Lecture Handouts

on

VECTORS AND CLASSICAL MECHANICS
(MTH-622)

Virtual University of Pakistan
Pakistan

About the Handouts

The following books have been mainly followed to prepare the slides and handouts:

1. Spiegel, M.R., *Theory and Problems of Vector Analysis: And an Introduction to Tensor Analysis*. 1959: McGraw-Hill.
2. Spiegel, M.S., *Theory and problems of theoretical mechanics*. 1967: Schaum.
3. Taylor, J.R., *Classical Mechanics*. 2005: University Science Books.
4. DiBenedetto, E., *Classical Mechanics: Theory and Mathematical Modeling*. 2010: Birkhäuser Boston.
5. Fowles, G.R. and G.L. Cassiday, *Analytical Mechanics*. 2005: Thomson Brooks/Cole.

The first two books were considered as main text books. Therefore the students are advised to read the first two books in addition to these handouts. In addition to the above mentioned books, some other reference book and material was used to get these handouts prepared.

Contents

Module No. 1

Introduction to the Course and Mathematics

Introduction to Mathematics

"Mathematics is the branch of science which deals with the study of relations and patterns, and means to represent and communicate them."

Introduction to the Course

There are two main portions of this course:

- Vectors
- Classical Mechanics

Module No. 2

Scalar and Vector Fields

Scalar Point Function

If to each point (x, y, z) of a region R in space there corresponds a scalar $\varphi(x, y, z)$, then φ is called a scalar point function in R.

Scalar Field

Scalar field is a function define on space whose value at each point is a scalar quantity

The set of all values of scalar point function φ in R together forms a Scalar field.

Examples of Scalar Fields

1. The temperature $T(x, y, z)$ within a body A is a scalar point function because there exist only one temperature at each point of A.
2. The pressure and potential due to gravity of the air in the earth's atmosphere define scalar field.

Vector Point Function

If to each point (x, y, z) of a region R in space there exist a unique vector $\vec{A}\,(x, y, z)$, then $\vec{A}$ is called a vector point function in R.

Vector Field

A function of a space whose value at each point is a vector quantity is called vector field.

Mathematically, we can write it as

$$\vec{A}=\vec{A}\,(x, y, z) = A_1\,(x, y, z) + A_2\,(x, y, z) + A_3\,(x, y, z)$$

The set of all values of $\vec{A}$ in R constitute a vector field.

Examples of Vector Field

1. $\vec{A}=\vec{A}\,(x,y,z) = xy\hat{\imath} - 2yz^3\hat{\jmath} + y^2z\hat{k}$ defines a vector point function and hence is a vector field.
2. The motion of a moving fluid at any time define vector field.
3. The set of tangent vector of a curve C and the set of normal vectors of a surface S are examples of vector field.

Module No. 3

The Operator Del and Gradient of Function

The Del Operator

The vector differential operator del, symbolize as ∇, is defined by

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

The symbol ∇ called **de**l or **nabla** is used to symbolize del operator. It is only applied as defined derivative on one-dimensional function, and for more dimensions it may be applied as partial derivative on the function. The del operator is not a particular operator but when we applied it on a scalar point function or a vector point function; this may be known as the gradient, the divergence, and the curl.

Gradient: grad (f) = $\nabla\varphi$

Curl: curl $\vec{A} = \nabla \times \vec{A}$

Divergence: div $(\overrightarrow{A}) = \nabla . \vec{A}$

We will discuss here the first application of del operator as gradient.

Gradient Function

Let $\varphi(x, y, z)$ be a scalar point function defined on a specific region on R and also differentiable on the same domain. The we can apply del operator on φ in order to obtain gradient of scalar function φ of grad(φ) written as $\nabla\varphi$ is defined by

$$\nabla\varphi = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\widehat{k}\right)\varphi$$

$$= \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\widehat{k}$$

it is to be noted that $\nabla\varphi$ defined a vector field. Also $\nabla\varphi = 0$ if and only if φ is constant.

Module No. 4

Properties of the Gradient

If φ and ψ are Scalar point functions as well as differentiable on a specific domain and C is a constant then we can show that the gradient holds the following properties.

i. $\nabla(C\varphi) = C\nabla\varphi$

ii. $\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi$

iii. $\nabla(\varphi\,\psi) = \varphi\nabla\psi + \psi\nabla\varphi$

iv. $\nabla\left(\frac{\varphi}{\psi}\right) = \frac{\psi\nabla\varphi - \varphi\nabla\psi}{\psi^2}, \psi \neq 0$

Proof:

i. $\boldsymbol{\nabla(C\varphi) = C\nabla\varphi}$

L.H.S $= \nabla(C\varphi) = \frac{\partial(C\varphi)}{\partial x}\hat{\imath} + \frac{\partial(C\varphi)}{\partial y}\hat{\jmath} + \frac{\partial(C\varphi)}{\partial z}\hat{k}$

$$= C\frac{\partial\varphi}{\partial x}\hat{\imath} + C\frac{\partial\varphi}{\partial y}\hat{\jmath} + C\frac{\partial\varphi}{\partial z}\hat{k}$$

$$= C\left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right)$$

$$= C\nabla\varphi = R.H.S$$

ii. $\boldsymbol{\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi}$

$$L.H.S = \nabla(\varphi + \psi) = \frac{\partial(\varphi + \psi)}{\partial x}\hat{\imath} + \frac{\partial(\varphi + \psi)}{\partial y}\hat{\jmath} + \frac{\partial(\varphi + \psi)}{\partial z}\hat{k}$$

$$= \left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right) + \left(\frac{\partial\psi}{\partial x}\hat{\imath} + \frac{\partial\psi}{\partial y}\hat{\jmath} + \frac{\partial\psi}{\partial z}\hat{k}\right)$$

$$= \nabla\varphi + \nabla\psi = R.H.S$$

iii. $\boldsymbol{\nabla(\varphi\,\psi) = \varphi\nabla\psi + \psi\nabla\varphi}$

$$L.H.S = \nabla(\varphi\,\psi) = \frac{\partial(\varphi\,\psi)}{\partial x}\hat{\imath} + \frac{\partial(\varphi\,\psi)}{\partial y}\hat{\jmath} + \frac{\partial(\varphi\,\psi)}{\partial z}\hat{k}$$

$$= \left(\varphi \frac{\partial \psi}{\partial x} + \psi \frac{\partial \varphi}{\partial x}\right)\hat{\imath} + \left(\varphi \frac{\partial \psi}{\partial y} + \psi \frac{\partial \varphi}{\partial y}\right)\hat{\jmath} + \left(\varphi \frac{\partial \psi}{\partial z} + \psi \frac{\partial \varphi}{\partial z}\right)\hat{k}$$

$$= \varphi \left(\frac{\partial \psi}{\partial x}\hat{\imath} + \frac{\partial \psi}{\partial y}\hat{\jmath} + \frac{\partial \psi}{\partial z}\hat{k}\right) + \psi\left(\frac{\partial \varphi}{\partial x}\hat{\imath} + \frac{\partial \varphi}{\partial y}\hat{\jmath} + \frac{\partial \varphi}{\partial z}\hat{k}\right)$$

$$= \varphi\nabla\psi + \psi\nabla\varphi = R.H.S$$

iv. $\boldsymbol{\nabla\left(\frac{\varphi}{\psi}\right) = \frac{\psi\nabla\varphi - \varphi\nabla\psi}{\psi^2}}$

$$L.H.S = \nabla\left(\frac{\varphi}{\psi}\right) = \nabla\left(\varphi\frac{1}{\psi}\right) = \varphi\nabla\left(\frac{1}{\psi}\right) + \psi\nabla\left(\frac{1}{\varphi}\right)$$

$$= \varphi\left(\frac{-1}{\psi^2}\right)\nabla\psi + \psi\nabla\left(\frac{1}{\varphi}\right)$$

$$= \frac{\psi\nabla\varphi - \varphi\nabla\psi}{\psi^2} = R.H.S$$

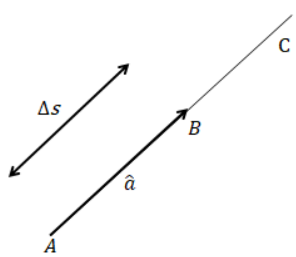
Module No. 5

Directional Derivative

The procedure to determine the derivative in a specific direction, other than the coordinate axes (x, y, z) is called **directional derivative**.

Let $\varphi(x, y, z)$ be a scalar point function defined on a specific region on R and also differentiable on the same domain. The first partial derivatives of $\varphi(x, y, z)$ are the rate of change of φ in the direction of coordinate axes (x, y, z). It is a restricted way to calculate the rate change is given function. Maybe one ought to need the derivative in a specific direction. Therefore the idea of directional derivative introduced.

To define the directional derivative we choose a point $A(x, y, z)$ in space and a direction at P, given by a unit vector $\hat{a}$. Let C be the ray drawn from P in the direction of $\hat{a}$, and let $A(x + \Delta x, y + \Delta y, z + \Delta z)$ denoted by B be a neighboring point on C, whose distance from P is Δs as shown in figure,



The value of given scalar point function is $\varphi(x, y, z)$ and $\varphi(+\Delta x, y + \Delta y, z + \Delta z)$ at P and P' respectively.

Then the limit

$$\lim_{\Delta s \to 0} \frac{\Delta \varphi}{\Delta s} = \lim_{\Delta s \to 0} \frac{\varphi(P') - \varphi(P)}{\Delta s}$$

if it exists, is called the directional derivative of φ at P in the direction of $\hat{a}$ and is denoted by $\frac{\partial \varphi}{\partial s}$.

Obviously,

$$\frac{\partial\varphi}{\partial s} = \frac{\partial\varphi}{\partial x}\frac{dx}{ds} + \frac{\partial\varphi}{\partial y}\frac{dy}{ds} + \frac{\partial\varphi}{\partial z}\frac{dz}{ds}$$

$$= (\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k})(\frac{dx}{ds}\hat{\imath} + \frac{dy}{ds}\hat{\imath} + \frac{dz}{ds}\hat{\imath})$$

$$= \nabla\varphi.\frac{d\vec{r}}{ds} = \nabla\varphi.\hat{a} \qquad (1)$$

Since $\hat{a}$ is a unit vector, directional derivative of φ (i.e. $\frac{\partial\varphi}{\partial s}$) is the component of $\nabla\varphi$ in the direction of this unit vector.

From equation (1), we have the operator equivalence,

$$\frac{\partial}{\partial s} = \nabla.\hat{a}$$

This means that the operator $\nabla.\hat{a}$ applied to the scalar function φ differentiates it w.r.t the distance s in the direction of $\hat{a}$.

Deductions

In particular, if we assume $\hat{a}$ has the direction of the positive x-axis, the $\hat{a} = \hat{\imath}$ then equation (1) will become $\nabla\varphi.\,\hat{\imath}$

$$= \left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right).\hat{\imath}$$

$$\nabla\varphi.\ \hat{\imath} = \frac{\partial\varphi}{\partial x}$$

Similarly,

$$\nabla\varphi.\ \hat{\jmath} = \frac{\partial\varphi}{\partial y},$$

$$\nabla\varphi.\ \hat{k} = \frac{\partial\varphi}{\partial z}$$

Module No. 6

Theorem related Directional Derivative

Statement

Show that the maximum value of the directional derivative of $\varphi(x, y, z)$ is equal to the magnitude of $\varphi(x, y, z)$is equal to the magnitude of $\nabla\varphi$ (i. e $|\nabla\varphi\ |$) and it takes place in the direction of $\nabla\varphi$.

Proof

We know that

$$\frac{\partial\varphi}{\partial s} = \nabla\varphi.\hat{a}$$

$$= |\nabla\varphi||\hat{a}|\cos\theta$$

where θ is the angle between $\nabla\varphi$ and $\hat{a}$. Since $-1 \leq \cos\theta \leq 1$, therefore $\frac{\partial\varphi}{\partial s}$ is maximum when $\cos\theta = 1$ or $\theta = 0°$ i.e. when the direction of $\hat{a}$ is the direction of $\nabla\varphi$ and max $\left(\frac{\partial\varphi}{\partial s}\right) = \nabla\varphi$.

Thus the maximum value of directional derivative takes place in the direction of $\nabla\varphi$ and has the magnitude $|\nabla\varphi|$.

It is important to be note that directional derivative $\frac{\partial\varphi}{\partial s}$ is zero, when $\theta = 90°$ i.e when $\nabla\varphi$ and $\hat{a}$ are orthogonal to each other.

Module No. 7

Example of the Directional Derivative

Problem Statement:

Find the directional derivative of $\varphi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\hat{\imath} — \hat{\jmath} — 2\hat{k}$.

Solution:

As we studied the equation

$$\text{Directional derivative} = \nabla\varphi.\hat{a}$$

$$\nabla\varphi = \nabla(x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\hat{\imath} + (x^2z)\,\hat{\jmath} + x^2y + 8xz)\hat{k}$$

$\nabla\varphi$ at $(1, -2, -1) = 8\hat{\imath} — \hat{\jmath} — 10\hat{k}$

The unit vector in the given direction can be calculated as

$$\hat{a} = \frac{\vec{a}}{|a|} = \frac{2\hat{\imath} — \hat{\jmath} — 2\hat{k}.}{3} = \frac{2}{3}\square - \frac{1}{3}\hat{\jmath} - \frac{2}{3}\hat{k}$$

then the required directional unit vector is

$$\nabla\varphi.\hat{a} = \left(8\hat{\imath} — \hat{\jmath} — 10\hat{k}\right).\left(\frac{2}{3}\hat{\imath} - \frac{1}{3}\hat{\jmath} - \frac{2}{3}\hat{k}\right)$$

$$= \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$$

Since this is positive, φ is increasing in this direction.

Module No. 8

Related Problem 1 of the Directional Derivative

Problem Statement:

Find the directional derivative of $\varphi\ = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ in the direction $2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}$.

Solution:

As we studied the equation

$$\text{Directional derivative} = \nabla\varphi.\hat{a}$$

$$\nabla\varphi = \nabla(4xz^3 - 3x^2y^2z\ )$$

$$= (4z^3 - 6xy^2z)\hat{\imath} + (-6x^2yz)\,\hat{\jmath} + (12xz^2 - 6x^2y^2)\hat{k}$$

$\nabla\varphi$ at $(2,-1,2) = 8\hat{\imath} + 24\hat{\jmath} +\ 84\hat{k}$

The unit vector in the given direction can be calculated as

$$\hat{a} = \frac{\vec{a}}{|a|} = \frac{2\hat{\imath} — 3\hat{\jmath} + 6\hat{k}.}{7} = \frac{2}{7}\hat{\imath} - \frac{3}{7}\hat{\jmath} + \frac{6}{7}\hat{k}$$

then the required directional unit vector is

$$\nabla\varphi.\hat{a} = \left(8\hat{\imath} + 48\hat{\jmath} +\ 84\hat{k}\right).\left(\frac{2}{7}\hat{\imath} - \frac{3}{7}\hat{\jmath} + \frac{6}{7}\hat{k}\right)$$

$$= \frac{16}{7} - \frac{144}{7} + \frac{504}{7} = \frac{376}{7}$$

Since this is positive, φ is increasing in this direction.

Module No. 9

Related problem 2 of Directional Derivative

Problem Statement

a) In what direction from the point $(2,1,-1)$ is the directional derivative of $\varphi = x^2yz^3$ a maximum?
b) What is the magnitude of this maximum?

Solution

First of all, we calculate gradient of $\varphi = x^2yz^3$

$$\nabla\varphi = \nabla(x^2yz^3) = (2xyz^3)\hat{\imath} + (x^2z^3)\hat{\jmath} + (3x^2yz^2)\hat{k}$$

$\nabla\varphi$ at $(2,1,-1)$ is $-4\hat{\imath} - 4\hat{\jmath} + 12\hat{k}$

using the result that directional derivative is maximum in the direction of $\nabla\varphi$ and its maximum magnitude is $|\nabla\varphi|$

The directional derivative is a maximum in the direction $\nabla\varphi = -4\hat{\imath} - 4\hat{\jmath} + 12\hat{k}$

the magnitude of this maximum is $|\nabla\varphi| = |(-4)^2 + (-4)^2 + (12)^2| = \sqrt{176} = 4\sqrt{11}$

Module No. 10

Related Problem 3 of the Directional Derivative

Problem Statement:

Find the values of the constants a, b, c so that the directional derivative of $\varphi = axy^2 + byz + cz^2x^3$ at

$(1,2,-1)$ has a maximum of magnitude 64 in a direction parallel to the z-axis.

Solution:

Since $\varphi = axy^2 + byz + cz^2x^3$, therefore

$$\nabla\varphi = \nabla(axy^2 + byz + cz^2x^3)$$

$$= (ay^2 + 3cz^2x^2)\hat{\imath} + (2axy + bz)\hat{\jmath} + (by + 2czx^3)\hat{k}$$

At the point $(1,2,-1)$, the value of this gradient is

$$= (4a + 3c)\hat{\imath} + (4a - b)\hat{\jmath} + (2b - 2c)\hat{k}$$

We know that the maximum directional derivative takes place in the direction of $\nabla\varphi$ and has the magnitude of $\nabla\varphi$. The maximum directional derivative will be parallel to the z-axis if

$4a + 3c = 0$ (1)

$4a - b = 0$ (2)

Therefore

$\nabla\varphi = (2b - 2c)\hat{k}$

since the magnitude of this maximum directional derivative is 64, there fore

$2b - 2c = 64$

(3)

Solving equation (1), (2) and (3), we find

$$a = 6, b = 24, c = -8$$

which is the required solution.

