MTH 641

Functional Analysis

MODULE NO. 1 To

(MID TERM SYLLABUS)

THESE ARE JUST SHORT HINT FOR THE PREPARATION OF MTH 641

Don't look for someone who can solve your problems,
Instead go and stand in front of the mirror,
Look straight into your eyes,
And you will see the best person who can solve your problems!
Always trust yourself.

A gift from Unknown to Juniors VU Mathematics Students

Functional Analysis

Module No. 1

Introduction:

Its applications are in differential equations and numerical analysis, approximation theory and calculus of variations etc.

Course Outcomes:

To be able to understand basics concepts, principles and methods of functional analysis and its applications.

Module No. 2

Course Outline:

Topics:

Introduction, ***Metric space***, subspace, Triangle inequality, Axioms of a metric, Sequence space, Space B(A) of bounded functions, Some Inequalities, Ball and sphere, Continuous mapping, accumulation point, Dense set, separable space, Convergence of a sequence, limit, Cauchy sequence, completeness, Real line, complex plane, Uniform convergence, Discrete metric, Isometric mapping, isometric spaces, Homeomorphism, Normed Space, Banach Space, Further Properties of Normed Spaces, Finite Dimensional Normed Spaces and Subspaces, Compactness and Finite Dimension, Linear Operators, Bounded and Continuous Linear Operators, Linear Functional, Linear Operators and Functional onFinite Dimensional Spaces, Normed Spaces of Operators, Dual Space, Inner Product Space, Hilbert Space, Further Properties of Inner Product Spaces, Orthogonal Complements and Direct Sums, Orthonormal Sets and Sequences, Series Related to Orthonormal Sequences and Sets, Total Orthonormal Sets and Sequences, Legendre Hermite and Laguerre Polynomials, Representation of Functional on Hilbert Spaces, Hilbert Adjoint Operator, Self-Adjoint, Unitary and Normal Operators.

Module No. 3

Recommended Books:

Book Title: Introductory Functional Analysis with Applications
Citation:
Author: Erwin Kreyszig, John Wiley & Sons. Inc.
Edition: 2007
Publisher: Printed In USA

Book Title: Functional Analysis, Sobolev Spaces and Partial Differential Equations
Citation:
Author: HaimBrezis. Universitext, Springer.
Edition: 2010
Publisher: sPRINGERsCIENCE+ Business Media,LLC,233 , NY ,USA

Book Title: Introduction to Functional Analysis
Citation:
Author: Angus E. Taylor, John Wiley & Sons. Inc
Edition: 2006
Publisher: Alpha Science International Limited

Book Title: Elements of Functional Analysis
Citation:
Author: Robert Zimmer, University of Chicago Lecture Series.
Edition: 1990
Publisher: University of chicago Press

Module No. 4

In functional Analysis we shall study more general “spaces” and “Functions” defined on them.

Metric Spaces:

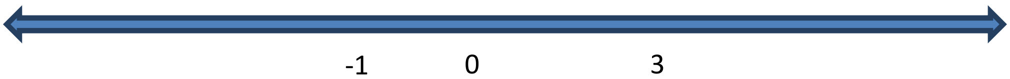
In functional analysis we shall study more general “spaces” and “functions” defined on them.

The given below is the real line

$x, y \in \mathbb{R}$

The distance function with two points x,yor usual metricon real line is $d(x, y) = |x - y|$.

Say we have two points -1 and 3 and if we want to measure distance between 3 and -1 then



$d(-1,3) = 4$ is same as $d(3,-1) = 4$

$3 - (-1) = 4$

For example: If we want to measure the distance between 1.8 and -3.5 then

$$d(1.8, -3.5) = |1.8 - (-3.5)| \qquad |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$= |1.8 + 3.5| = 5.3$$

MODULE NO. 5

METRIC SPACES:

Formal definition:

A metric space is a pair *(X,d)*,where X is a set and d is a metric on X (or distance function on X), that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have following four properties.

1*st* Property:	M_1	d *is real – valued, finite and non – negative*
2*nd* Property:	M_2	$d(x, y) = 0$ *if and only if* $x = y$
3rd Property:	M_3	$d(x, y) = d(y, x)$ (Symmetry)
4th Property:	M_4	$d(x, y) \le d(x, z) + d(z, y)$ (Triangle Inequality)

The above four properties called axioms of metric space. As metric space is ordered pair so we take $X \times X$ mean two elements from set X.

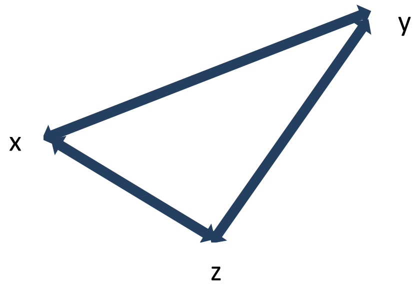
Explanation :

Let's we have three points x, y and z, then equality holds

if and only if all the three points are on the same line.

And in triangular inequality the distance between x and y is

always less than the sum of distances of zy and zy.



