $\int_1^3 \frac{1}{x} dx, \quad h = 1/5$