Module No. 11

Geometrical Interpretation of Gradient

Geometrically, gradient of a scalar function represents a normal vector to the surface.

$\nabla\varphi|_{(x,y,z)}$ represents the normal vector of the surface at (x, y, z).

Example

If $\varphi = x^2yz$

$$\nabla\varphi = 2xyz\hat{\imath} + x^2z\hat{\jmath} + x^2y\hat{k}$$

At (1,1,1)

$$\nabla\varphi|_{(1,1,1)} = 2\hat{\imath} + \hat{\jmath} + \hat{k}$$

which is normal to the surface $\varphi = x^2yz$

Module No. 12

Theorem Related to Gradient

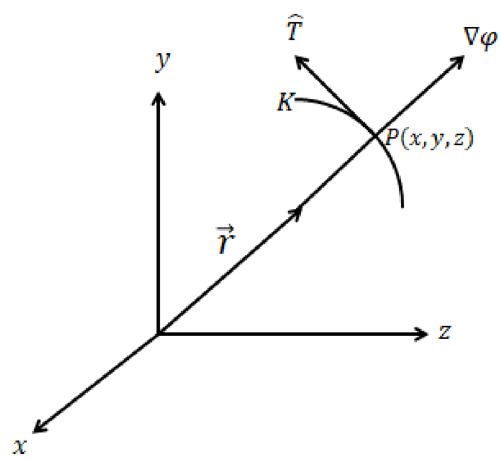
Theorem

Prove that $\nabla\varphi$ is a vector perpendicular to the surface $\varphi(x, y, z) = \mathrm{C}$, where C is a constant.

Proof

Let the parametric equation of the curve K be $x = x(s)$, $y = y(s)$, $z = z(s)$.

let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ be the position vector of the point $P(x, y, z)$, then $\hat{T} = \frac{dr}{ds}$ is unit tangent vector to the curve K at P as shown in figure.



Since the curve K lies on the level surface, together, the condition of any point on the curve must satisfy equation (1), and so $[x(s), y(s), z(s)] = C$.

Differentiating above equation w.r.t'‘s’ using the chain rule,

$$\frac{\partial\varphi}{\partial x}\frac{dx}{ds} + \frac{\partial\varphi}{\partial y}\frac{dy}{ds} + \frac{\partial\varphi}{\partial z}\frac{dz}{ds} = 0$$

or

$$= \left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right).\left(\frac{dx}{ds}\hat{\imath} + \frac{dy}{ds}\hat{\jmath} + \frac{dz}{ds}\hat{k}\right) = 0$$

or

$$\nabla\varphi.\frac{dr}{ds} = 0 \; or \; \nabla\varphi.\hat{T} = 0$$

Which implies that $\nabla\varphi$ is a vector perpendicular to the unit tangent vector $\hat{T}$and therefore on the surface $\varphi(x, y, z) = C$.

Module No. 13

Related Problem 1; Gradient

Problem Statement

If $\vec{A} = 2x^2 i - 3yzj + xz^2 k$ and $\varphi = 2z - x^3 y$

Find

i. $\vec{A}.\nabla\varphi$

ii. $\vec{A} \times \nabla\varphi$

Solution

Since $\varphi = 2z - x^3 y$ then

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$$

$$= \frac{\partial(2z - x^3 y\,)}{\partial x}\hat{\imath} + \frac{\partial(2z - x^3 y\,)}{\partial y}\hat{\jmath} + \frac{\partial(2z - x^3 y\,)}{\partial z}\hat{k}$$

$$\nabla\varphi = -3x^2 yi - x^3 j + 2k$$

$\nabla\varphi$ at $(1, -1,1)$

$$\nabla\varphi = 3i - j + 2k$$

Given vector $\vec{A} = 2x^2 i - 3yzj + xz^2 k$ at $(1, -1,1)$

$$\vec{A} = 2i + 3j + k$$

i. $\vec{A}.\nabla\varphi = (2i + 3j + k).(3i - j + 2k)$

$$(2)(3) - (3)(1) + (1)(2) = 5$$

ii. $\vec{A} \times \nabla\varphi = (2i + 3j + k) \times (\,3i - j + 2k)$

$$= \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= \big(6 - (-1)\big)i - (4 - 3)j + (-2 - 9)k$$

$$= 7i - j - 11k$$

Module No. 14

Related Problem 2; Gradient

Statement

Show that

$$\nabla f(r) = \frac{f'(r)\vec{r}}{r}$$

Proof

We know that

$$\nabla f(r) = \frac{\partial}{\partial x} f(r)\hat{\imath} + \frac{\partial}{\partial y} f(r)\hat{\jmath} + \frac{\partial}{\partial z} f(r)\hat{k}$$

$$= \frac{\partial f}{\partial x}\frac{\partial r}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\frac{\partial r}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\frac{\partial r}{\partial z}\hat{k}$$

$$= \frac{df}{dx}\frac{\partial r}{\partial x}\hat{\imath} + \frac{df}{dy}\frac{\partial r}{\partial y}\hat{\jmath} + \frac{df}{dz}\frac{\partial r}{\partial z}\hat{k} \qquad (1)$$

since $r = \sqrt{x^2 + y^2 + z^2}$, it follows that

$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

putting these values in (1), we get

$$= f'(r)\frac{x}{r}\hat{\imath} + f'(r)\frac{y}{r}\hat{\jmath} + f'(r)\frac{z}{r}\hat{k}$$

$$= \frac{f'(r)}{r}(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$

$$= \frac{f'(r)\vec{r}}{r}$$

Hence Proved

Module No. 15

Related Problem 3; Gradient

Statement

If $\nabla\varphi = 2xyz^3\hat{\imath} + x^2z^3\hat{\jmath} + 3x^2yz^2\hat{k}$,

Find

$\varphi(x, y, z)$ if $\varphi(1, -2,2) = 4$

Solution

Given equation is

$$\nabla\varphi = 2xyz^3i + x^2z^3j + 3x^2yz^2\hat{k} \qquad (1)$$

We know that

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} \qquad (2)$$

therefore by comparing equation(1) and (2), we get

$$\frac{\partial\varphi}{\partial x} = 2xyz^3 \qquad (3)$$

$$\frac{\partial\varphi}{\partial y} = x^2z^3 \qquad (4)$$

$$\frac{\partial\varphi}{\partial z} = 3x^2yz^2 \qquad (5)$$

Integrating Equation (3) w.r.t x , keeping y and z constants

$$\varphi = x^2yz^3 + f(y, z) \qquad (6)$$

Similarly, from equation (4) and (5), we get

$$\varphi = x^2yz^3 + g(x, z) \qquad (7)$$

$$\varphi = x^2yz^3 + h(x, y) \qquad (8)$$

Comparison of equation (6), (7) and (8) shows that there will be a common value of φ if we choose

$$f(y, z) = g(x, z) = h(x, y) = C$$

where C is an arbitrary constant.

Thus

$$\varphi = x^2yz^3 + C \qquad (9)$$

using $\varphi(1, -2,2) = 4$ in equation (9) we get

$$C = 20$$

hence from equation (9), we obtained

$$\varphi = x^2yz^3 + 20$$

Which is the required solution of φ.

Module No. 16

Divergence of a Vector Point Function

Definition

Divergence is a vector operator, when we applied this operator to the quantity of a vector field, it produces a scalar field.

Let $\vec{V}(x, y, z) = V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k}$ be defined and differentiable at each point (x, y, z) in a certain region of space (i.e. V defines a differentiable vector field). Then the divergence of V, written $\nabla.V$ or $div\ V$, is defined by

$$\nabla.\vec{V} = (\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}).(V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k})$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Some Deductions

It is important to be noted that $\nabla.\vec{V}$ is a scalar quantity. Also $\nabla.\vec{V} \neq \vec{V}.\nabla$

If $\vec{V}$ is a constant vector, then $\nabla.\vec{V} = 0$.

if $\nabla.\vec{V} = 0$ everywhere in some region of R, Then $\vec{V}$ is called **solenoid vector point function** in the region.

We can consider an example of solenoid to understand the concept.

Example

Determine the constant a so that the vector $\vec{V} = (x + 3y)\hat{\imath} + (y - 2z)\hat{\jmath} + (x + az)\hat{k}$ is solenoidal.

Solution

$$\nabla.V = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left((x + 3y)\hat{\imath} + (y - 2z)\hat{\jmath} + (x + az)\hat{k}\right)$$

$$= \frac{\partial(x + 3y)}{\partial x} + \frac{\partial(y - 2z)}{\partial y} + \frac{\partial(x + az)}{\partial z}$$

$$= 1 + 1 + a = 2 + a$$

$$a = -2$$

Hence if we substitute $a = -2$ in given vector field then given $\vec{V}$ will become solenoidal.

Module No. 17

Properties of the Divergence

Statement

If $\vec{A}$ and $\vec{B}$ are differentiable vector point functions, and φ is differentiable scalar point function, then prove that

i. $\nabla.\left(\vec{A}+\vec{B}\right)=\nabla.\vec{A}+\nabla.\vec{B}$

ii. $\nabla.\left(\varphi\vec{A}\right)=\varphi(\overrightarrow{\nabla.A})+\vec{\square}.(\nabla\varphi)$

Proof

Let

$\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$ and $\vec{B} = B_1\hat{\imath} + B_2\hat{\jmath} + B_3\hat{k}$, then

i. $\boldsymbol{\nabla.\left(\vec{A}+\vec{B}\right)=\nabla.\vec{A}+\nabla.\vec{B}}$

$$L.H.S=\left(\vec{A}+\vec{B}\right)=(A_1+B_1)\hat{\imath}+(A_2+B_1)\hat{\jmath}+(A_3+B_3)\hat{k}$$

hence

$$\nabla.\left(\vec{A}+\vec{B}\right)=\frac{\partial}{\partial x}(A_1+B_1)\hat{\imath}+\frac{\partial}{\partial y}(A_2+B_1)\hat{\jmath}+\frac{\partial}{\partial z}(A_3+B_3)\hat{k}$$

$$=(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z})+(\frac{\partial B_1}{\partial x}+\frac{\partial B_2}{\partial y}+\frac{\partial B_3}{\partial z})$$

$$=\nabla.\vec{A}+\nabla.\vec{B}=R.H.S$$

ii. $\boldsymbol{\nabla.\left(\varphi\vec{A}\right)=\varphi(\overrightarrow{\nabla.A})+\vec{A}.(\nabla\varphi)}$

$L.H.S=\nabla.\left(\varphi\vec{A}\right)$

We have $\varphi\vec{A}=\varphi A_1\hat{\imath}+\varphi A_2\hat{\jmath}+\varphi A_3\hat{k}$

Hence

$$\nabla.\left(\varphi\vec{A}\right)=\frac{\partial}{\partial x}(\varphi A_1)\hat{\imath}+\frac{\partial}{\partial y}(\varphi A_2)\hat{\jmath}+\frac{\partial}{\partial z}(\varphi A_3)\hat{k}$$

$$=\varphi\frac{\partial A_1}{\partial x}+A_1\frac{\partial\varphi}{\partial x}+\varphi\frac{\partial A_2}{\partial x}+A_2\frac{\partial\varphi}{\partial x}+\varphi\frac{\partial A_3}{\partial x}+A_3\frac{\partial\varphi}{\partial x}$$

$$= \varphi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + (A_1\frac{\partial \varphi}{\partial x} + A_2\frac{\partial \varphi}{\partial x} + A_3\frac{\partial \varphi}{\partial x})$$

$$= \varphi(\overrightarrow{\nabla . A)} + (\frac{\partial \varphi}{\partial x}\hat{\imath} + \frac{\partial \varphi}{\partial y}\hat{\jmath} + \frac{\partial \varphi}{\partial z}\hat{k})(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k})$$

$$\nabla.\left(\varphi\vec{A}\right) == \varphi(\overrightarrow{\nabla . A)} + (\nabla\varphi)\overrightarrow{. A} = \varphi(\overrightarrow{\nabla . A)} + \vec{A}.(\nabla\varphi)$$

$\nabla.\left(\varphi\vec{A}\right) = \varphi(\overrightarrow{\nabla . A)}$ if φ is constant.

Module No. 18

Laplacian

The Laplacian or Laplace operator is a second-order differential operator given by the divergence of the gradient of a given function defined over a space R. It is usually denoted by $\nabla.\nabla, or\ \nabla^2\ or$ Δ.

Thus if $\vec{u}$ is a twice differentiable function, then the Laplacian of $\vec{u}$ is defined by

$$\Delta\vec{u} = \nabla^2\vec{u} = \nabla.\nabla\vec{u}$$

In cartesian coordinate system, the Laplacian is given by the sum of second order partial derivatives of the function w.r.t each independent variable. Laplace is a Second order differential operator which is obtained by taking the divergence of gradient of any scalar point function.

It is denoted as

$$\nabla.\nabla = \nabla^2 = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

It is a scalar operator.

In one and two dimension, the Laplace operator reduces to

$$\nabla^2 = \frac{\partial^2}{\partial x^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

If $\varphi(x, y, z)$ is a scalar point function, then the divergence of gradient of φ written as $\nabla.\nabla\varphi = \nabla^2\varphi$ is called the Laplacian of φ and the equation $\nabla^2\varphi = 0$ is called Laplace's equation.

If a scalar function φ satisfies the Laplace equation $\nabla^2\varphi = 0$ in a Cartesian region R, then φ is said to be a harmonic function in the region R.

Mathematically, we can write it as

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

We can also express the Laplace operator in polar coordinate (r, θ) notation.

Two dimensional Laplace operators can be expressed as

$$\nabla^2\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\varphi}{\partial\theta^2}\right)$$

$$= \frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}$$

The Laplacian occurs in differential equation that describes many physical phenomena, such as Diffusion equation for heat and fluid flow, gravitational potentials and quantum mechanics.

Module No. 19

Example of Divergence

Statement

If $\vec{A} = x^2 z\hat{\imath} - 2y^3 z^2\hat{\jmath} + xy^2 z\hat{k}$, find $\nabla.\vec{A}$ (or div A) at the point $(1,-1,1)$.

Solution

As we know del operator ∇ is

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

Since given vector point function $\vec{A}$ is

$$\vec{A} = x^2 z\hat{\imath} - 2y^3 z^2\hat{\jmath} + xy^2 z\hat{k}$$

$$\nabla.\vec{A} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(x^2 z\hat{\imath} - 2y^3 z^2\hat{\jmath} + xy^2 z\hat{k}\right)$$

$$= \frac{\partial(x^2 z)}{\partial x} + \frac{\partial(-2y^3 z^2)}{\partial y} + \frac{\partial(xy^2 z)}{\partial z}$$

$$= 2xz - 6y^2 z^2 + xy^2$$

As we calculated the expression $\nabla.\vec{A}$, we can easily determine its value at $(1,-1,1)$.

$$\nabla.\vec{A}\ at\ (1,-1,1) = 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2$$

$$= 2 - 6 + 1 = -3$$

$$\nabla.\vec{A}\ at\ (1,-1,1) = -3$$

which is the required Result.

Module No. 20

Related Problem 1: Divergence

Statement

If $\vec{A} = 3xyz^2\hat{\imath} + 2xy^3\hat{\jmath} - x^2yz\hat{k}$ and $\varphi = 3x^2 - yz$,

find

i. $\nabla.\vec{A}$

ii. $\vec{A}.\nabla\varphi$

iii. $\nabla.\left(\varphi\vec{A}\right)$

iv. $\nabla.(\nabla\varphi)$

at the point $(1,-1,1)$.