Equality: $d(x, y) = d(x, z) + d(z, y)$

Inequality: $d(x, y) \le d(x, z) + d(z, y)$

Now if we have more than three points say $x_1, x_2, x_3, ...x_n$

then distance between any two point say x_1 *and* x_2 is

$$d(x_1, x_n) \le d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + ... + d(x_{n-1}, x_n)$$

is the generalized triangle inequality.

MODULE NO. 6

SUBSPACE:

Formal definition:

A subspace (Y,d) of (X,d) is obtained if we take a subset $Y \subset X$ and restrict d to $Y \times Y$. Thus the metric on Y is the restriction

$$\tilde{d} = d|_{Y \times Y}$$

$\tilde{d}$ is called the metric induced on Y by d.

MODULE NO. 7

METRIC SPACE:

- **Real line** $\mathbb{R}$
- **Euclidean plane** $\mathbb{R}^2$

Real line $\mathbb{R}$

Example 1:

Let x and y be two real points on real line, then

$$d(x,y) = |x-y| \quad ; \quad x, y \in \mathbb{R}$$

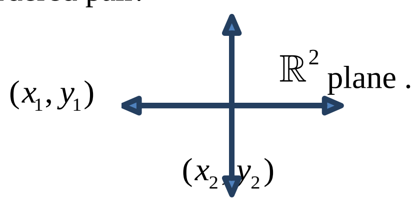
Now we prove all the four properties (axioms) of metric space.

$$\begin{aligned} d(x,y) &= |x-y| \\ d(x,y) &= |x-z+z-y| \quad ; \quad z \in \mathbb{R} \\ d(x,y) &\le |x-z| + |z-y| \\ &= d(x,z) + d(z,y) \end{aligned}$$

Euclidean plane $\mathbb{R}^2$

Euclidean space mean that the points are taken from $\mathbb{R}^2$ in ordered pair.

$$\mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$$



Example 2:

Suppose that one point is (x_1, y_1) and the other point is (x_2, y_2),

thenthe distance d between these two points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

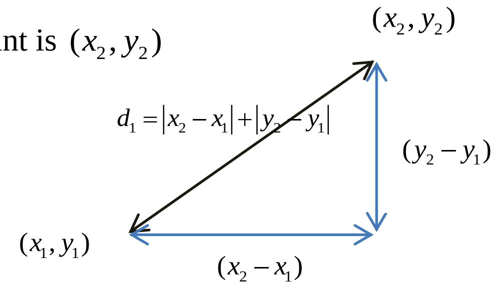
Thus $(\mathbb{R}^2, d)$ is a metric space

Example 3:

Suppose that one point is (x_1, y_1) and the other point is (x_2, y_2)

the distance d between these two points is

$$d_1 = |x_2 - x_1| + |y_2 - y_1|$$



d and d_1 measures the same distance.

Thus $(\mathbb{R}^2, d_1)$ is a metric space

So, we can define any distance function according to our requirement and it should satisfied the four axioms of metric space.

Module No. 8

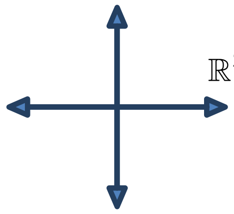
Examples Metric Space:

Real line $\mathbb{R}$ **:**

$\mathbb{R}$

Euclidean plane $\mathbb{R}^2$

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$



$\mathbb{R}^2$ plane

Three dimensional Euclidean Space $\mathbb{R}^3$

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

$\mathbb{R}^3$ plane

In $\mathbb{R}^3$ set the elements are in ordered triple form whose all entries are real numbers.Suppose u and v be two points in $\mathbb{R}^3$ such as

$$u = \{\xi_1, \xi_2, \xi_3\} \quad and \quad v = \{\eta_1, \eta_2, \eta_3\} \ , \qquad \xi_i, \eta_i \in \mathbb{R},$$
$$(where\ \xi\ is\ exai\ and\ \ \eta\ is\ eta)$$

The distance between u and v is

$$d(u,v) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2}$$

Thus $(\mathbb{R}^3, d)$ satisfy all four properties of metric space and is a metric space.

nTuples Euclidean Space $\mathbb{R}^n$

In $\mathbb{R}^n$ set the elements are in ordered n tuples form.Suppose u and v be two points in $\mathbb{R}^n$ such as

$$u = \{\xi_1, \xi_2, ...\xi_n\} \quad and \quad v = \{\eta_1, \eta_2, ..., \eta_n\} \ , \qquad \xi_i, \eta_i \in \mathbb{R},$$
$$(where\ \xi\ is\ exai\ and\ \ \eta\ is\ eta)$$

The distance between u and v is

$$d(u,v) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + ... + (\xi_n - \eta_n)^2}$$

Thus $d\Big|_{(u,v)}$ satisfy all four properties of metric space and is a metric space.

Unitary Space C^n

$$C^n = \{\xi_1, ...\xi_n) \mid \xi_i \in C\} \qquad wherer\ C\ is\ complex\ no.$$

(note: In C^2 both the first and second elements are from complex numbers as $(1+\iota, 1-\iota)$ and also in C^n all the n tuples from $\xi_i, \eta_i \in C$ to ξ_n are all complex elements).

Let $z = a + ib$, $w = c + id$ then

$$|z| = \sqrt{a^2 - b^2} \quad , \quad |w| = \sqrt{c^2 - d^2} \quad \text{and}$$

as z-w=(a-c)+$i(b-d)$, $|z-w| = \sqrt{(a-c)^2 + (b-d)^2}$

Suppose u and v be two points in C^n such as

$$u = \{\xi_1, \xi_2, ...\xi_n\} \quad and \quad v = \{\eta_1, \eta_2, ..., \eta_n\} \ , \quad \xi_i, \eta_i \in \mathbb{R},$$
$$(where\ \xi\ is\ exai\ and\ \eta\ is\ eta\ and\ are\ complex\ numbers)$$

The distance between *u* and *v* is

$$d(u,v) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + ... + (\xi_n - \eta_n)^2} \qquad \xi_i, \eta_i \in C$$

Thus $d\Big|_{(u,v)}$ satisfy all four properties of metric space and is a metric space.

<u>Complex Plane C (n=1) special case</u>

Instead of n if (n=1) special case then there is only one complex no. in u and v such that

$$u = a + ib \ \ and \ \ v = c + id$$

then it is Complex plane C.

The distance between u and v is

$$d(u,v) = |u - v| \qquad u, v \in \mathbb{R}$$

Now we have four different examples of metric space,

i): $\mathbb{R}^3$ ii): $\mathbb{R}^n$

iii): C^n iv): C

now first we have a set then we define a distance function in above cases.

Module No. 9

Examples Metric Space:

Here we discussed examples of metric space other than usual $\mathbb{R}, \mathbb{R}^2, ...\mathbb{R}^n$ or more general form.

Sequence Space l^∞ **:**

As a set X we take the set of all bounded sequences of complex numbers.These bounded sequences may be real or complex but we take here complex numbers. Collection of all complex number.

If we take all sequences of complex number which are bounded in a set then the set is called l^∞.

Let X be a sequence space and x be the element of that space then $x \in X$

$$x = \{\xi_1, \xi_1, \xi_1, ...\} \quad ;$$

We can write this as

$$x=(\xi_i) \qquad \text{where i=1,2,3,.......}$$

The sequence is bounded means if we calculate the value of ξ_i that value is less than $C_x \Rightarrow$

$$|\xi_i| \le C_x \qquad , \qquad \xi_i \in C$$

If we take any sequence from this space it is bounded, it means C_x is depending on sequence. Now we are going to define d, on any two elements from this sequence space such that

$$x, y \in X \qquad , \qquad x = (\xi_i) \qquad y = (\eta_i)$$

The distance function is

$$d(x, y) = \sup_{i \in \mathbb{N}} |\xi_i - \eta_i|$$

The supremum of all differences of $|\xi_i - \eta_i|$ is the distance d between x and y. This sequence of complex numbers space, d(x,y) form a metric space. We take the difference of all points and then its supremum which is distance of that sequence. Here $\mathbb{N}$ is the domain of sequence. Function has any domain but domain of sequence is $\mathbb{N}$.

Module No. 10

Examples Metric Space:

- ***Function Space C[a, b]***
- ***Discrete Metric Space***

i): Function Space C[a, b]

As a set X we take the set of all real-valued functions x, y,……...Which are functions of an independent real variable t. And are define and continuous on a given closed interval

$$J=[a, b]$$

To define distance function:

Say we have two unique points x and y such that $x, y \in C[a,b]$. Here C[a,b] is a function space and x and y are functions on t variable and also real valued (its value is always real value) as

$$x: x(t) \in \mathbb{R} \qquad \textit{and} \quad y: y(t) \in \mathbb{R} \qquad \text{(Real values and continuous).}$$

Domain is fixed from a to b. x and y are function from interval [a,b] to $\mathbb{R}$.

$$x:[a,b] \to \mathbb{R}, \qquad y:[a,b] \to \mathbb{R}$$
$$d(x,y) = \max_{t \in J} |x(t) - y(t)| \quad \textit{where} \quad j = [a,b]$$

We will calculate the difference of two functions x(t)-y(t) at each value of t from J. The maximum of the all the values of difference between two functions is the distance between the functions. Here we have defined the distance between two functions.

ii): Discrete Metric Space

In discrete metric space let X be a set, which could be real number, R^3, R^n , function or set of sequence etc. then we need a distance function.

Distance function is generalized, that if we take two elements from X and those two elements are same then its distance is zero.

d(x , y) =0 if x and y are same

and d(x , y) =1 if x and y are different.

On the other hand if we take two different elements then distance is 1, we fixed. It means that we have fixed the set X with two options 0(same elements) and 1(different elements).This definition forms a metric space and is called a discrete metric space.

MODULE NO. 11

EXAMPLES METRIC SPACE:

Sequence Space s:

The previous example consists of only bounded set but this space consists of all (bounded or unbounded) sequences of complex numbers.