Solution

Since $\vec{A} = 3xyz^2\hat{\imath} + 2xy^3\hat{\jmath} - x^2yz\hat{k}$

also

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

then

i. $\boldsymbol{\nabla}.\vec{A}$

$$\nabla.\vec{A} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(3xyz^2\hat{\imath} + 2xy^3\hat{\jmath} - x^2yz\hat{k}\right)$$

$$= 3yz^2 + 6xy^2\hat{\jmath} - x^2y$$

$\nabla.\vec{A}$ at point $(1,-1,1) is$

$$= 3(-1)(1)^2 + 6(1)(-1)^2 - (1)^2(-1)$$

$$= -3 + 6 - 1 = 2$$

ii. $\vec{A}.\boldsymbol{\nabla\varphi}$

Since $\varphi = 3x^2 - yz$

then

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$$

$$= \frac{\partial(3x^2 - yz)}{\partial x}\hat{\imath} + \frac{\partial(3x^2 - yz)}{\partial y}\hat{\jmath} + \frac{\partial(3x^2 - yz)}{\partial z}\hat{k}$$

$$= 6x\hat{\imath} - z\hat{\jmath} - y\hat{k}$$

Now,

$$\vec{A}.\nabla\varphi = \left(3xyz^2\hat{\imath} + 2xy^3\hat{\jmath} - x^2yz\hat{k}\right).\left(6x\hat{\imath} - z\hat{\jmath} - y\hat{k}\right)$$

$$= (18x^2yz^2 - 2xy^3z + x^2y^2z)$$

$\vec{A}.\nabla\varphi$ at point $(1,-1,1)$is

$$= [18(1)^2(-1)(1)^2 - 2(1)(-1)^3(1) + (1)^2(-1)^2(1)]$$

$$= -18 + 2 + 1 = -15$$

iii. $\boldsymbol{\nabla}.\left(\boldsymbol{\varphi}\vec{A}\right)$

then

$$\varphi\vec{A} = (3x^2 - yz)\left(3xyz^2\hat{\imath} + 2xy^3\hat{\jmath} - x^2yz\hat{k}\right)$$

$$= (9x^3yz^2 - 3xy^2z^3)\hat{\imath} + (6x^3y^3 - 2xy^4z)\hat{\jmath} + (3x^4yz - x^2y^2z^2)\hat{k}$$

Now

$$\nabla.\left(\varphi\vec{A}\right) = \nabla.((9x^3yz^2 - 3xy^2z^3)\hat{\imath} + (6x^3y^3 - 2xy^4z)\hat{\jmath} + (3x^4yz - x^2y^2z^2)\hat{k})$$

$$= \frac{\partial(9x^3yz^2 - 3xy^2z^3)}{\partial x} + \frac{\partial(6x^3y^3 - 2xy^4z)}{\partial y} + \frac{\partial(3x^4yz - x^2y^2z^2)}{\partial z}$$

$$= 27x^2yz^2 - 3y^2z^3 + 18x^3y^2 - 8xy^3z + 3x^4y - 2x^4yz$$

$\nabla.\left(\varphi\vec{A}\right)$ at point $(1,-1,1)$is

$$\nabla.\left(\varphi\vec{A}\right) = -27 - 3 + 18 + 8 - 3 + 2 = -5$$

iv. $\boldsymbol{\nabla}.(\boldsymbol{\nabla\varphi})$

$$= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(\frac{\partial\Box}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right)$$

$$= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(\frac{\partial(3x^2 - yz)}{\partial x}\hat{\imath} + \frac{\partial(3x^2 - yz)}{\partial y}\hat{\jmath} + \frac{\partial(3x^2 - yz)}{\partial z}\hat{k}\right)$$

$$= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\widehat{k}\right).\left(6x\hat{\imath} - z\hat{\jmath} - y\widehat{k}\right) = 6$$

Module No. 21

Related Problem 2: Divergence

Problem Statement

Given

$\varphi = 2x^3y^2z^4$.

To find

i. $\nabla.\nabla\varphi$ (or div gradφ).

ii. Show that $\nabla.\nabla\varphi = \nabla^2\varphi$, where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ denotes the Laplacian operator.

Solution

i. $\nabla.\nabla\varphi$

As we know the grad of φ

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$$

$$= \frac{\partial(2x^3y^2z^4)}{\partial x}\hat{\imath} + \frac{\partial(2x^3y^2z^4)}{\partial y}\hat{\jmath} + \frac{\partial(2x^3y^2z^4)}{\partial z}\hat{k}$$

$$= 6x^2y^2z^4\hat{\imath} + 4x^3yz^4\hat{\jmath} + 8x^3y^2z^3\hat{k}$$

Then the divergence of the grad of φ

$$\nabla.\nabla\varphi = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(6x^2y^2z^4\hat{\imath} + 4x^3yz^4\hat{\jmath} + 8x^3y^2z^3\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)$$

$$= \frac{\partial(6x^2y^2z^4)}{\partial x} + \frac{\partial(4x^3yz^4)}{\partial y} + \frac{\partial(8x^3y^2z^3)}{\partial z}$$

$$= 12xy^2z^4 + 4x^3yz^4 + 24x^3y^2z^2$$

ii. $\nabla.\nabla\varphi = \nabla^2\varphi$

L.H.S= $\nabla.\nabla\varphi$

$$= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial z}\right)$$

$$= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi$$

$$= \nabla^2 \varphi$$

Hence proved.

Module No. 22

Related Problem 3: Laplacian

Problem Statement

Show that

i. $\nabla.(r^3\vec{r})$

ii. $\nabla.\left[\vec{r}\nabla(\frac{1}{r^3})\right] = \frac{3}{r^4}$

iii. $\nabla.\left[\nabla.(\frac{\vec{r}}{r})\right] = -\frac{2\vec{r}}{r^3}$

where $\vec{r}$ is position vector.

Solution

First of all we will derive a general relation for $\nabla.[\vec{r}f(r)]$ then we will substitute our given values to evaluate our required result.

$\nabla.[\vec{r}f(r)] = f(r)(\nabla.\vec{r}) + \vec{r}.(\nabla f(r))$

$$\therefore\ \nabla.\vec{r} = 3$$

We have

$$= 3f(r) + \vec{r}f'(r)\hat{r} = 3f(r) + \vec{r}.\frac{f'(r)\vec{r}}{r}$$

$$\therefore \hat{r} = \frac{\vec{r}}{r}$$

$$= 3f(r) + \frac{f'(r)}{r}\vec{r}.\vec{r}$$

$$\therefore \vec{r}.\vec{r} = r^2$$

$$\nabla.[\vec{r}f(r)] = 3f(r) + rf'(r)$$

Now setting $f(r) = r^n$ in above relation, we obtain

$$\nabla.(\vec{r}r^n) = 3r^n + r(nr^{n-1})$$

$$\nabla.(\vec{r}r^n) = 3r^n + (nr^n)$$

$$\nabla.(r^n\vec{r}) = (3+n)r^n$$

i. We have $\nabla.(r^3\vec{r})$ to evaluate, it is of the form $\nabla.[\vec{r}f(r)]$, where $\square(r) = r^3$

Put $n = 3$ in above relation, we will obtain the required result

$$\nabla.(r^3\vec{r}) = (3+3)r^3 = 6r^3$$

Is the required solution.

ii. $L.H.S = \nabla.\left[r\nabla(\frac{1}{r^3})\right]$

We can also write it as

$$= \nabla.[r(\nabla r^{-3})] = \nabla.[r(-3r^{-5}\vec{r})]$$

where we used the result

$$\nabla f(r) = \frac{f'(r)\vec{r}}{r}$$

We have $\nabla.[r(-3r^{-5}\vec{r})]$ to evaluate, it is of the form $\nabla.[\vec{r}f(r)]$, where $f(r) = r^{-4}\vec{r}$ thus,

$$\nabla.\left[\vec{r}\nabla(\frac{1}{r^3})\right] = -3\nabla.(r^{-4}\vec{r}) = -3(-4+3)r^{-4} = \frac{3}{r^4} = R.H.S$$

is the required solution.

iii. $L.H.S = \nabla.\left[\nabla.(\frac{\vec{r}}{r})\right]$

frist we evaluate $\nabla.(\frac{\vec{r}}{r})$ using formula $\nabla.[\vec{r}f(r)]$, where $f(r) = r^{-1}\vec{r}$

$$\nabla.(r^n\vec{r}) = (3+n)r^n$$

put $n = 1$, we have

$$\nabla.(r^{-1}\vec{r}) = (3-1)r^{-1} = \frac{2}{r}$$

therefore

$$= \nabla.\left(\frac{2}{r}\right) = -2r^{-3}\vec{r} = -\frac{2\vec{r}}{r^3} = R.H.S$$

Module No. 23

Curl of a vector Point Function

Introduction

The infinitesimal rotation of a vector field is described by the curl. The curl of a specific point is shows by a vector at every point of the vector field. The attributes of this vector (length and direction) characterize the rotation at that point.

The direction of the curl is the axis of rotation, as determined by the right-hand rule, and the magnitude of the curl is the magnitude of rotation.

Definition

If $\vec{V}(x, y, z)$ is a differentiable vector field in a certain region of space, then the curl or rotation of V, written $\nabla \times \vec{V}$, curl V or rot V, is defined by

$$\nabla \times \vec{V} = (\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}) \times (V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k}).$$

Where V_1, V_2, V_3 are the components of vector field along x, y and z-axis.

We can write this expression in matrix from

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_2 & V_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_1 & V_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_1 & V_2 \end{vmatrix} \hat{k}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{\imath} - \left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\hat{k}$$

A vector field whose curl is zero is called irrotational.

Identities of Curl

Some identities of curl are stated below

A gradient has zero curl:

$$\nabla \times \nabla\varphi = 0.$$

A curl has zero divergence:

$$\nabla \cdot (\nabla \times \vec{V}) = 0$$

Note: $\nabla \cdot (\nabla \times \vec{V}) = (\nabla \times \nabla) . \vec{V} = 0$ because $\nabla \times \nabla = 0$.

Module No. 24

Properties of the Curl

If $\vec{A}$ and $\vec{B}$ are differentiable vector functions, and φ is differentiable scalar functions of position (x, y, z), then

i. $\nabla \times (\vec{A} + \vec{B}) = \nabla\times\vec{A} + \nabla\times\vec{B}$ or curl $(\vec{A} + B)$ = curl $\vec{A}$ + curl $\vec{B}$

ii. $\nabla\times(\varphi\vec{A}) = \varphi(\nabla\times\vec{A}) + (\nabla\varphi)\times\vec{A}$

iii. $\nabla\times(\nabla\varphi) = 0$ The curl of the gradient of φ is zero.

iv. $\nabla.(\nabla\times\vec{A}) = 0$ The divergence of the curl of $\vec{A}$ is zero.

Proof

Let $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$ and $\vec{B} = B_1\hat{\imath} + B_2\hat{\jmath} + B_3\hat{k}$, then

i. curl $(\vec{A}$ +B) = curl $\vec{A}$ + curl $\vec{B}$

L.H. S= curl $(\vec{A} + \vec{B})$

$$\vec{A} + \vec{B} = (A_1 + B_1)\hat{\imath} + (A_2 + B_2)\hat{\jmath} + (A_3 + B_3)\hat{k}$$

Hence by using the definition of curl

$$\nabla \times (\vec{A} + B) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 + B_1 & A_2 + B_2 & A_3 + B_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$= \nabla\times\vec{A} + \nabla\times\vec{B}$

Hence proved

ii. $\nabla\times(\varphi\vec{A}) = \varphi(\nabla\times\vec{A}) + (\nabla\varphi)\times\vec{A}$

L.H.S= $\nabla\times(\varphi\vec{A})$

$$\varphi\vec{A} = \varphi\left(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}\right) = \varphi A_1\hat{\imath} + \varphi A_2\hat{\jmath} + \varphi A_3\hat{k},$$

Then by using the definition of curl

$$\nabla\times(\varphi\vec{A}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi A_2 & \varphi A_2 & \varphi A_3 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(\varphi A_3) - \frac{\partial}{\partial z}(\varphi A_2)\right)\hat{\imath} - \left(\frac{\partial}{\partial x}(\varphi A_3) - \frac{\partial}{\partial z}(\varphi A_1)\right)\hat{\jmath} + \left(\frac{\partial}{\partial x}(\varphi A_2) - \frac{\partial}{\partial y}(\varphi A_1)\right)\hat{k}$$

$$= \left[\varphi\frac{\partial(A_3)}{\partial y} + A_3\frac{\partial(\varphi)}{\partial y} - \varphi\frac{\partial(A_2)}{\partial z} - A_2\frac{\partial(\varphi)}{\partial z}\right]\hat{\imath} - \left[\varphi\frac{\partial(A_3)}{\partial x} + A_3\frac{\partial(\varphi)}{\partial x} - \varphi\frac{\partial(A_1)}{\partial z} - A_1\frac{\partial(\varphi)}{\partial z}\right]\hat{\jmath}$$
$$+ \left[\varphi\frac{\partial(A_2)}{\partial x} + A_2\frac{\partial(\varphi)}{\partial x} - \varphi\frac{\partial(A_1)}{\partial y} - A_1\frac{\partial(\varphi)}{\partial y}\right]\hat{k}$$

$$= \varphi\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)\hat{\jmath} + (\frac{\partial(A_2)}{\partial x} - \frac{\partial A_1}{\partial y})\hat{k}\right] + \left[(A_3\frac{\partial\varphi}{\partial y}A_2\frac{\partial\varphi}{\partial z})\hat{\imath} + \left(A_3\frac{\partial\varphi}{\partial x} - A_1\frac{\partial\varphi}{\partial z}\right)\hat{\jmath} + (A_2\frac{\partial\varphi}{\partial x} - A_1\frac{\partial\varphi}{\partial y})\hat{k}\right]$$

$$= \varphi\,(\nabla\times\vec{A}) + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \varphi\,(\nabla\times\vec{A}) + (\nabla\varphi)\times\vec{A}$$

Hence Proved

iii. $\nabla\times(\nabla\boldsymbol{\varphi}) = 0$

$$\text{L.H.S} = \nabla\times(\nabla\varphi) = \nabla\times\left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial \square} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}\right)\hat{\imath} - \left(\frac{\partial^2\varphi}{\partial z\partial x} - \frac{\partial^2\varphi}{\partial x\partial z}\right)\hat{\jmath} + \left(\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y\partial x}\right)\hat{k} \qquad (1)$$

We assume that φ has continuous second order partial derivative so the order of differentiation can be neglected.

i.e.

$$\frac{\partial^2\varphi}{\partial y\partial z} = \frac{\partial^2\varphi}{\partial z\partial y}, \frac{\partial^2\varphi}{\partial z\partial x} = \frac{\partial^2\varphi}{\partial x\partial z}, \frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial^2\varphi}{\partial y\partial x}$$

⟹equation (1) must be equal to zero

Hence the expression.

iv. $\nabla.(\nabla\times\vec{A}) = 0$

L.H.S=$\nabla.(\nabla\times\vec{A})$

Since

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{\imath} - \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)\hat{\jmath} + (\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\hat{k}$$

Hence $\nabla.(\nabla\times\vec{A}) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{\imath} - \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)\hat{\jmath} + (\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y})\hat{k}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= \left(\frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z}\right) - \left(\frac{\partial^2 A_3}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y \partial z}\right) + \left(\frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y}\right)$$

$$= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} + \frac{\partial^2 A_3}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y \partial z} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y} = 0$$

Assuming that $\vec{A}$ has continuous second order derivative.

Hence the theorem.

Module No. 25

Example of Curl

Statement

If $\vec{A} = xz^3\, i - 2x^2yzj + 2yz^4k,$

Find

$\nabla \times \vec{A}$ (or curl $\vec{A}$) at the point $(1, -1, 1)$.

Solution

$$\nabla \times \vec{A} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \times (xz^3\,\hat{\imath} - 2x^2yz\hat{\jmath} + 2yz^4\hat{k})$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2\square z^4 \end{vmatrix}$$

$$= \left(\frac{\partial(2yz^4)}{\partial y} - \frac{\partial(-2x^2yz)}{\partial z}\right)\hat{\imath} + \left(\frac{\partial(2yz^4)}{\partial x} - \frac{\partial(xz^3)}{\partial z}\right)\hat{\jmath} + (\frac{\partial(-2x^2yz)}{\partial x} - \frac{\partial(xz^3)}{\partial y})\hat{k}$$

$$\nabla \times \vec{A} = (2z^4 + 2x^2yz)\hat{\imath} + 3xz^2\hat{\jmath} - 4xyz\hat{k}$$

Now we calculate the value of $\nabla \times \vec{A}$ at $(1, -1, 1)$

$$\nabla \times \vec{A} = (2 - 2)\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$$

$$\nabla \times \vec{A} = 3\hat{\jmath} + 4\hat{k}$$

is the required solution.

Module No. 26

Related Problem 1: Curl

Problem Statement

If $A = 2yz\,i - x^2y\,j + xz^2\,k$ and $\varphi = 2x^2yz^3$,

Find

i. $(A \times \nabla)\varphi$

ii. $A \times (\nabla\varphi)$

Also show whether they are identical or not.