Here the distance function is changed from previous one, the metric d defined by

$$d(x,y)=\sum_{j=1}^{\infty}\frac{1}{2j}\frac{|\xi_j-\eta_j|}{1+|\xi_j-\eta_j|}$$

where $x=(\xi_j)$ *and* $y=(\eta_j)$ and are all complex nos.

Domain of sequences ($(\xi_1,\xi_2,\xi_3,.......)$ or $(\eta_1,\eta_2,\eta_3........)$ is rational numbers.

For distance function we just need to check four axioms. 1st, 2nd and third axioms do yourself, here is only 4th axiom is proved.

4th axiom: $M_4 \quad d(x,y)\le d(x,z)+d(z,y)$ (Triangle Inequality)

let $$f(t)=\frac{t}{1+t}$$

Differentiating w.r.t. t $$f'(t)=\frac{1}{(1+t)^2}$$

As the derivative of the above function is positive this means that it is increasing sequence.

$$|a+b|\le|a|+|b|$$

we know that if $a\le b$ then $f(a)\le f(b)$

the inequality sign does not change.

So, applying above triangular inequality $f(|a+b|)\le f(|a|+|b|)$

$$\frac{|a+b|}{1+|a+b|}\le\frac{|a|+|b|}{1+|a|+|b|}$$

$$\Rightarrow \quad \frac{|a+b|}{1+|a+b|}\le\frac{|a|}{1+|a|+|b|}+\frac{|b|}{1+|a|+|b|}$$

$$OR \quad \frac{|a+b|}{1+|a+b|}\le\frac{|a|+|b|}{1+|a|+|b|}=\frac{|a|}{1+|a|+|b|}+\frac{|b|}{1+|a|+|b|}$$

Now if we remove denominator $|b|$ from $\frac{|a|}{1+|a|+|b|}$ it becomes $\frac{|a|}{1+|a|}$

Andremoving denominator $|a|$ from $\frac{|b|}{1+|a|+|b|}$ it becomes $\frac{|b|}{1+|b|}$

so, the remaining values will be increasedwhich result as

$$\frac{|a+b|}{1+|a+b|} \le \frac{|a|+|b|}{1+|a|+|b|} = \frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|} \le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

Simply we can write

$$\frac{|a+b|}{1+|a+b|} \le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

Using above expression $a = \xi_i - \alpha_i \in \mathrm{x}$; $b = \alpha_i - \eta_i \in \mathrm{y}$

In triangular inequality we use three elements, so we use new sequence zor $z = (\alpha_i)$.

Putting values in above expression

$$\frac{|\xi_i - \alpha_i + \alpha_i - \eta_i|}{1+|\xi_i - \alpha_i + \alpha_i - \eta_i|} \le \frac{|\xi_i - \alpha_i|}{1+|\xi_i - \alpha_i|} + \frac{|\alpha_i - \eta_i|}{1+|\alpha_i - \eta_i|}$$

$$\Rightarrow \quad \frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|} \le \frac{|\xi_i - \alpha_i|}{1+|\xi_i - \alpha_i|} + \frac{|\alpha_i - \eta_i|}{1+|\alpha_i - \eta_i|}$$

We want to change the above equation in this form $\frac{1}{2j}\frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|}$

So, we multiply by $\frac{1}{2i}$ on both sides.

$$\frac{1}{2i}.\frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|} \le \frac{1}{2i}.\frac{|\xi_i - \alpha_i|}{1+|\xi_i - \alpha_i|} + \frac{1}{2i}.\frac{|\alpha_i - \eta_i|}{1+|\alpha_i - \eta_i|}$$

Taking summation of all values

$$\sum \frac{1}{2i}.\frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|} \le \sum \frac{1}{2i}.\frac{|\xi_i - \alpha_i|}{1+|\xi_i - \alpha_i|} + \sum \frac{1}{2i}.\frac{|\alpha_i - \eta_i|}{1+|\alpha_i - \eta_i|}$$

Hence we have proved that 4^{th} axiom $d(x, y) \le d(x, z) + d(z, y)$ (Triangle Inequality)

For metric space we have proved all four axioms. Above we have proved only 4^{th} axiom.

Module No. 12

Examples Metric Space:

Last example of Sequence Space s:

- ***Space*** l^p

➢ The Hilbert Sequence Space l^2

Space l^p

Let $p \geq 1$ be a fixed real number.

By definition, each element in the space l^p is a sequence $x = (\xi_i) = (\xi_1, \xi_2, ...)$ of the numbers such that $|\xi_1|^p + |\xi_2|^p + ...$ converges.

Thus $\sum_{j=1}^{\infty} |\xi_j|^p < \infty$

and the metric is defined by

$$d(x, y) = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p\right)^{\frac{1}{p}}$$

$$where \qquad y = (\eta_j) \quad and \quad \sum_{j=1}^{\infty} |\eta_j|^p < \infty$$

The elements ξ_j *and* η_j are complex numbers.Distance function *d(x,y)* of the set is a metric space. We are not proving all four axioms because it is complicated but it satisfied the axioms of metric space.