Solution

Since $A = 2yz\,i - x^2y\,j + xz^2\,k$ and $\varphi = 2x^2yz^3$

i. $(\boldsymbol{A} \times \boldsymbol{\nabla})\boldsymbol{\varphi}$

$$(A \times \nabla)\varphi = \left[(2yz\,i - x^2y\,j + xz^2\,k) \times \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\right]\varphi$$

$$(A \times \nabla\varphi) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2yz & -x^2y & xz^2 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}\varphi$$

$$= \left[\left(-x^2y\frac{\partial}{\partial z} - xz^2\frac{\partial}{\partial y}\right)\hat{\imath} - \left(2yz\frac{\partial}{\partial z} - xz^2\frac{\partial}{\partial x}\right)\hat{\jmath} + \left(2yz\frac{\partial}{\partial y} + x^2y\frac{\partial}{\partial x}\right)\hat{k}\right]\varphi$$

Now substituting φ

$$= \left[\left(-x^2y\frac{\partial\varphi}{\partial z} - xz^2\frac{\partial\varphi}{\partial y}\right)\hat{\imath} - \left(2yz\frac{\partial\varphi}{\partial z} - xz^2\frac{\partial\varphi}{\partial x}\right)\hat{\jmath} + \left(2yz\frac{\partial\varphi}{\partial y} + x^2y\frac{\partial\varphi}{\partial x}\right)\hat{k}\right]$$

Since $\varphi = 2x^2yz^3$, substituting value

$$= \left[\left(-x^2y\frac{\partial(2x^2yz^3)}{\partial z} - xz^2\frac{\partial(2x^2yz^3)}{\partial y}\right)\hat{\imath} - \left(2yz\frac{\partial(2x^2yz^3)}{\partial z} - xz^2\frac{\partial(2x^2yz^3)}{\partial x}\right)\hat{\jmath}\right.$$
$$\left. + \left(2yz\frac{\partial(2x^2yz^3)}{\partial y} + x^2y\frac{\partial(2x^2yz^3)}{\partial x}\right)\hat{k}\right]$$

$$= \left[\{-x^2y(6x^2yz^2) - xz^2(2x^2z^3\}\hat{\imath} - \{2yz(6x^2yz^2) - xz^2(4xyz^3\}\hat{\jmath} + \{2yz(2x^2z^3) + x^2y(4xyz^3\}\hat{k}\right]$$

$$(A \times \nabla)\varphi = -(6x^4y^2z^2 + 2x^3z^5)\hat{\imath} - (12x^2y^2z^3 - 4x^2yz^5)\hat{\jmath} + (4x^2yz^4 + 4x^3y^2z^3)\hat{k}$$

(1)

ii. $\boldsymbol{A} \times (\boldsymbol{\nabla\varphi})$

Here $A = 2yz\, i - x^2y\, j + xz^2\, k$ and $\varphi = 2x^2yz^3$

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$$

$$= \frac{\partial(2x^2yz^3)}{\partial x}\hat{\imath} + \frac{\partial(2x^2yz^3)}{\partial y}\hat{\jmath} + \frac{\partial(2x^2yz^3)}{\partial z}\hat{k}$$

$$\nabla\varphi = 4xyz^3\hat{\imath} + 2x^2z^3\hat{\jmath} + 6x^2yz^2\hat{k}$$

Now

$$A \times (\nabla\varphi) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2yz & -x^2y & xz^2 \\ 4xyz^3 & 2x^2z^3 & 6x^2yz^2 \end{vmatrix}$$

$$= [(-x^2y)(6x^2yz^2) - (xz^2)(2x^2z^3)]\hat{\imath} - [(2yz)(6x^2yz^2) - (xz^2)(4xyz^3)]\hat{\jmath} + \big((2yz)(2x^2z^3) - (-x^2y)(4xyz^3)\big)\hat{k}$$

$$A \times (\nabla\varphi) = -(6x^4y^2z^2 + 2x^3z^5)\hat{\imath} - (12x^2y^2z^3 - 4x^2yz^5)\hat{\jmath} + (4x^2yz^4 + 4x^3y^2z^3)\hat{k}$$

(2)

From equation (1) and (2), we conclude that

$$A \times (\nabla\varphi) = (A \times \nabla)\varphi$$

Module No. 27

Related Problem 2: Curl

Problem Statement

A vector V is called irrotational if $curlv = 0$.

i. Find constants a, b, c so that

$$\vec{A} = (x + 2y + az)\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4x + cy + 2z)\hat{\imath}$$

is irrotational.

ii. Show that $\vec{A}$ can be expressed as the gradient of a scalar function.

Solution

Since give vector field is

$$\vec{A} = (x + 2y + az)\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4x + cy + 2z)\hat{\imath}$$

i. Also for a vector to be irrotational, we have expression

$$\text{curl}\vec{A} = \nabla \times \vec{A} = 0$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}(bx - 3y - z)\right)\hat{\imath} - \left(\frac{\partial}{\partial x}(4x + cy + 2z) - \frac{\partial}{\partial z}(x + 2y + az)\right)\hat{\jmath}$$

$$+ \left(\frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(x + 2y + az)\right)\hat{k}$$

$$= (c - 1)\hat{\imath} - (4 - a)\hat{\jmath} + (b - 2)\hat{k}$$

or

$$= (c - 1)\hat{\imath} + (a - 4)\hat{\jmath} + (b - 2)\hat{k}$$

According to the given condition

$$(c - 1)\hat{\imath} + (a - 4)\hat{\jmath} + (b - 2)\hat{k} = 0$$

$\Longrightarrow c - 1 = 0 \Longrightarrow c = 1$

$\Longrightarrow a - 4 = 0 \Longrightarrow a = 4$

$\Longrightarrow b - 2 = 0 \Longrightarrow b = 2$

and thus

$$\vec{A} = (x + 2y + 4z)\hat{\imath} + (2x - 3y - z)\hat{\jmath} + (4x + y + 2z)\hat{\imath} \quad (1)$$

is irrotational vector field.

ii. Assume that

$$\vec{A} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\widehat{k} \quad (2)$$

By comparing equation (1) and (2), we obtain

$$\frac{\partial\varphi}{\partial x} = x + 2y + 4z \quad (3)$$

$$\frac{\partial\varphi}{\partial y} = 2x - 3y - z \quad (4)$$

$$\frac{\partial\varphi}{\partial z} = 4x + y + 2z \quad (5)$$

Integrating equation (3) w.r.t x, keeping y and z constants,

$$\varphi = \frac{x^2}{2} + 2xy + 4xz + \mathrm{f}(y, z) \quad (6)$$

$$\varphi = 2xy - \frac{3y^2}{2} - yz + \mathrm{g}(x, z) \quad (7)$$

$$\varphi = 4xz - yz + z^2 + \mathrm{h}(x, y) \quad (8)$$

Comparison of equation (6), (7) and (8) shows that there will be a common value of φ if we choose

$$\mathrm{f}(y, z) = -\frac{3y^2}{2} + z^2$$

$$\mathrm{g}(x, z) = \frac{x^2}{2} + z^2$$

$$\mathrm{h}(x, y) = \frac{x^2}{2} - \frac{3y^2}{2}$$

so that

$$\varphi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + \text{constant}$$

Note that we can also add any constant to φ. In general if $\nabla \times \vec{A} = 0$, then we can find φ so that $\vec{A} = \nabla\varphi$.A vector field $\vec{A}$ which can be derived from a scalar field φ so that $\vec{A} = \nabla\varphi$ is called a conservative vector field and φ is called the scalar potential.

Note that conversely if $\vec{A} = \nabla\varphi$, then $\nabla \times \vec{A} = 0$

Module No. 28

Related Problem 3: Curl

Problem Statement

If $v = \omega \times r$, prove $\omega \;=\; r\;curl\;v$ where ω is a constant vector.

Solution

$$curl\;v \;=\; \nabla \times v \;=\; \nabla \times (\omega \times r)$$

$$= \nabla \times \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \nabla \times \left[(\omega_2 z - \omega_3 y)\hat{\imath} - (\omega_1 z - \omega_3 x)\hat{\jmath} + (\omega_1 y - \omega_2 x)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_3 x - \omega_1 z)\right)\hat{\imath} - \left(\frac{\partial}{\partial x}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_2 z - \omega_3 y)\right)\hat{\jmath}$$

$$+ \left(\frac{\partial}{\partial x}(\omega_3 x - \omega_1 z) - \frac{\partial}{\partial y}(\omega_2 z - \omega_3 y)\right)\hat{k}$$

$$= (\omega_1 + \omega_1)\hat{\imath} - (-\omega_2 - \omega_2)\hat{\jmath} + (\omega_3 - (-\omega_3))\hat{k}$$

$$= 2(\omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k})$$

$$\Longrightarrow \nabla \times (\omega \times r) = 2\vec{\omega}$$

$$\Longrightarrow \nabla \times v = 2\vec{\omega}$$

$$\Longrightarrow \vec{\omega} = \frac{1}{2}\nabla \times v$$

$$\Longrightarrow \vec{\omega} = \frac{1}{2}\text{curl } v$$

This problem indicates that the curl of a vector field is linked with rotational properties of the field. We might say that if curl $\vec{A} = 0,$ then there would be no rotation and the field is called irrotational field. A field which is not irrotational is sometimes called a vortex field.

Module No. 29

Vector Identities

Problem Statement

Evaluate $\nabla.(\vec{A}\times r)\ if\ \nabla\times\vec{A} = 0$.

Solution

Let $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$ and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, then

$$\vec{A}\times r = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$

$$= (A_2z - A_3y)\hat{\imath} - (A_1z - A_3x)\hat{\jmath} + (A_1y - A_2x)\hat{k}$$

And

$$\nabla.\left(\vec{A}\times r\right) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left((A_2z - A_3y)\hat{\imath} - (A_1z - A_3x)\hat{\jmath} + (A_1y - A_2x)\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(A_2z - A_3y) - \frac{\partial}{\partial y}(A_1z - A_3x) + \frac{\partial}{\partial z}(A_1y - A_2x)$$

$$= z\frac{\partial A_2}{\partial x} - y\frac{\partial A_3}{\partial x} - z\frac{\partial A_1}{\partial y} + x\frac{\partial A_3}{\partial y} + y\frac{\partial A_1}{\partial z} - x\frac{\partial A_2}{\partial z}$$

$$= x\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + y\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + z\left(\frac{\partial A_2}{\partial x} - z\frac{\partial A_1}{\partial y}\right)$$

$$= \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).\left(\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}\right)$$

$$= \vec{r}.\left(\nabla\times\vec{A}\right) = \vec{r}.curl\ \vec{A}$$

If $\nabla\times A = 0$ this reduces to zero.

Statement

Prove that

$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla.\left(\nabla.\vec{A}\right) - \nabla^2 \vec{A}$

Proof

Let $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$ be a vector point function

Then

L.H.S=$\nabla \times (\nabla \times \vec{A})$

Since

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

Hence $\nabla \times (\nabla \times \vec{A}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)\right]\hat{\imath} - \left[\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\right]\hat{\jmath}$$
$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\right]\hat{k}$$

$$= \left(\frac{\partial^2 A_2}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_3}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z^2}\right)\hat{\imath} + \left(-\frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_1}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial z \partial y} - \frac{\partial^2 A_2}{\partial z^2}\right)\hat{\jmath}$$
$$+ \left(\frac{\partial^2 A_1}{\partial x \partial z} - \frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} + \frac{\partial^2 A_2}{\partial y \partial z}\right)\hat{k}$$

$$= \left(-\frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right)\hat{\imath} + \left(-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial z^2}\right)\hat{\jmath} + \left(\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2}\right)\hat{k} + \left(\frac{\partial^2 A_2}{\partial y \partial x} + \frac{\partial^2 A_3}{\partial z \partial x}\right)\hat{\imath}$$
$$+ \left(\frac{\partial^2 A_1}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial z \partial y}\right)\hat{\jmath} + \left(\frac{\partial^2 A_1}{\partial x \partial z} + \frac{\partial^2 A_2}{\partial y \partial z}\right)\hat{k}$$

$$= \left(-\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right)\hat{\imath} + \left(-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2}\right)\hat{\jmath} + \left(\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_3}{\partial z^2}\right)\hat{k}$$

$$+ \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_2}{\partial y \partial x} + \frac{\partial^2 A_3}{\partial z \partial x}\right)\hat{\imath} + \left(\frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_1}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial z \partial y}\right)\hat{\jmath}$$

$$+ \left(\frac{\partial^2 A_3}{\partial z^2} + \frac{\partial^2 A_1}{\partial x \partial z} + \frac{\partial^2 A_2}{\partial y \partial z}\right)\hat{k}$$

$$= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}\right) + \frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$+ \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= -\nabla^2\vec{A} + \nabla.(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z})$$

$$= -\nabla^2\vec{A} + \nabla.\left(\nabla.\vec{A}\right) = \nabla.\left(\nabla.\vec{A}\right) - \nabla^2\vec{A}$$

Hence the result.

Problem Statement

Find $curl\ (\vec{r}f(r))$ where $f(r)$ is differentiable.

Solution

Let $\vec{r} = xi + yj + zk$ be the given vector

then

$$curl\left(\vec{r}f(r)\right) = \nabla \times \left(\vec{r}f(r)\right)$$

$$= \nabla \times \left(xf(r)\hat{\imath} + yf(r)\hat{\jmath} + zf(r)\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}zf(r) - \frac{\partial}{\partial z}yf(r)\right]\hat{\imath} + \left[\frac{\partial}{\partial z}xf(r) - \frac{\partial}{\partial x}zf(r)\right]\hat{\jmath} + \left[\frac{\partial}{\partial x}yf(r) - \frac{\partial}{\partial y}xf(r)\right]\hat{k}$$

$$= \left[z\frac{\partial f(r)}{\partial y} - y\frac{\partial f(r)}{\partial z}\right]i + \left[x\frac{\partial f(r)}{\partial z} - z\frac{\partial f(r)}{\partial x}\right]\hat{\jmath}$$

$$+ \left[y\frac{\partial f(r)}{\partial x} - x\frac{\partial f(r)}{\partial y}\right]\hat{k} \qquad (1)$$

As we know $r = \sqrt{x^2 + y^2 + z^2}$, therefore

$$\frac{\partial f(r)}{\partial x} = \frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial f(r)}{\partial r}\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = f'(r)\frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{f'(r)x}{r}$$

Similarly,

$$\frac{\partial f(r)}{\partial y} = \frac{f'(r)y}{r}$$

And

$$\frac{\partial f(r)}{\partial z} = \frac{f'(r)z}{r}$$

By substituting these values in equation (1), we obtain

$$= \left[z\frac{f'(r)y}{r} - y\frac{f'(r)z}{r}\right]i + \left[x\frac{f'(r)z}{r} - z\frac{f'(r)x}{r}\right]\hat{j} + \left[y\frac{f'(r)x}{r} - x\frac{f'(r)y}{r}\right]\hat{k} = 0$$

Hence the result.

Module No. 30

Line Integral

An integral where the function is evaluated along a curve is called line integral. It is also named as path integral, curve integral, and curvilinear integral; contour integral as well. The function to be integrated may be a scalar field or a vector field. The value of the line integral is the sum of values of the field at all points on the curve, weighted by some scalar function on the curve. A line integral is a natural generalization of definite integral. Line integral can be transformed into double integral and surface integrals and vice versa.

The symbolic form of line integral is

$$\int_C \vec{A}.d\vec{r}$$

Where $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$ is called the differential displacement vector. The integrals that involves differential displacement vector $d\vec{r}$ care called line integrals.

Line integral can also be expressed as

$$\int_C \vec{A}.d\vec{r} = \int_C (A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}).(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k})$$

$$= \int_C (A_1dx + A_2dy + A_3dz)$$

The line integral $\int_C \vec{A}.d\vec{r}$ is sometimes called a scalar line integral of a vector field $\vec{A}$.

If C is a closed curve which we shall assume a simple closed curve (a curve which does not intersect itself anywhere), the line integral around C is denoted by

$$\oint_C \vec{A}.d\vec{r} = \oint_C (A_1dx + A_2dy + A_3dz)$$

Applications

Some applications of line integrals are:

i. If the vector field to be integrated $\vec{A}$ is the force $\vec{F}$ on a particle move along C, then this line integral show the work done by the force.

ii. In fluid mechanics, if the vector field to be integrated $\vec{A}$ represents the velocity of some fluid then the line integral is called the circulation of $\vec{A}$ about C.

iii. In general, we can say that any integral which is to be calculated along a curve is called a line integral.

Module No. 31

Other forms and General Properties of line Integrals

Other forms of Line integrals

Generally, we define line integral by the vector function $\vec{A}$ by

$$\int_C \vec{A}.d\vec{r}$$

As we studied previously, that the function to be integrated can be either scalar point function or vector point function.

Here, we will define line integral using scalar function,

$$\int_C \varphi d\vec{r} = \int_C \varphi\left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right)$$

$$= \int_C \left(\varphi dx\hat{\imath} + \varphi dy\hat{\jmath} + \varphi dz\hat{k}\right)$$

$$\int_C \varphi.d\vec{r} = \hat{\imath}\int_C \varphi dx + \hat{\jmath}\int_C \varphi dy + \hat{k}\int_C \varphi dz$$

Also the general line integral indicates the dot product (".") between the given vector field and the differential displacement vector. However we can also express it as a cross product (×) of both the vector according to our requirement.

$$\int_C \vec{A}\times d\vec{r} = \hat{\imath}\int_C (A_2dz - A_3dy) + \hat{\jmath}\int_C (A_3dx - A_1dz) + \hat{k}\int_C (A_1dy - A_2dx)$$

Thus any integral that involves differential displacement vector $d\vec{r}$ are called line integrals.

General Properties of Line Integral

The following are the properties of the line integrals that are useful in computational subjects and application

i. $\int_C K\vec{A}.d\vec{r} = K\int_C \vec{A}.d\vec{r}$ (K is any real constant)

ii. $\int_C (\vec{A}+\vec{B}).d\vec{r} = \int_C \vec{A}.d\vec{r} + \int_C \vec{B}.d\vec{r}$

iii. $\int_C \vec{A}.d\vec{r} = \int_{C_1} \vec{A}.d\vec{r} + \int_{C_2} \vec{A}.d\vec{r}$

Where the path C is subdivided into two arcs C_1 and C_2 that has the same orientations as C. If the sense of orientation along C is reversed, the value of the integral is multiplied by '-1'.

iv. If C is piece-wise smooth, consisting of smooth curves $C_1, C_2, \ldots \ldots \ldots .., C_n$ then the line integral of $\vec{A}$ over C is defined as the sum of the line integrals of $\vec{A}$ over each of the smooth curve making up C:

$$\int_C \vec{A}.d\vec{r} = \int_{C_1} \vec{A}.d\vec{r} + \int_{C_2} \vec{A}.d\vec{r} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots . + \int_{C_n} \vec{A}.d\vec{r}$$

In the sum, the orientation along C must be maintained over curves $C_1, C_2, \ldots \ldots \ldots .., C_n$. That is the initial point of C_j is the terminal point of C_{j-1}.

Module No. 32

Example of Line Integral

Problem Statement

If $\varphi = 2xyz^2, \vec{F} = xy\hat{\imath} - z\hat{\jmath} + x^2\hat{k}$ and C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0 \, to \, t = 1.$

Evaluate the line integrals

i. $\int_C \varphi . d\vec{r}$

ii. $\int_C F \times d\vec{r}$

Solution

i. $\int_C \boldsymbol{\varphi . d\vec{r}}$

By substituting the given values of x, y, z, we obtain

$$\varphi = 2xyz^2 = 2(t^2)(2t)(t^3)^2 = 4t^9$$

and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\Longrightarrow \vec{r} = t^2\hat{\imath} + 2t\hat{\jmath} + t^3\hat{k}$

Also $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$

$\Longrightarrow d\vec{r} = (2t\hat{\imath} + 2\hat{\jmath} + 3t^2\hat{k})dt$

As we studied the relation

$$\int_C \varphi . d\vec{r} = \hat{\imath}\int_C \varphi dx + \hat{\jmath}\int_C \varphi dy + \hat{k}\int_C \varphi dz$$

$$\Longrightarrow \int_C \varphi d\vec{r} = \int_0^1 4t^9(2t\hat{\imath} + 2\hat{\jmath} + 3t^2\hat{k})\,dt = \hat{\imath}\int_0^1 8t^{10}dt + \hat{\jmath}\int_0^1 8t^9 dt + \hat{k}\int_0^1 12t^{11}dt$$

$$\int_C \varphi d\vec{r} = \frac{8}{11}\hat{\imath} + \frac{8}{10}\hat{\jmath} + \hat{k}$$

Hence the required solution.

ii. $\int_C \boldsymbol{F \times d\vec{r}}$

Since $\vec{F} = xy\hat{\imath} - z\hat{\jmath} + x^2\hat{k}$, first of all we will calculate the value of F using parametric equations of x,y,z

$$\vec{F} = 2t^3\hat{\imath} - t^3\hat{\jmath} + t^4\hat{k}$$

Also we calculated, $d\vec{r} = \left(2t\hat{\imath} + 2\hat{\jmath} + 3t^2\hat{k}\right)dt$

Then

$$\vec{F} \times d\vec{r} = \left(2t^3\hat{\imath} - t^3\hat{\jmath} + t^4\hat{k}\right) \times \left(2t\hat{\imath} + 2\hat{\jmath} + 3t^2\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} = (-3t^5 - 2t^4)\hat{\imath} - (6t^5 - 2t^5)\hat{\jmath} + (4t^3 + 2t^4)\hat{k}$$

Using the expression

$$\int_C \vec{A} \times d\vec{r} = \hat{\imath} \int_C (A_2 dz - A_3 dy) + \hat{\jmath} \int_C (A_3 dx - A_1 dz) + \hat{k} \int_C (A_1 dy - A_2 dx)$$

Substituting values, we get

$$\int_C \vec{F} \times d\vec{r} = \hat{\imath} \int_0^1 (-3t^5 - 2t^4)dt - \hat{\jmath} \int_0^1 (6t^5 - 2t^5)dt + \hat{k} \int_0^1 (4t^3 + 2t^4)dt$$

$$= -\frac{9}{10}\hat{\imath} - \frac{2}{3}\hat{\jmath} + \frac{7}{5}\hat{k}$$

Module No. 33

Example of Line Integral

Problem statement

Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\hat{\imath} - 5z\hat{\jmath} + 10x\hat{k}$ along the curve $x = t^2 + 1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$.

Solution

Since $\vec{F} = 3xy\hat{\imath} - 5z\hat{\jmath} + 10x\hat{k}$,

Substituting values $x = t^2 + 1, y = 2t^2, z = t^3$, in $\vec{F}$.

$$\vec{F} = 3(t^2 + 1)(2t^2)\hat{\imath} - 5(t^3)\hat{\jmath} + 10(t^2 + 1)\hat{k}$$

$$\vec{F} = (6t^4 + 6t^2)\hat{\imath} - 5(t^3)\hat{\jmath} + (10t^2 + 10)\hat{k}$$

Also, $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$

$\Rightarrow d\vec{r} = \left(2t\hat{\imath} + 4t\hat{\jmath} + 3t^2\hat{k}\right)dt$

Total work done $= \int_C \vec{F}.d\vec{r} =$

$$= \int_1^2 [(6t^4 + 6t^2)\hat{\imath} - 5(t^3)\hat{\jmath} + (10t^2 + 10)\hat{k}].\left(2t\hat{\imath} + 4t\hat{\jmath} + 3t^2\hat{k}\right)dt$$

$$= \int_1^2 2t(6t^4 + 6t^2) - 4t(5t^3) + 3t^2(10t^2 + 10)dt$$

$$= \int_1^2 (12t^5 + 12t^3) - (20t^4) + (30t^4 + 30t^2)dt$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)dt$$

$$= \int_1^2 12t^5dt + \int_1^2 10t^4dt + \int_1^2 12t^3dt + \int_1^2 30t^2dt$$

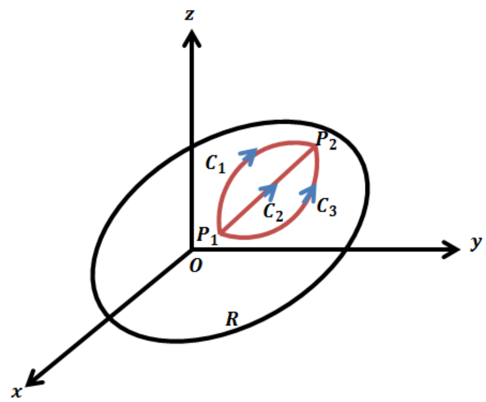
$$= \left|\frac{12t^6}{6}\right|_1^2 + \left|\frac{10t^5}{5}\right|_1^2 + \left|\frac{12t^4}{4}\right|_1^2 + \left|\frac{30t^3}{3}\right|_1^2$$

$$= 2[(2)^6 - (1)^6] + 2[(2)^5 - (1)^5] + 3[(2)^4 - (1)^4] + 10[(2)^3 - (1)^3]$$
$$= 303$$

Module No. 34

Line Integral Dependent on Path

The value of line integrals $\int_C \vec{A}.d\vec{r}$ generally depends not only on the end points P_1 and P_2 of the curve C but also the geometric shape of the path C i.e. if we integrate form point P_1 to P_2 along different paths, we in general, obtain different values of the integral.



We will illustrate this concept by the following example

If $A = (3x^2 + 6y)\hat{\imath} - 14yz\hat{\jmath} + 20xz^2\hat{k}$, evaluate $\int_C \vec{A}.d\vec{r}$ from (0,0,0) to (1,1,1) along the following paths C

i. The straight lines from (0,0,0) to (1,0,0), then to (1,1,0), and then to (1,1,1).

ii. The straight line joining (0,0,0) and (1,1,1).

Solution

i. Along the straight line from (0,0,0)to (1,0,0)$y = 0, z = 0, dy = 0, dz = 0$ while x varies from 0 to 1. Then

the integral over this part of the path is

$$\int_C \vec{A}.d\vec{r} = \int_C (A_1dx + A_2dy + A_3dz)$$

$$= \int_0^1 3x^2 + 6(0)\, dx - 14y(0)dy + 20x(0)^2 = \int_0^1 3x^2\, dx = 1$$

Along the straight line from (1,0,0) to (1,1,0) $x - 1, z = 0, dx = 0, dz = 0$ while y varies from 0 to 1.

Then the integral over this part of the path is

$$\int_0^1 3(1)^2 + 6y\,dx - 14y(0)dy + 20x(0)^2 = \int_0^1 3 + 6y\,dx = 0$$

Along the straight line from (1,1,0) to (1,1,1) $x = 1, y = 1, dx = 0, dy = 0$ while z varies from 0 to 1.

Then the integral over this part of the path is

$$\int_0^1 3(1)^2 + 6(1)\,(0) - 14(1)z(0) + 20(1)z^2dz = \int_0^1 20z^2dz = \frac{20}{3}$$

By adding all the results, we have

$$\int_0^1 \vec{A}.d\vec{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$$

ii. The straight line joining (0,0,0) and (1,1,1) is given in parametric form by $x = t, y = t, z = t$. Then

$$\int_C \vec{A}.d\vec{r} = \int_C (3x^2 + 6y)\hat{\imath} - 14yz\hat{\jmath} + 20xz^2\hat{k}.d\vec{r}$$

$$= \int_0^1 (3t^2 + 6t) - 14t^2 + 20t^3\,dt = \int_0^1 6t - 11t^2 + 20t^3\,dt = \frac{13}{3}$$

In example, we discuss two cases here. In each case the end points of the curve were same but we chose different path to obtain results. The results obtained by both the cases are different. Hence we conclude that the line integral not only depends on the end points of the curve but also on the path taken by the particle.

Module No. 35

Independence of Path

Definition

The line integral $\int_C \vec{V}.d\vec{r}$ is said to be independent of the path C in a given region R, if the value of the line integral $\int_{P_1}^{P_2} \vec{V}.d\vec{r}$ is the same for all paths C joining any two points P_1 and P_2 in R.

Theorem

Prove that a necessary and sufficient condition for $\int_{P_1}^{P_2} \vec{V}.d\vec{r}$ to be independent of the path joining any two points P_1 and P_2 (i.e. $\vec{V}$ to be conservative) in a given region is that $\oint_C \vec{V}.d\vec{r} = 0$ for all closed path C in the region.

Proof

Let C be any simple closed curve, and let P_1 and P_2 be any two points on C as shown.

Then since by the supposition, the integral is independent of path (i.e. $\vec{V}$ to be conservative), we have

$$\int_{P_1 A\, P_2} \vec{V}.d\vec{r} = \int_{P_1 B P_2} \vec{V}.d\vec{r}$$

Reversing the direction of integration in the integral on the right, we have

$$\int_{P_1 A\, P_2} \vec{V}.d\vec{r} = -\int_{P_2 B P_1} \vec{V}.d\vec{r}$$

or

$$\int_{P_1 A\, P_2} \vec{V}.d\vec{r} + \int_{P_2 B P_1} \vec{V}.d\vec{r} = 0$$

or

$$\oint_C \vec{V}.d\vec{r} = 0$$

Conversely, if $\oint_C \vec{V}.d\vec{r} = 0$,

then

$$\int_{P_1 A\, P_2} \vec{V}.\,d\vec{r} + \int_{P_2 B P_1} \vec{V}.\,d\vec{r} = 0$$

$$\int_{P_1 A\, P_2} \vec{V}.\,d\vec{r} = -\int_{P_2 B P_1} \vec{V}.\,d\vec{r}$$

Again changing the direction in the integral, we have

$$\int_{P_1 A\, P_2} \vec{V}.\,d\vec{r} = \int_{P_1 B P_2} \vec{V}.\,d\vec{r}$$

Which shows the line integral is independent of the path joining P_1 and P_2 as required.

Module No. 36

Theorems on line Integral

Scalar Potential Function

A scalar potential function φ is a single-valued function for which there exists a continuous vector field $\vec{V}$ in a simply connected region R that satisfies the relation $\vec{V} = \nabla$□ .

Theorem 1 Statement

Prove that a necessary and sufficient condition for $\int_{P_1}^{P_2} \vec{V}.d\vec{r}$ to be independent of the path joining any two points $P_1(x, y, z)$ and $P_2(x, y, z)$ (i.e. $\vec{V}$ to be conservative) is that there exist a scalar function φ such that $\vec{V} = \nabla\varphi$, where φ is single valued and has continuous partial derivatives.

Proof

Let $\vec{V} = \nabla\varphi$, then

$$\int_C \vec{V}.d\vec{r} = \int_{P_1}^{P_2} \vec{V}.d\vec{r} = \int_{P_1}^{P_2} \nabla\varphi.d\vec{r}$$

$$= \int_{P_1}^{P_2} \left(\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}\right).(dxi + dyj + dzk)$$

$$= \int_{P_1}^{P_2} \left(\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz\right)$$

$$= \int_{P_1}^{P_2} d\varphi = \varphi(P_2) - \varphi(P_1) = \varphi(x_2, y_2, z_2) - \varphi(x_1, y_1, z_1)$$

Thus the line integrals only depends only on points P_1 and P_2 and not on the path joining them i.e. $\vec{V}$ is conservative.

Conversely, let $\int_C \vec{V}.d\vec{r}$ be the independent of the path C joining any two points, We choose these points as a fixed point $P_1 \equiv (x_1, y_1, z_1)$ and a variable point $P_2 \equiv (x, y, z)$, so that the result is a function of only of the coordinates (x, y, z) of the variable end points. Then

$$\varphi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \vec{V}.d\vec{r} = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \vec{V}.\frac{d\vec{r}}{ds}ds$$

By taking derivative, we have

$$\frac{d\varphi}{ds} = \vec{V}.\frac{d\vec{r}}{ds} \qquad (1)$$

But

$$\frac{d\varphi}{ds} = \frac{\partial\varphi}{\partial s} = \nabla\varphi.\frac{d\vec{r}}{ds} \qquad (2)$$

From (1) and (2) we have

$$\vec{V}.\frac{d\vec{r}}{ds} = \nabla\varphi.\frac{d\vec{r}}{ds} = \left(\vec{V} - \nabla\varphi\right).\frac{d\vec{r}}{ds} = 0$$

Since $\frac{d\vec{r}}{ds}$ is a unit tangent vector, therefore $\frac{d\vec{r}}{ds} \neq 0$

$$\Longrightarrow \vec{V} - \nabla\varphi = 0$$

Or

$$\vec{V} = \nabla\varphi$$

Hence Proved.

Theorem 2 Statement

Prove that a necessary and sufficient condition that a vector field $\vec{V}$ be conservative is that $\nabla \times \vec{V} = 0$(i.e. $\vec{V}$ is irrotational).

Proof

If $\vec{V}$ is conservative field then by the previous theorem, we have $\vec{V} = \nabla\varphi$.

Thus

$$\nabla \times \vec{V} = \nabla \times \nabla\varphi = 0$$

Conversely, if $\nabla \times \vec{V} = 0$, then

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = 0$$

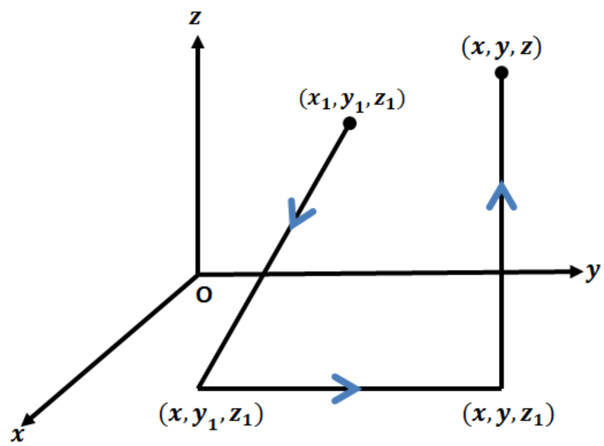
and thus

$$\frac{\partial V_3}{\partial y} = \frac{\partial V_2}{\partial z}, \frac{\partial V_1}{\partial z} = \frac{\partial V_3}{\partial x}, \frac{\partial V_2}{\partial x} = \frac{\partial V_1}{\partial y}$$

we must prove that $\vec{V} = \nabla\varphi$ follows as a consequence of this,

Now,

$$\int_C \vec{V}.d\vec{r} = \int_C (V_1(x,y,z)dx + V_2(x,y,z)dy + V_3(x,y,z)dz)$$



where C is path joining (x_1, y_1, z_1) and (x_2, y_2, z_3). Let us choose as a particular path, the straight lone segment from (x_1, y_1, z_1) to (x, y_1, z_1) to (x, y, z) and $\varphi(x, y, z)$ the value of the integral along this particular path. Then omitting the integrand, we have

$$\varphi(x,y,z) = \int_{(x_1,y_1,z_1)}^{(x,y_1,z_1)} [\quad] + \int_{(x,y_1,z_1)}^{(x,y,z_1)} [\quad] + \int_{(x,y,z_1)}^{(x,y,z)} [\quad] \qquad (1)$$

i. Along the straight line (x_1, y_1, z_1) to (x, y_1, z_1) , $y = \text{constant} = y_1, z = \text{constant} = z_1$, so that $dy = 0, dz = 0$, while x varies from x_1 to x.

ii. Along the straight line (x, y_1, z_1) to (x, y, z_1) , $x = \text{constant}$, $z = \text{constant}$, so that $dx = 0, dz = 0$, while y varies from $\square_1$ to y.

iii. Along the straight line (x, y, z_1) to (x, y, z) , $x = \text{constant}$, $y = \text{constant}$, so that $dx = 0, dy = 0$, while y varies from z_1 to z.