Space l^p

The real Space l^p

If the elements ξ_j *and* η_j are not complex numbers but from real numbers then the space is called real space l^p .

The complex Space l^p

If the elements ξ_j *and* η_j are complex numbers then the space is called complex space l^p

Above both have the same condition that summation of $|\xi_1|^p + |\xi_2|^p + ...$ that should be converges and $\sum_{j=1}^{\infty} |\xi_1|^p < \infty$, and the distance function is one by one entry difference with power p and overall power 1/p.

The Hilbert Sequence Space l^2

Now the case p=2 (fixed)

The Hilbert sequence space l^2 with the metric defined by

$$d(x,y)=\left(\sum_{j=1}^{\infty}\left|\xi_j-\eta_j\right|^2\right)^{\frac{1}{2}}$$
$$=\sqrt{\sum_{j=1}^{\infty}\left|\xi_j-\eta_j\right|^2}$$

(note: Check video lecture value is wrong)It is also satisfied the four axioms of metric space.

Now we have done that if we have a set x, then we define a function and last we have proved all the four axioms of metric space. If the set satisfied the four axioms then it make the metric space otherwise it is not metric space.

Module No. 13

Open Set, Closed Set

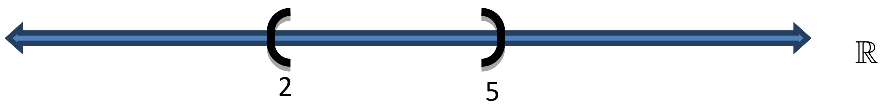
- ***Open/Closed Ball***
- ***Sphere***

Open Ball in $\mathbb{R}^n$ We start with real line.

Open Set and Closed Set on Real Line $\mathbb{R}$

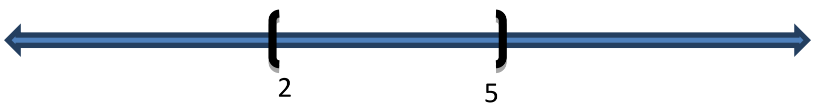
Open Set on Real Line $\mathbb{R}$

On real line we have open set not open ball.



(2,5) is an open set. It includes all values between 2 and 5 but does not include 2 and 5.

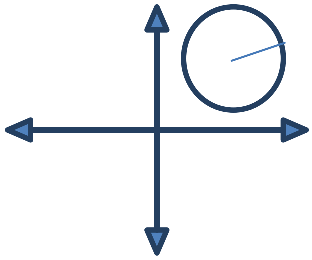
Closed Set on Real Line $\mathbb{R}$



$\mathbb{R}$

[2,5] is a closed set. It includes all values between 2 and 5 including 2 and 5.

Open Set and Closed Set on Real Line $\mathbb{R}^2$



In $\mathbb{R}^2$ we have open ball.

Here open ball has center x_o and radius r.

It includes all values but does not include boundaries

Points on the Boundary:

$$d(x_o, x) = r \quad ; \; x \text{ is on } boundary$$

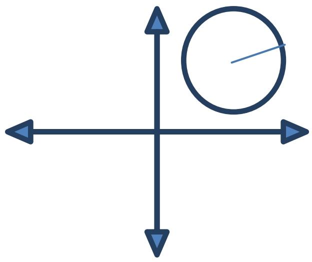
All the points are lies on the boundary if

the difference of that point$x$ from the center x_o is r.

$$x_o - x = r$$

$x$is a boundary point.

Points inside the Boundary



All the points are lies inside the boundary if

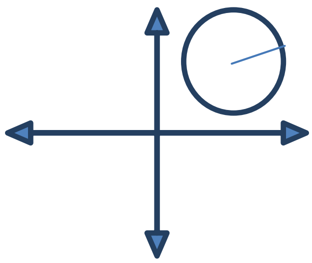
thedistance between x and the center x_o

(i-e. difference of that point x from the center x_o) is less than r.

$$d(x_o,\, x) < r$$

$x$lies inside the boundary.

Points Outside the Bounday



All the points are lies outside the boundary if

the distance between x and the center x_o

(i-e. the difference of that point x from the center x_o) is greater than r.

$$d(x_o,\, x) > r$$

x lies outside the boundary.

Open Ball and Closed Ball in $\mathbb{R}^2$

Open Ball:

In $\mathbb{R}^2$ if boundary is not included then it is open ball. It means all points inside the boundary are included. $\Rightarrow$ $d(x_o,\ x) < r$

Closed Ball:

In $\mathbb{R}^2$ if boundary is included then it is Closed ball. It means all points inside the boundary and on the boundary are included. $\Rightarrow$ $d(x_o,\ x) \le r$

Open Sphere, Closed Sphere in $\mathbb{R}^3$

In $\mathbb{R}^3$ we have open sphere, closed sphere.