Thus we can write equation (1) as

$$\varphi(x,y,z) = \int_{x_1}^{x} V_1(x,y_1,z_1)dx + \int_{y_1}^{y} V_2(x,y,z_1)dy + \int_{z_1}^{z} V_3(x,y,z)dz$$

it follows that

$$\frac{\partial \varphi}{\partial z} = V_3(x, y, z)$$

$$\frac{\partial \varphi}{\partial y} = V_2(x, y, z_1) + \int_{z_1}^{z} \frac{\partial V_3}{\partial y}(x, y, z)dz$$

$$= V_2(x, y, z_1) + \int_{z_1}^{z} \frac{\partial V_2}{\partial z}(x, y, z)dz = V_2(x, y, z_1) + |V_2(x, y, z)|_{z_1}^{z} = V_2(x, y, z)$$

$$\frac{\partial \varphi}{\partial x} = V_1(x, y_1, z_1) + \int_{y_1}^{y} \frac{\partial V_2}{\partial x}(x, y, z_1)dy + \int_{z_1}^{z} \frac{\partial V_3}{\partial x}(x, y, z)dz$$

$$= V_1(x, y_1, z_1) + \int_{y_1}^{y} \frac{\partial V_1}{\partial y}(x, y, z_1)dy + \int_{z_1}^{z} \frac{\partial V_1}{\partial z}(x, y, z)dz$$

$$= V_1(x, y_1, z_1) + |V_1(x, y, z_1)|_{y_1}^{y} + |V_1(x, y, z_1)|_{z_1}^{z} = V_2(x, y, z)$$

$$= V_1(x, y_1, z_1) + V_1(x, y, z_1) - V_1(x, y_1, z_1) + V_1(x, y, z) - V_1(x, y, z_1)$$

$$\frac{\partial \varphi}{\partial x} = V_1(x, y, z)$$

Then we have,

$$\vec{V} = V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k} = \frac{\partial \varphi}{\partial x}\hat{\imath} + \frac{\partial \varphi}{\partial y}\hat{\jmath} + \frac{\partial \varphi}{\partial z}\hat{k} = \nabla\varphi$$

hence $\vec{V}$ is conservative i.e $\vec{V} = \nabla\varphi$.

Module No. 37

Selected Example/Problem 1

Problem Statement

If $\vec{A}(t) = t\hat{\imath} - t^2\hat{\jmath} + (t-1)\hat{k}$ and $\vec{B}(t) = 2t^2\hat{\imath} + 6t\hat{k}$,

Evaluate

i. $\int_0^2 \vec{A}.\vec{B}dt$

ii. $\int_0^2 \vec{A} \times \vec{B}dt$

Solution

i. $\int_0^2 \vec{A}.\vec{B}dt$

Since $\vec{A}(t) = t\hat{\imath} - t^2\hat{\jmath} + (t-1)k$ and $\vec{B}(t) = 2t^2\hat{\imath} + 6t\hat{k}$

So,

$$\vec{A}.\vec{B} = (t\hat{\imath} - t^2\hat{\jmath} + (t-1)k).\left(2t^2\hat{\imath} + 0\hat{\jmath} + 6t\hat{k}\right)$$

$$= 2t^3 + 6t(t-1)$$

Now

$$\int_0^2 \vec{A}.\vec{B}dt = \int_0^2 [2t^3 + 6t(t-1)]dt$$

$$= \int_0^2 2t^3dt + \int_0^2 6t^2dt - \int_0^2 6tdt$$

$$= \left|\frac{t^4}{2} + 2t^3 - 3t^2\right|_0^2$$

$$= 8 + 16 - 12 = 12$$

Hence

$$\int_0^2 \vec{A}.\vec{B}dt = 12$$

is required solution.

ii. $\int_0^2 \vec{A} \times \vec{B} dt$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ t & -t^2 & t-1 \\ 2t^2 & 0 & 6t \end{vmatrix}$$

$$= [(-t^2)(6t) - (t-1)(0)]\hat{\imath} - [(t)(6t) - (t-1)(2t^2)]\hat{\jmath} + [(t)(0) - (-t^2)(2t^2)]\hat{k}$$

$$= -6t^3\hat{\imath} - (6t^2 + 2t^2 - 2t^3)\hat{\jmath} + 2t^4 k$$

$$= -6t^3\hat{\imath} - (8t^2 - 2t^3)\hat{\jmath} + 2t^4 k$$

Now,

$$\int_0^2 \vec{A} \times \vec{B} dt = \int_0^2 (-6t^3\hat{\imath} - (8t^2 - 2t^3)\hat{\jmath} + 2t^4\hat{k}) dt$$

$$= -\hat{\imath}\int_0^2 6t^3 dt - \hat{\jmath}\int_0^2 (8t^2 - 2t^3) dt + \hat{k}\int_0^2 2t^4 dt$$

$$= \left| -\hat{\imath}\frac{3}{2}t^4 - \hat{\jmath}\left(\frac{8}{3}t^3 - \frac{t^4}{2}\right) + \hat{k}\frac{2}{5}t^5 \right|_0^2$$

$$= -\hat{\imath}\frac{3(16)}{2} - \hat{\jmath}\left(\frac{8(8)}{3} - \frac{16}{2}\right) + \hat{k}\frac{2(32)}{5}$$

$$= -\hat{\imath}24 - \hat{\jmath}\frac{80}{6} + \hat{k}\frac{64}{5}$$

$$\int_0^2 \vec{A} \times \vec{B} dt = -24\hat{\imath} - \frac{40}{3}\hat{\jmath} + \frac{64}{5}\hat{k}$$

is the required result.

Module No. 38

Selected Example/Problem 2: Line Integrals

Problem Statement

If $F = 3xy\hat{\imath} — y^2\hat{\jmath}$, evaluate work done by the force on the curve C in the xy plane, y $= 2x^2$, from (0,0) to (1,2).

Solution

As we know total work done on a curve is

$$\int_C \vec{F}\,.\,d\vec{r}$$

Also,

$$d\vec{r} = dx\hat{\imath} + dy\hat{\jmath}$$

By substituting

$$= \int_0^1 (3xy\hat{\imath} — y^2\hat{\jmath})\,.\,(dx\hat{\imath} + dy\hat{\jmath})$$

$$= \int_0^1 (3xy\,dx — y^2 dy)$$

As y $= 2x^2$ is given, this implies $dy = 4xdx$, now substituting these values in above integral, we obtain

$$= \int_0^1 (3x(2x^2)dx — (2x^2)^2 4xdx)$$

$$= \int_0^1 6x^3 dx - \int_0^1 16x^5 dx$$

$$= \left|\frac{3}{2}x^4 - \frac{8}{3}x^6\right|_0^1$$

$$= \frac{3}{2} - \frac{8}{3} = \frac{-7}{6}$$

Note that if the curve were traversed in the opposite sense, i.e. from (1,2) to (0,0), the value of the integral would have been 7/6 instead of — 7/6.

Module No. 39

Selected Example/Problem 3: Line Integrals

Problem Statement

Find the work done in moving a particle once around a circle C in the xy plane, if the circle has center at the origin and radius 3 and if the force field is given by

$$\vec{F} = (2x - y + z)\hat{\imath} + (x + y - z^2)\hat{\jmath} + (3x - 2y + 4z)\hat{k}$$

Solution

In the plane $z = 0$, $\vec{F} = (2x - y)\hat{\imath} + (x + y)\hat{\jmath}$ and $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath}$

So, that the work done on the curve is

$$\text{Total work done} = \int_C \vec{F}.d\vec{r}$$

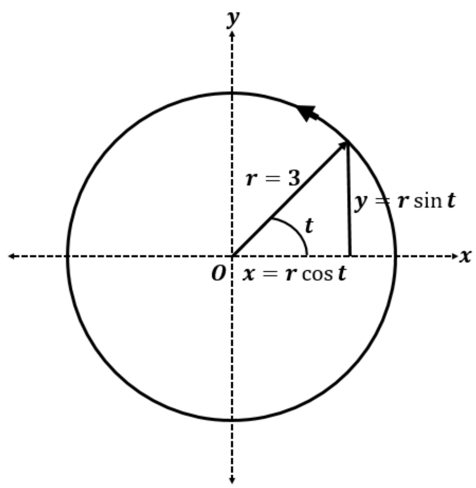
$$= \int_C ((2x - y)\hat{\imath} + (x + y)\hat{\jmath}) . (dx\hat{\imath} + dy\hat{\jmath})$$

$$= \int_C ((2x - y)dx + (x + y)dy)$$

As radius of the circle is $r = 3$, we choose the parametric equations of the circle as

$$x = 3\cos t \implies dx = -3sint\ dt$$

$$y = 3sint \implies dy = 3\cos t\ dt$$



where t varies from 0 to 2π. Then the line integral

$$= \int_0^{2\pi} ((2(3\cos t) - 3sint)(-3sint)dt) + (3\cos t \ + 3sint\ )(3\cos t\ )dt$$

$$= \int_0^{2\pi} (6\cos t - 3sint)(-3sint) + 9(\cos t)^2 + 9sint\cos t \ dt$$

$$= \int_0^{2\pi} -18\cos t sint + 9\sin^2 t + 9\cos^2 t + 9sint\cos t \ dt$$

$$= \int_0^{2\pi} 9 - 9\cos t sint\, dt$$

$$= \left| 9t - \frac{9}{2}\sin^2 t \right|_0^{2\pi}$$

$$= 18\pi$$

In traversing C we have chosen the counterclockwise direction indicated in the adjoining figure. We call this the positive direction, or say that C has been traversed in the positive sense. If C were traversed in the clockwise (negative) direction the value of the integral would be —18π.

Module No. 40

Selected Example/Problem 4: Line Integral

Problem Statement

Given a force vector $\vec{F} = (2xy + z^3)\hat{\imath} + x^2\,\hat{\jmath} + 3xz^2\hat{k}$.

i. Show that $\vec{F}$ a conservative force field

ii. Find the scalar potential

iii. Find the work done in moving an object in this field from $(1,-2,1)\ to\ (3,1,4)$.

Solution

i. We derived a necessary and sufficient condition that a force will be conservative is that

$curl\vec{F} = \nabla \times \vec{F} = 0.$

Now

$$curl\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(x^2)\right)\hat{\imath} - \left(\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(2xy + z^3)\right)\hat{\jmath}$$

$$+ \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z^3)\right)\hat{k}$$

$$= 0\hat{\imath} - (3z^2 - 3z^2)\hat{\jmath} + (2x - 2x)\hat{k} = 0$$

$$curl\vec{F} = 0$$

Hence $\vec{F}$ is conservative.

ii. As $\vec{F}$ is conservative, so $\vec{F} = \nabla\varphi$

and

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k}$$

this implies=

$$\vec{F} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial z}\hat{k} = (2xy\ +\ z^3)\hat{\imath}\ +\ x^2\,\hat{\jmath}\ +\ 3xz^2\hat{k}$$

By comparing, we get

$$\frac{\partial\varphi}{\partial x} = 2xy\ +\ z^3 \qquad (1)$$

$$\frac{\partial\varphi}{\partial y} = x^2 \qquad (2)$$

$$\frac{\partial\varphi}{\partial z} = 3xz^2 \qquad (3)$$

By integrating equation (1), (2) and (3), we get

$$\varphi = x^2y + xz^3 + \mathrm{f}(y,z)$$

$$\varphi = x^2y + \mathrm{g}(x,z)$$

$$\varphi = xz^3 + \mathrm{h}(x,y)$$

These equation will agree if we choose

$\mathrm{f}(y,z) = 0$, $\mathrm{g}(x,z) = xz^3$ and $\mathrm{h}(x,y) = x^2y$

so that $\varphi = x^2y + xz^3$ to which may be added any constant.

iii. From part (ii), we have

$$\varphi = x^2y + xz^3 + K$$

Where K is any constant,

Then work done = $\varphi(P_2) - \varphi(P_1)$

$$= \varphi(x_2, y_2, z_2) - \varphi(x_1, y_1, z_1) \qquad (4)$$

$$\varphi(x_2, y_2, z_2) = \varphi(3,1,4) = (3)^2(1) + 3(4)^3 = 9 + 192 = 201$$

$$\varphi(x_1, y_1, z_1) = \varphi(1,-2,1) = (1)^2(-2) + 1(1)^3 = -2 + 1 = -1$$

Now substituting these values in equation (4)

Work done = $\varphi(P_2) - \varphi(P_1) = \varphi(x_2, y_2, z_2) - \varphi(x_1, y_1, z_1) = 201 - (-1) = 202$

202 is the required work done.

Module No. 41

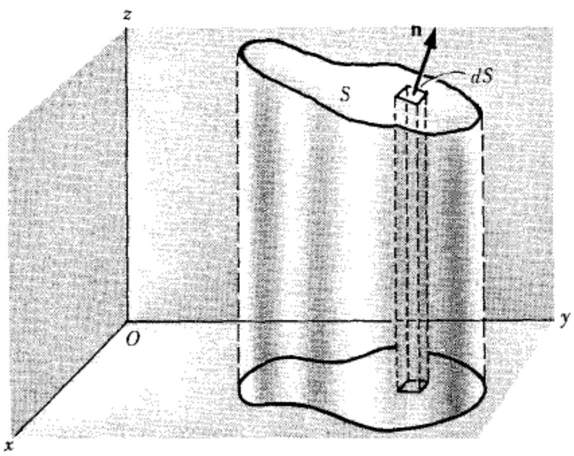
Surface Integral

Definition

Integrals that involves the differential surface elements $d\vec{S}$ are called surface integrals.
Mathematically, we can represents it as

$$\iint_S \vec{A}.d\vec{S}$$

Let S be a two-sided surface (open or closed), and $\vec{A}(x,y,z) = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$ be be a defined and continuous vector function over the surface S. Let one side of S be considered arbitrarily as the positive side (if S is a closed surface this is taken as the outer side). A unit normal $\hat{n}$ to any point of the positive side of S is called a positive or outward drawn unit normal.



[1]

Associate with the differential of surface area dS a vector $d\vec{S}$ whose magnitude is dS and whose direction is that of n. Then $d\vec{S} = \hat{n}\, dS$.

$$\iint_S \vec{A}.\hat{n}dS = \iint_S \vec{A}.d\vec{S}$$

The surface integral of a vector field $\vec{A}$ is called the flux of $\vec{A}$ through S.
The other forms of surface integrals are

$$\iint_S \varphi\, d\vec{S} \text{ and } \iint_S \vec{A} \times d\vec{S}$$

Where φ is a scalar point function.

If S is a closed surface, then the surface integrals may be written as

$$\oint \vec{A}.d\vec{S},$$

$$\oint \vec{A} \times d\vec{S},$$

Or

$$\oint \varphi\, d\vec{S}$$

Properties of Surface Integrals

Like the double integrals, surface integrals have the following properties:

i. $\iint_S K\vec{A}.d\vec{S} = K\iint_S \vec{A}.d\vec{S}$ (K is any real constant)

ii. $\iint_S (\vec{A}+\vec{B}).d\vec{S} = \iint_S \vec{A}.d\vec{S} + \iint_S \vec{B}.d\vec{S}$

iii. $\iint_S \vec{A}.d\vec{S} = \iint_{S_1} \vec{A}.d\vec{S} + \iint_{S_2} \vec{A}.d\vec{S}$

Where the surface S is subdivided into two smooth surfaces S_1 and S_2 having atmost a curve in common.

iv. If the surface S is partitioned by the smooth curves into a finite number of non-overlapping smooth patches $S_1, S_2, \dots \dots \dots, S_n$ (i.e. is S is piece-wise smooth), then the normal surface integral of $\vec{A}$ over S is the sum of the normal surface integrals of $\vec{A}$ over all the smooth patches, i.e.

$$\iint_S \vec{A}.d\vec{S} = \iint_{S_1} \vec{A}.d\vec{S} + \iint_{S_2} \vec{A}.d\vec{S} + \cdots \dots \dots \dots \dots \dots \dots . + \iint_{S_n} \vec{A}.d\vec{S}$$

Module No. 42

Evaluation of the Surface Integral

To evaluate the surface integral, it is convenient to express them as double integrals taken over the projected area of the surface S on one of the coordinate planes.
Evaluation of surface integrals can be made by done by the following result

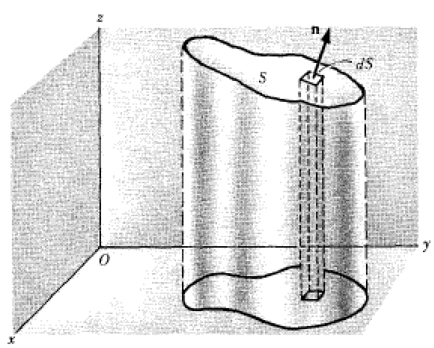
Theorem Statement

Let R ne the projection of the surface S on the xy-plane, then prove that

$$\iint_S \vec{A}.\hat{n}dS = \iint_R \vec{A}.\hat{n}\frac{dxdy}{|\hat{n}.\hat{k}|}$$

Proof

Let the surface S and its projection R on the xy-plane be as shown in figure.
Divide R into a rectangles of area $\Delta A_{k,}\ k = 1,2, \dots \dots, n$ and erect a vertical column on each of these sub-regions to intersects S in an element of surface area ΔS_k.