Ball and Sphere (General Form)

Open Ball: $B(x_o;r) = \{x \in X \mid d(x,x_o) < r\}$

Closed Ball: $\tilde{B}(x_o;r) = \{x \in X \mid d(x,x_o) \le r\}$

Sphere: $S(x_o : r) = \{x \in X \mid d(x,x_o) = r\}$

Sphere includes all those points which are exactly lies on the boundary or on the radius r. It does not have any inside or outside points.

In all three cases, x_o is called the center and r the radius.

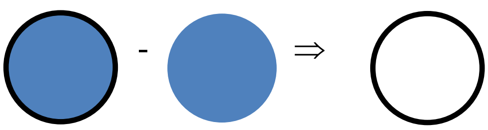
Warning

In discrete space, we have defined distance function.sphere can be empty.

$$d(x,x) = 0$$
$$d(x,y) = 1 \quad ; \quad x \ne y$$
$$d(x,x_c) < r$$
$$d(x,x_c) \le r$$

$$\text{Sphere} = \phi$$

For boundary we can subtract open ball from closed ball.



$$S(x_0,\ r) = \tilde{B}(x_0,\ r) - B(x_0,\ r)$$

Open Set:

A subset M of a metric space X is said to be open if it contains a ball about each of its points.

Closed Set:

A subset K of X is said to be closed if ***its complement (in X) is open.***

that is $K^c = X - K$ is open.

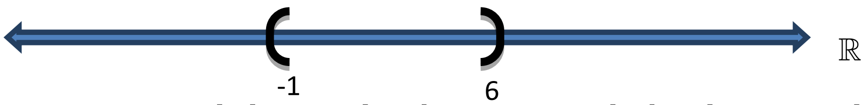
In $\mathbb{R}$ we have two intervals, open and closed interval. Any point in open interval (however it is very near to boundary) we can take another open interval, However in closed interval we can take an open interval beside the boundary.

Module No. 14

Examples Open Ball, Closed Ball:

Example 1:

On real line $\mathbb{R}$ we have open set not open ball.If we find an open ball against each point then it is open sets otherwise it closed (compliment of open) is closed.



(-1, 6) is an open set. It includes all values between -1 and 6 but does not include-1 and 6.

In metric space language, here we can find an open interval against each point.

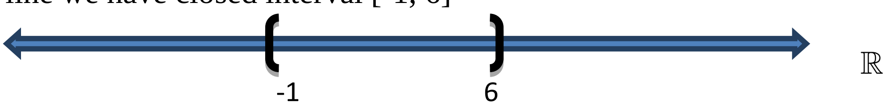
Point 2 has open interval (1, 3) and many mores intervals.

Similarly point 5.99 has an open interval(1, 5.999) and many mores intervals.

Now for point 5.999 has an open interval (1, 5.9999) and many mores intervals.

For closed interval

On real line we have closed interval [-1, 6]



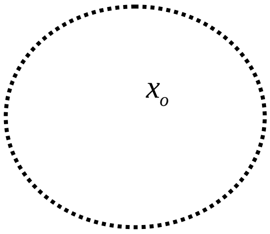
It includes all values between -1 and 6 including -1 and 6. For inside point this condition is true, for each inside point we can find an open interval, but for any point on boundary we ***cannot find any open interval. e.g. for point 6 we can't find any open interval.***

In $\mathbb{R}^2$ we have open ball, if we take any open ball against that ball then we can find an open ball containing that ball because the boundary is not closed.

In $\mathbb{R}^2$ we have closed ball, then points on boundary will not give us any open ball.

MODULE NO. 15

NEIGHBORHOOD OF A POINT:



We can find an open ball around each point in Open set.

"An open ball $B(x_o, \varepsilon)$ of radius ε is often called an ε-neighborhood of x_o."

By a neighborhood of x_o, we mean any subset of ***X*** which contains an ε-neighborhood of x_o.

Difference between Radius r and ε.

Radius r .

For radius r means larger values, 0.1, 0.5, 10,40 while radius ε means very small values like 0.002, 0.000003 etc.

Radius ε.

If we take a point x then all the points around it make a ball whose radius is very small ε or ε-neighborhood of x_o.

Interior Point:

We call x_o an interior point of a set $M \subset X$ if M is a neighborhood of x_o.

The interior of M is the set of all interior points of M.

Int(M) is open and is the largest open set contained in M.

Collection of all open balls is an open set whether the radius ε of open ball is greater or smaller.

MODULE NO. 16

TOPOLOGICAL SPACE:

Definition:

Let $\Im$ be collection of all open subsets of X. Then (X, $\Im$) is said to be a topological space if it satisfies following properties:

T1): $\phi \in \Im$ *and* $X \in \Im$

T2): The union of any member of $\Im$ is a member of $\Im$.

T3): The intersection of finitely many members of $\Im$ is a member of $\Im$.

It holds.!

that $\Im$ is the collection of all open subsets of X.

i) itsimpliy satisfied, empty set is open because there is no point so condition is automatically satisfied. i.e ϕ is always open. Also X belongs to $\Im$.

ii) for second property that union of any member of $\Im$ is a member of $\Im$.
U=union of open subsets

Let say there is at least one open subset M of X who contains x such that
$x \in U$; *at least* $x \leftarrow M$ where M contains a ball B whose radius is x about X.

$M \in union$ $\quad$ x $\subset$ M ,

$B \subset U \Rightarrow U$ *is open*

iii): $y \in \bigcap_{i=1}^{m} M_i \quad \forall$ i=1,......m

$$B_1(y,\varepsilon_1) \subset M_1$$
$$B_2(y,\varepsilon_2) \subset M_2$$
$$.$$
$$.$$
$$.$$
$$B_m(y,\varepsilon_m) \subset M_m$$

We take minimum of all radiussay ε ,as

$$\varepsilon = \min\{\varepsilon_1, ..., \varepsilon_m\}$$
$$B(y,\varepsilon) \subset M_i \quad \forall \quad i \quad , \quad B(y,\varepsilon) \text{ contains in all M}$$
$$B(y,\varepsilon) \in \bigcap_{i=1}^{m} M_i$$

Hence we take a ball from $y \in \bigcap_{i=1}^{m} M_i \quad \forall$ i=1,......m and prove that there exist a ball whose radius is ε which is minimum of all radii. That means M_i containing that ball so that this intersection is also open. Hence a metric space is a topological space, because metric space contains open intervals and open intervals satisfied all the conditions of topological space.

MODULE NO. 17

CONTINUOUS MAPPINGS:

First see the definition of continuous function then proceed next.

Definition:

Let $X = (X, d) \quad and \quad Y = (Y, \tilde{d})$ be metric spaces.

A mapping $T : X \to Y$ is said to be continuous at a point $x_o \in X$

if for every $\varepsilon > 0$ there is a $\delta > 0$ *such that* $\tilde{d}(Tx, Tx_o) < \varepsilon$ for all x satisfying $d(x, x_o) < \delta$

Here we have two spaces X and Y whose distances are d and $\tilde{d}$. Tx same as T(x).

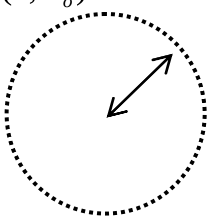
$\tilde{d}(Tx, Tx_o)$ is basically open disk whose radius is ε , center is Tx_o and Tx is any point on the disk.

$d(x, x_o) < \delta$ is also a open disk whose radius is δ and center is x_o .

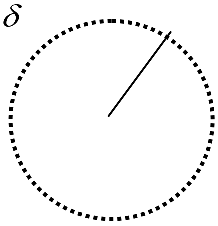
$$T : X \to Y$$

$$\tilde{d}(Tx, Tx_o) < \varepsilon$$

$$d(x, x_o) < \delta \qquad , \quad \text{for every } \varepsilon > 0 \;\; \exists \;\; \delta$$



$\delta - neighborhood$



$\varepsilon - neighborhood$

X space Y space

x is mapping on Tx, $x \to Tx$

x_o is mapping on Tx_o, $x_o \to Tx_o$

MODULE NO. 18

CONTINUOUS MAPPING:

First see the definition of continuous function and continuous mapping then now another definition of continuous mapping.

Theorem (Continuous Mapping):

A mapping T of a metric space X into a metric space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X.

It says that inverse image is open then metric space is continuous. As it is if and only if condition then we suppose continuous condition, then we prove that inverse image of open subset is open.

Conversely we consider inverse image of open subset is open and prove that it is continuous.

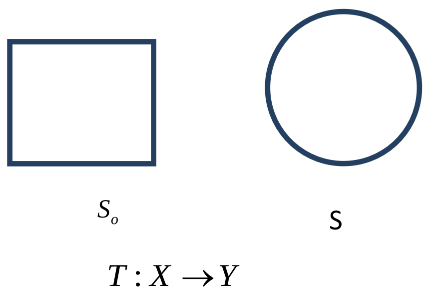
Proof: Suppose a mapping $T: X \to Y$ and T is continuous.

Now we will prove that inverse image of any open subset in Y is open in X.

Let $S \subset Y$ be open subset. Let S_o be the inverse image of S.

Space X $\to$ spaceY

We have to prove that this S_o open. For this we have two cases.



1st case:

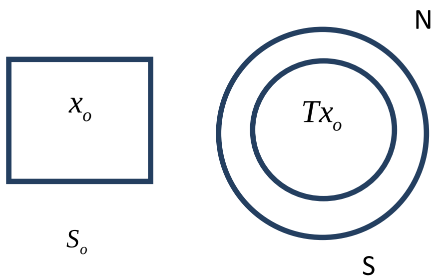
Suppose that we have chosen the element has no inverse image then

$S_o = \phi \quad \Rightarrow \quad$ open (because empty set is always open)

2nd case:

Now suppose S_o is not empty then there is at least one

Point x_o such that



$S_o \neq \phi \quad \Rightarrow \quad x_o \in S_o$

$y_o = Tx_o$ $\qquad T : X \to Y$

S is open , there exist an ε neighborhood N of y_o

since T is continuous $\exists$ S_o neighborhood of x_o which is mapped into N.