[1]

Choose a point (x_k, y_k, z_k) on each surface element ΔS_k and draw the unit normal $\hat{n}_k$ to this element at this point. Let γ_k be the acute angle between this unit normal $\hat{n}_k$ and the positive z-axis.
If this surface element is sufficiently small, it can be regarded as a plane.
We know from geometry, that if two planes intersect at an acute angle, an area in one plane may be projected into the other by multiplying the cosine of the included angle as shown in figure.
Since the angle between two planes is the angle between their normal, therefore

$$\Delta S_k \cos \gamma_k \approx \Delta A_k$$

or

$$\Delta S_k \approx \sec \gamma_k \, \Delta A_k \; = \frac{\Delta x_k \Delta y_k}{\left|\hat{n}.\hat{k}\right|}$$

Thus the sum (1) in the definition of unit normal surface integrals

$$\sum_{k=1}^{n} A_k.\hat{n}_k \Delta S_k \approx \sum_{k=1}^{n} A_k.\hat{n}_k \frac{\Delta x_k \Delta y_k}{\left|\hat{n}.\hat{k}\right|}$$

and the limit of this sum can be written as

$$\iint_S \vec{A}.\hat{n} dS = \iint_R \vec{A}.\hat{n} \frac{dxdy}{\left|\hat{n}.\hat{k}\right|}$$

Similarly, we can prove that if R is the projection of the surface S on the yz-plane, then

$$\iint_S \vec{A}.\hat{n} dS = \iint_R \vec{A}.\hat{n} \frac{dydz}{|\hat{n}.\hat{\imath}|}$$

And if R is the projection of S on the xz-plane, then

$$\iint_S \vec{A}.\hat{n} dS = \iint_R \vec{A}.\hat{n} \frac{dzdx}{|\hat{n}.\hat{\jmath}|}$$

Module No. 43

Example to the previous topic: Surface Integrals

Problem Statement

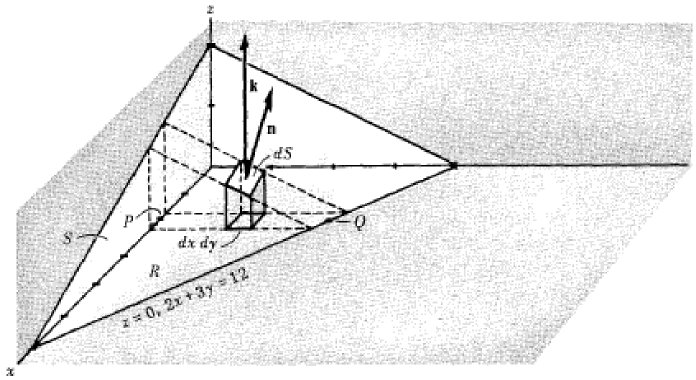
Evaluate

$\iint_S \vec{A}.\hat{n}dS$ where $A = 18z\hat{\imath} - 12\hat{\jmath} + 3y\hat{k}$ and S is that part of the plane

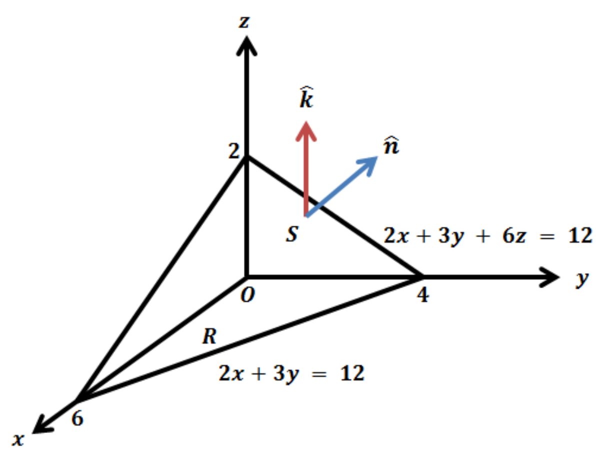
$2x + 3y + 6z = 12$ which is located in the first octant.

Solution

The surface S and its projection R on the xy plane are shown in the figure below.



[1]



We know that

$$\iint_{S} \vec{\boldsymbol{A}}.\hat{\boldsymbol{n}}\boldsymbol{dS} = \iint_{R} \vec{\boldsymbol{A}}.\hat{\boldsymbol{n}}\frac{\boldsymbol{dxdy}}{|\hat{\boldsymbol{n}}.\hat{\boldsymbol{k}}|}$$

Also we know that a normal vector to the surface $2x + 3y + 6z = 12$ is given by

$$\nabla(2x + 3y + 6z) = 2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$$

Then a unit normal $\hat{n}$ to any point of S is $\hat{n} = \frac{2\hat{\imath}+3\hat{\jmath}+6\hat{k}}{7} = \frac{2}{7}\hat{\imath} + \frac{3}{7}\hat{\jmath} + \frac{6}{7}\hat{k}$

Thus $\hat{n}.\hat{k} = \left(\frac{2}{7}\hat{\imath} + \frac{3}{7}\hat{\jmath} + \frac{6}{7}\hat{k}\right).\hat{k} = \frac{6}{7}$

and so

$$\frac{dxdy}{|\hat{n}.\hat{k}|} = \frac{7}{6}dxdy$$

Also

$$\vec{\boldsymbol{A}}.\hat{n} = (18z\hat{\imath} - 12\hat{\jmath} + 3y\hat{k}).\left(\frac{2}{7}\hat{\imath} + \frac{3}{7}\hat{\jmath} + \frac{6}{7}\hat{k}\right)$$

$$= \frac{36z - 36 + 18y}{7}$$

Now from the equation of the surface S $2x + 3y + 6z = 12$,

$$z = \frac{12 - 2x - 3y}{6}$$

Therefore

$$\vec{A}.\hat{n} = \frac{6(12 - 2x - 3y) - 36 + 18y}{7} = \frac{36 - 12x}{7}$$

Then

$$\iint_{S} \vec{A}.\hat{n}dS = \iint_{R} \vec{A}.\hat{n}\frac{dxdy}{|\hat{n}.\hat{k}|}$$

$$= \iint_{R} \frac{36 - 12x}{7}\frac{7}{6}dxdy$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (6 - 2x)dydx$$

$$= \int_{0}^{6}(6 - 2x)\left(\frac{12 - 2x}{3}\right)dx$$

$$= \int_0^6 \left(24 - 12x + \frac{4}{3}x^2\right) dx$$

$$= \left| 24x - 6x^2 + \frac{4}{9}x^3 \right|_0^6$$

$$= 144 - 216 + 96 = 24$$

If we had chosen the positive unit normal $\hat{n}$ opposite to that in the figure above, we would have obtained the result — 24 .

Module No. 44

Further Example on Surface Integral

Problem Statement

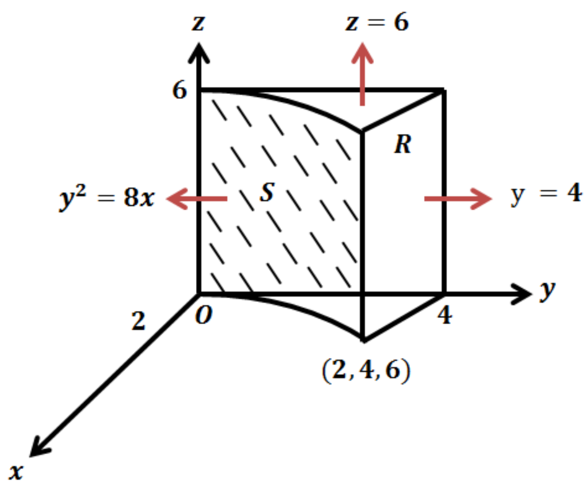
If $\vec{F} = 2y\hat{\imath} - z\hat{\jmath} + x^2\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$,

Evaluate

$$\iint_S \vec{F}.\hat{n}dS$$

Solution

The surface S and its projection R on the yz-plane are shown in figure.



[1]

A vector normal to S is

$$\nabla(8x - y^2) = 8\hat{\imath} - 2y\hat{\jmath}$$

and therefore

$$\hat{n} = \frac{8\hat{\imath} - 2y\hat{\jmath}}{\sqrt{8^2 + (2y)^2}} = \frac{8\hat{\imath} - 2y\hat{\jmath}}{\sqrt{64 + 4y^2}} = \frac{8\hat{\imath} - 2y\hat{\jmath}}{\sqrt{4(16 + y^2)}} = \frac{4\hat{\imath} - y\hat{\jmath}}{\sqrt{16 + y^2}} = \frac{4\hat{\imath}}{\sqrt{16 + y^2}} + \frac{-y\hat{\jmath}}{\sqrt{16 + y^2}}$$

Also,

$$\hat{n}.\hat{\imath} = \frac{4}{\sqrt{16 + y^2}}$$

and

$$\vec{F}.\hat{n} = (2y\hat{\imath} - z\hat{\jmath} \; + \; x^2\hat{k}).\left(\frac{4\hat{\imath}}{\sqrt{16+y^2}} + \frac{-y\hat{\jmath}}{\sqrt{16+y^2}}\right)$$

$$= \frac{8y + zy}{\sqrt{16+y^2}}$$

Thus

$$\iint_S \vec{A}.\hat{n}dS = \iint_R \vec{A}.\hat{n}\frac{dzdy}{|\hat{n}.\hat{\imath}|}$$

$$= \int_0^4 \int_0^6 \frac{8y + zy}{\sqrt{16+y^2}} \frac{\sqrt{16+y^2}}{4} dzdy$$

$$= \frac{1}{4}\int_0^4 \int_0^6 (8y + zy)\, dzdy = \frac{1}{4}\int_0^4 66ydy$$

$$= \left|\frac{33}{4}y^2\right|_0^4 = \frac{33}{4}4^2 = 132$$

$$\iint_S \vec{F}.\hat{n}dS = 132$$

is the required result.

Module No. 45

Selected Example/Problem 1: Surface Integrals

Problem Statement

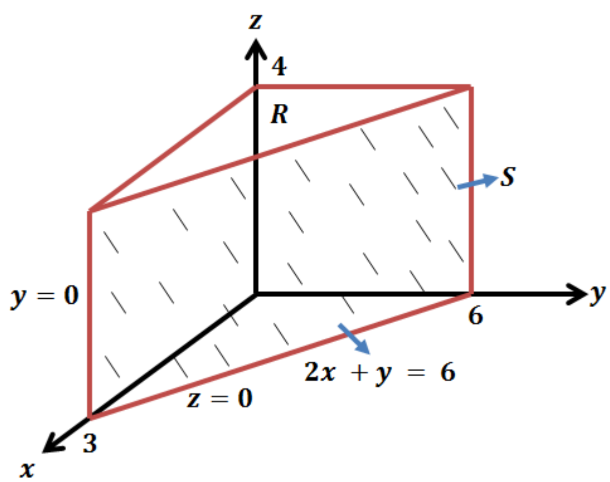
Evaluate

$$\iint_S \vec{A}.\hat{n}dS$$

for $\vec{A} = y\hat{\imath} + 2x\hat{\jmath} — z\hat{k}$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$

Solution

The Surface S and its projection R in the yz-plane are shown in figure.



[1]

A vector normal to S is given by

$$\nabla(2x + y) = 2\hat{\imath} + \hat{\jmath}$$

Therefore,

$$\hat{n} = \frac{2\hat{\imath} + \hat{\jmath}}{\sqrt{2^2 + (1)^2}} = \frac{2\hat{\imath} + \hat{\jmath}}{\sqrt{4 + 1}} = \frac{2\hat{\imath} + \hat{\jmath}}{\sqrt{5}}$$

Also,

$$\hat{n}.\hat{\imath} = \frac{2}{\sqrt{5}}$$

and

$$\vec{A}.\hat{n} = \left(y\hat{\imath} \ + \ 2x\hat{\jmath} - \ z\hat{k}\right).\left(\frac{2\hat{\imath}}{\sqrt{5}} + \frac{\hat{\jmath}}{\sqrt{5}}\right)$$

$$= \frac{2y}{\sqrt{5}} + \frac{2x}{\sqrt{5}} = \frac{2}{\sqrt{5}}(x + \square)$$

and

$$\iint_S \vec{A}.\hat{n}dS = \iint_R \vec{A}.\hat{n}\frac{dzdy}{|\hat{n}.\hat{\imath}|}$$

$$= \int_{y=0}^{6}\int_{z=0}^{4} \frac{2}{\sqrt{5}}(x+y)\frac{\sqrt{5}}{2}dzdy$$

$$2x \ + y \ = \ 6 \Longrightarrow x = \frac{6-y}{2}$$

$$= \int_{y=0}^{6}\int_{z=0}^{4} y + \left(\frac{6-y}{2}\right)dzdy = \int_{y=0}^{6}\int_{z=0}^{4}\left(3+\frac{y}{2}\right)dzdy$$

$$= \int_{y=0}^{6}\left(3+\frac{y}{2}\right)|z|_0^4 \, dy$$

$$= 4\int_{y=0}^{6}\left(3+\frac{y}{2}\right)dy$$

$$= \left|3y + \frac{y^2}{4}\right|_0^6$$

$$= 4(18+9) = 108$$

Module No. 46

Selected Example/Problem 2: Surface Integral

Problem Statement

Evaluate

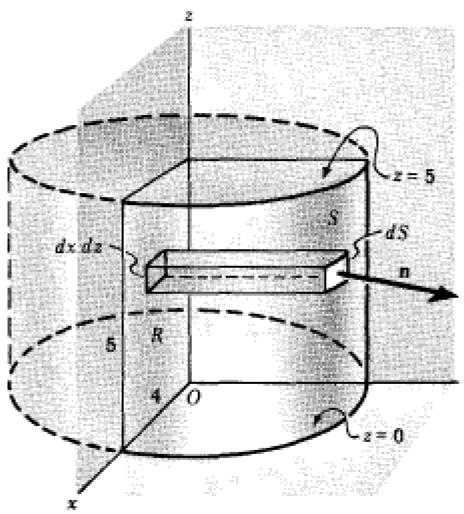
i. $\iint_S \vec{A}.\hat{n}dS$

ii. $\iint_S \varphi\, \hat{n}dS$

where $\vec{A} = z\hat{\imath} + x\hat{\jmath} — 3y^2z\hat{k}$ and $\varphi = \frac{3}{8}xyz$ S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Solution

i. The surface S and its projection R on the xz-plane are shown in figure.



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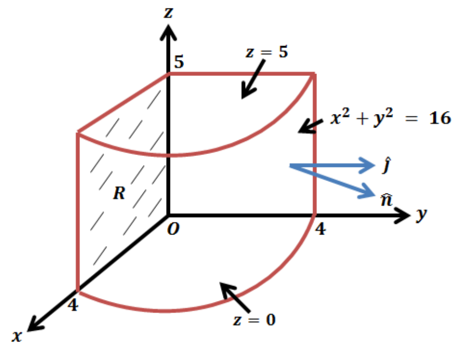


[1]

Then

$$\iint_S \vec{A}.\hat{n}dS = \iint_R \vec{A}.\hat{n}\frac{dxdz}{|\hat{n}.\hat{j}|} \quad (1)$$

A normal vector to $x^2 + y^2 = 16$ is $\nabla(x^2 + y^2) = 2x\hat{\imath} + 2y\hat{\jmath}$

Thus the unit normal $\hat{n}$ to S is

$$\hat{n} = \frac{2x\hat{\imath} + 2y\hat{\jmath}}{\sqrt{(2x)^2 + (2y)^2}} = \frac{2x\hat{\imath} + 2y\hat{\jmath}}{\sqrt{4x^2 + 4y^2}} = \frac{2x\hat{\imath} + 2y\hat{\jmath}}{\sqrt{4(x^2 + y^2)}} = \frac{2x\hat{\imath} + 2y\hat{\jmath}}{\sqrt{4(x^2 + y^2)}} = \frac{2x\hat{\imath} + 2y\hat{\jmath}}{\sqrt{4(16)}}$$

$$= \frac{2x\hat{\imath} + 2y\hat{\jmath}}{\sqrt{64}} = \frac{2(x\hat{\imath} + y\hat{\jmath})}{8} = \frac{(x\hat{\imath} + y\hat{\jmath})}{4}$$

Using $x^2 + y^2 = 16$ on S

Now,

$$\vec{A}.\hat{n} = (z\hat{\imath} + x\hat{\jmath} - 3y^2z\hat{k}).\left(\frac{x\hat{\imath} + y\hat{\jmath}}{4}\right)$$

$$= \frac{1}{4}(zx + xy)$$

And

$$\hat{n}.\hat{j} = \left(\frac{x\hat{\imath} + y\hat{\jmath}}{4}\right).\hat{\jmath} = \frac{y}{4}$$

Thus putting obtained values in equation (1)

$$\iint_S \vec{A}.\hat{n}dS = \iint_R \frac{1}{4}(zx + xy)\frac{4}{y}dxdz$$

$$= \int_{z=0}^{5}\int_{x=0}^{4} \frac{zx + xy}{y}dxdz$$

$$= \int_{z=0}^{5}\int_{x=0}^{4} \frac{zx}{y} + x\,dxdz \quad (2)$$

Utilizing the equation of the surface of the cylinder

$x^2 + y^2 = 16 \Longrightarrow y^2 = 16 - x^2 \Longrightarrow y = \sqrt{16 - x^2}$

Substituting the value of y in equation (2)

$$= \int_{z=0}^{5} \int_{x=0}^{4} \frac{zx}{\sqrt{16 - x^2}} + x\, dxdz$$

$$= \int_{0}^{5} \left| -z\sqrt{16 - x^2} + \frac{x^2}{2} \right|_0^4 dz$$

$$= \int_{0}^{5} (4z + 8)dz = |2z^2 + 8z|_0^5 = 90$$

ii. $\iint_S \varphi\, \hat{n} dS$

$$\iint_S \varphi . \hat{n} dS = \iint_R \varphi \hat{n} \frac{dxdz}{|\hat{n}.\hat{j}|}$$

Substituting values, we get

$$\iint_S \varphi . \hat{n} dS = \iint_R \left(\frac{3}{8} xyz\right) \left(\frac{x\hat{\imath} + y\hat{\jmath}}{4}\right) \frac{4}{y} dxdz$$

$$= \frac{3}{8} \int_{z=0}^{5} \int_{x=0}^{4} xz\,(x\hat{\imath} + y\hat{\jmath}) dxdz$$

Using the value of $y = \sqrt{16 - x^2}$ in above integral,

$$= \frac{3}{8} \int_{z=0}^{5} \int_{x=0}^{4} \left(x^2 z\hat{\imath} + xz\sqrt{16 - x^2}\hat{\jmath} \right) dxdz$$

$$= \int_{0}^{5} \left[z\hat{\imath} \left| \frac{x^3}{3} \right|_0^4 + z\hat{\jmath} \left| \frac{-1}{3} (16 - x^2)^{3/2} \right|_0^4 \right] dz$$

$$= \int_{0}^{5} \left(\frac{64}{3} z\hat{\imath} + \frac{64}{3} z\hat{\jmath} \right) dz = 8 \left| \frac{z^2}{2}\hat{\imath} + \frac{z^2}{2}\hat{\jmath} \right|_0^5$$

$$= 8\left(\frac{25}{2}\hat{\imath} + \frac{25}{2}\hat{\jmath} \right) = 100\hat{\imath} + 100\hat{\jmath}$$

is required solution.

Module No. 47

Selected Example/Problem 3: Surface Integral

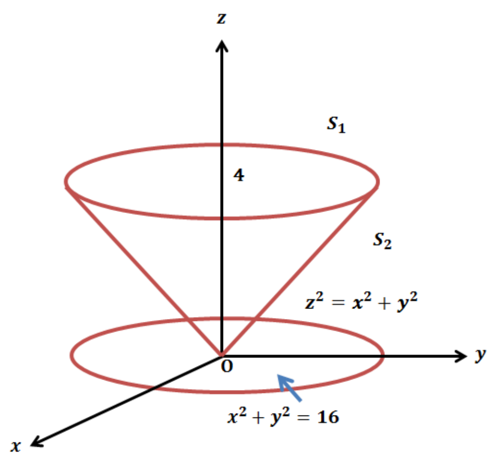
Problem Statement

Evaluate $\iint_S \vec{A}.\hat{n}dS$

over the entire surface of the region above the xy plane bounded by the cone S $z^2 = x^2 + y^2$ and the plane $z = 4$, if $\vec{A} = 4xz\,\hat{\imath} + xyz^2\hat{\jmath} + 3z\hat{k}$.

Solution

The surface S and its projection R on the xy-plane are shown in figure.



[1]

Then

$$\iint_S \vec{A}.\hat{n}dS = \iint_{S_1} \vec{A}.\hat{n}dS_1 + \iint_{S_2} \vec{A}.\hat{n}dS_2$$

For $S_1, z = 4, \hat{n} = \hat{k}$ and $\hat{n}.\hat{k} = 1$

Then

$$\iint_{S_1} \vec{A}.\hat{n}dS = \iint_{S_1} 12\, dS_1 = 12 \iint_R dxdy$$

$$= 12 \times \pi(4)^2 = 192\pi$$

For S_2, the normal vector is given by

$$\nabla(x^2+y^2-z^2) = 2x\hat{\imath} + 2y\hat{\jmath} - 2z\hat{k}$$

Therefore,

$$\hat{n} = \frac{2x\hat{\imath} + 2y\hat{\jmath} - 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2(x\hat{\imath} + y\hat{\jmath} - z\hat{k})}{2\sqrt{x^2+y^2+z^2}} = \frac{(x\hat{\imath} + y\hat{\jmath} - z\hat{k})}{\sqrt{z^2+z^2}} = \frac{(x\hat{\imath} + y\hat{\jmath} - z\hat{k})}{\sqrt{2z^2}}$$

$$\hat{n} = \frac{(x\hat{\imath} + y\hat{\jmath} - z\widehat{k)}}{z\sqrt{2}}$$

Also,

$$\vec{A}.\hat{n} = \frac{1}{\sqrt{2}}(4x^2 + xy^2z - 3z) \text{ and } \hat{n}.\hat{k} = \frac{-1}{\sqrt{2}}$$

Therefore

$$\iint_{S_2} \vec{A}.\hat{n}dS_2 = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \frac{1}{\sqrt{2}}(4x^2 + xy^2z - 3z).\sqrt{2}\,dydx$$

Using $z^2 = x^2 + y^2$ in the above integral, we obtain

$$= \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left(4x^2 + xy^2\sqrt{x^2+y^2} - 3\sqrt{x^2+y^2}\right) dydx$$

Now we will further solve it by using polar coordinates

Let $x = r\cos\theta, y = r\sin\theta$, therefore $0 \le r \le 4$ and $0 \le \theta \le 2\pi$

and

$$\iint_{S_2} \vec{A}.\hat{n}dS_2 = \int_0^{2\pi}\int_0^4 (4r^2\cos^2\theta + r^4\cos\theta\sin^2\theta - 3r)rdrd\theta\theta$$

$$= \int_0^{2\pi}\int_0^4 (4r^3\cos^2\theta + r^5\cos\theta\sin^2\theta - 3r^2)drd\theta$$

integerating w.r.t r

$$= \int_0^{2\pi} \left| r^4\cos^2\theta + \frac{r^6}{6}\cos\theta\sin^2\theta - r^3 \right|_0^4 d\theta$$

$$= \int_0^{2\pi} \left(256\cos^2\theta + \frac{4^6}{6}\cos\theta\sin^2\theta - 4^3\right) d\theta$$

$$= \int_0^{2\pi} \left(256\cos^2\theta + \frac{4^6}{6}\cos\theta\sin^2\theta - 4^3\right)d\theta$$

$$= \int_0^{2\pi} \left(128(1+\cos 2\theta) + \frac{2048}{3}\cos\theta\sin^2\theta - 64\right)d\theta$$

Now integrating w.r.t θ , we get

$$= \left|128\theta + 64\sin 2\theta + \frac{2048}{3}\sin^3\theta - 64\theta\right|_0^{2\pi}$$

$$\iint_{S_2} \vec{A}.\hat{n}dS_2 = 128\pi$$

Thus,

$$\iint_S \vec{A}.\hat{n}dS = \iint_{S_1} \vec{A}.\hat{n}dS_1 + \iint_{S_2} \vec{A}.\hat{n}dS_2$$

$$\iint_S \vec{A}.\hat{n}dS = 192\pi + 128\pi = 320\pi$$

is the required solution.

Module No. 48

Volume Integral

A volume integral refers to an integral over a 3-dimensional domain, that is, it is a special case of multiple integrals.

Definition

Let $\vec{A}$ be a given vector point function which is defined and continuous in a closed region R.Then

$$\iiint_R \vec{A} dV$$

If $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$, then the above integral may be written as

$$\iiint_R \vec{A} dV = \hat{\imath} \iiint_R A_1 dV + \hat{\jmath} \iiint_R A_2 dV + \hat{k} \iiint_R A_3 dV$$

If we have a scalar point function $\varphi(x, y, z)$ defined as continuous over the region R, then the volume integral becomes

$$\iiint_R \varphi dV$$

In rectangular coordinate system, $dV = dxdydz$ so the volume integral becomes

$$\iiint_R \varphi dV = \iiint_R \varphi(x, y, z) dxdydz$$

Which is ordinary triple integral of $\varphi(x, y, z)$ over the region R.

If $\varphi(x, y, z) = 1$, the volume V of the region R is given by

$$\iiint_R dxdydz = \iiint_R d\,V$$

Notations for other than Cartesian system

Volume integral can be expressed also in cylindrical and spherical coordinates as

i. Volume integral in cylindrical coordinates:

$$\iiint_R g(\rho, \varphi, z)\rho d\rho d\varphi dz$$

ii. Volume integral in cylindrical coordinates:

$$\iiint_R g(r, \theta, \varphi)r^2 \sin\varphi \, dr d\theta d\varphi$$

Which are equivalent to ordinary triple integrals in cylindrical and spherical coordinates.

Module No. 49

Example on Volume Integral

Problem Statement

Let $\vec{F} = 2xzi - xj + y^2k$.

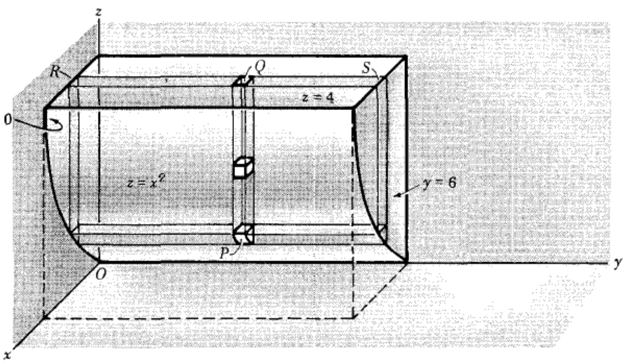
Evaluate

$$\iiint_R \vec{F}\, dV$$

where V is the region bounded by the surfaces $x = 0, y = 0, y = 6, z = x^2, z = 4$.

Solution

We will evaluate the given problem through following scheme.



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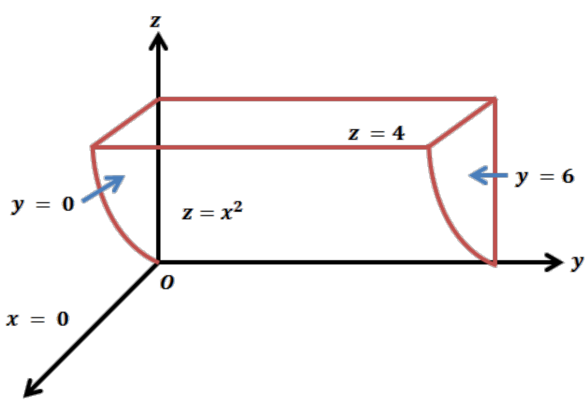
The region V is covered (i) by keeping x and y fixed and integrating from $z = x^2$ to $z = 4$ (base to top of column PQ),



[1]

(ii) Then by keeping χ fixed and integrating from y $= 0$ to y $= 6$ (R to S in the slab),

(iii) Finally integrating from $x = 0$ to $x = 2$ (where z = x2 meets $z = 4$). Then the required integral is

$$\iiint_R \vec{F}dV = \int_0^2 \int_0^6 \int_{x^2}^4 (2xzi - xj + y^2k)dzdydx$$

$$= \hat{\imath}\int_0^2 \int_0^6 \int_{x^2}^4 2xz\, dzdydx - \hat{\jmath}\int_0^2 \int_0^6 \int_{x^2}^4 xdzdydx + \hat{k}\int_0^2 \int_0^6 \int_{x^2}^4 y^2dzdydx$$

Solving integrals turn by turn

$$\int_0^2 \int_0^6 \int_{x^2}^4 2xz\, dzdydx = \int_0^2 \int_0^6 x(16 - x^4)dydx = \int_0^2 x(16 - x^4)|y|_0^6 dx = 6\int_0^2 x(16 - x^4)dx$$

$$= 6\int_0^2 (16x - x^5)dx = 6\left|\frac{16x^2}{2} - \frac{x^6}{6}\right|_0^2 = 6\left|\frac{64}{2} - \frac{64}{6}\right| = 128$$

$$\int_0^2 \int_0^6 \int_{x^2}^4 xdzdydx = \int_0^2 \int_0^6 x|z|_{x^2}^4 dydx = \int_0^2 \int_0^6 x(4 - x^2)dydx = \int_0^2 x(4 - x^2)|y|_0^6 dx$$

$$= 6\int_0^2 x(4 - x^2)dx = 6\left|2x^2 - \frac{x^4}{4}\right|_0^2 = 6\left|8 - \frac{16}{4}\right| = 24$$

$$\int_0^2 \int_0^6 \int_{x^2}^4 y^2dzdydx = \int_0^2 \int_0^6 y^2|z|_{x^2}^4 dydx = \int_0^2 \int_0^6 y^2(4 - x^2)dydx = \int_0^2 \left|\frac{y^3}{3}\right|_0^6 (4 - x^2)dx$$

$$= 72\int_0^2 (4 - x^2)dx = 72\left|4x - \frac{x^3}{3}\right|_0^2 = 72(8 - \frac{8}{3}) = 384$$

Hence

$$\iiint_R \vec{F}dV = \hat{\imath}\int_0^2 \int_0^6 \int_{x^2}^4 2xz\, dzdydx - \hat{\jmath}\int_0^2 \int_0^6 \int_{x^2}^4 xdzdydx + \hat{k}\int_0^2 \int_0^6 \int_{x^2}^4 y^2dzdydx$$

$$\iiint_R \vec{F}dV = 128\hat{\imath} - 24\hat{\jmath} + 384\hat{k}$$

is the required solution.

Module No. 50

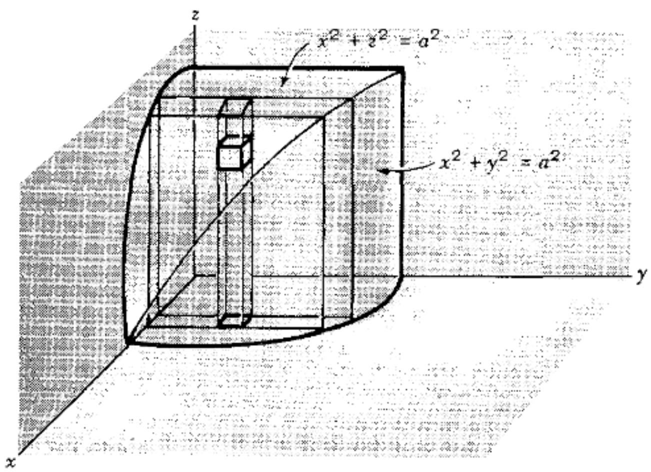
Selected Example/Problem 1: Volume Integral

Problem Statement

Find the volume of the region common to the intersecting cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Solution

Let M be the required common region then



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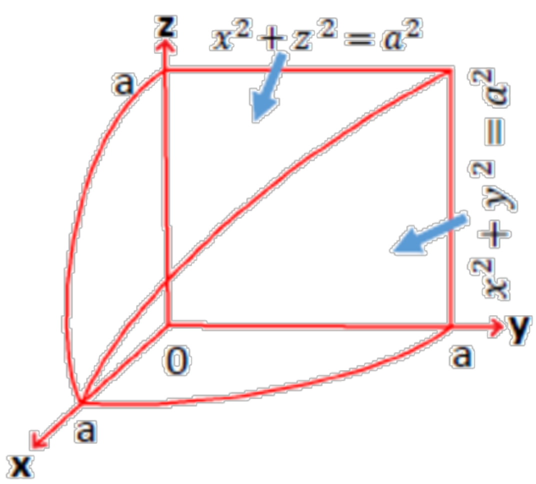

[1]

Required volume = 8 times volume of region shown in above figure.

$$\iiint_V M dV = 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2}} dzdydx$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} zdydx = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2} dydx = \int_{x=0}^{a} (a^2 - x^2) dx$$

$$= \frac{16a^3}{3}$$