Since for $N \subset S$ we have $N_o \subset S_o$ so,

S_o is open on $x_o \in S_o$ and has δ neighborhood.

Conversely we also prove that T is continuous.

For every $x_o \in X$ and ε-neighborhood N of Tx_o, the inverse image N_o of N is open.

Since N is open and N_o contains x_o, N_o also contains a δ neighborhood of x_o (being open) which is mapped into N because N_o is mapped into N.

By definition T is continuous at x_o.

Module No. 19

Accumulation Point (limit point):

Definition:

If M is a subset of a metric space X then x_o is a limit point of M. if it is the limit of an eventually non-constant sequence (a_i) of points of M (or limit point of M)if every neighborhood of x_o contains at least one point $y \in M$ distinct from x_o.

Translation in the form of metric space:

Let M be a subset of a metric space X, then a point x_o of X (which may or may not be a point of M) is called an accumulation point of M (or limit point of M) if every neighborhood of x_o contains at least one point $y \in M$ distinct from x_o.

Example (1) $\qquad \mathbb{R} : d(x, y) = |x - y|$

M(0,1)

$0 \notin M = (0,1)$ is a limit point of M.

1 is also a limit point of M.

as $\lim_{x\to\infty}\frac{1}{n}$, $1+\frac{1}{n}$

Another example:

The set of integers has no limits points, $\mathbb{Z}\subset\mathbb{R}$ has no limit point, e.g. any sequence in $\mathbb{Z}$ converging to any integer is eventually constant.

Example (2):

Let in $\mathbb{R}^2$ $\quad d(x, y)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Open disk $\{(x, y)\in\mathbb{R}^2 \mid x^2+y^2<1\}$

All those points from $\mathbb{R}^2$ such that $x^2+y^2<1$,

All those point which are on the boundary of this open ball are accumulation points or limit points.

Closure of M:

The set consisting of the points of M and the accumulation points of M is called the closure of M and is denoted by $\bar{M}$.

Example (3):

M=(0,1) has limit points are 0, 1

when we collect these points then it transforms to.

$$[0,1]=(0,1)\cup\{0,1\}=\bar{M}$$

Closure of M= Points of M U limits points of M

Module No. 20

Dense Set:

Definition:

A subset M of a metric space X is said to be dense in X if $\bar{M}=X$

Closure set is a set along with its limits points.

Example (1):

The rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$.

we have infinite sets $\mathbb{Q}$ and $\mathbb{R}$,

Let $x \in \mathbb{R}$, and x can be a integer or fraction,

as $x=n+r$ *(e.g 2.123=2+0.123)*

$n \in \mathbb{Z}, \quad 0 \leq r < 1$ if r is between 0 and 1 then it is fraction.

here, $r = 0.r_1r_2r_3..........$

set $x_k = n + 0.r_1r_2r_3..........r_k$

So, each x_k is a rational number as we fix the fractional part $r = 0.r_1r_2r_3..........$ when we fix the fractional part then it gives you rational number. Real number may be rational(fraction part is fix and not continue) or irrational (fraction part is not fix and continue).

$$\lim_{x \to \infty} x_k = x$$

here x_k is a rational number and at $x \to \infty$ and it gives irrational number x. so all rational numbers cover all irrational numbers, $\bar{Q} = \mathbb{R}$ this shows that Q is dense in $\mathbb{R}$, as real set $\mathbb{R}$ contains rational and irrational numbers, if Q gives a rational number then it is also in Q and also in $\mathbb{R}$ but if Q gives an irrational number then it is also present in $\mathbb{R}$.

Separable Space

A metric space X is said to be separable if it has a countable subset which is dense in X.

It has two conditions; First its subset is dense and second is countable.

Module No. 21

If $\bar{M} = X$ then M is dense in X.

Separable Spaces:

A metric space X is said to be separable if it has a countable subset which is dense in X.

Examples (Separable Spaces):

- ***The Real Line*** $\mathbb{R}$

- ***The Complex Plane*** $\mathbb{C}$
- ***Discrete Metric Space***

Example 1:

1st The real line $\mathbb{R}$

$(\mathbb{R}, \text{d})$, $\text{d(x,y)}=|x-y|$

Now $\mathbb{Q}$ is subset of $\mathbb{R}$, such that closure of $\mathbb{Q}$ is equal to $\mathbb{R}$. So, it satisfy the both conditions of $\mathbb{Q}$ is subset of $\mathbb{R}$ (means $\mathbb{Q} \subset \mathbb{R}$),

$\mathbb{Q}$ is dense in $\mathbb{R}$ (means $\overline{\mathbb{Q}} = \mathbb{R}$) and $\mathbb{Q}$ is countable.

Hence $\mathbb{R}$ is a separable.

2nd $\mathbb{R}^n$

$(\mathbb{R}, d)$ is a space where $d(\underline{x}, \underline{y})$ is the distance function.

$$d(\underline{x}, \underline{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + ... + (y_n - x_n)^2}$$

The elements of $\mathbb{R}^n$ are vectors, $\underline{x}$ and $\underline{y}$ are vectors and we have represent as underscore such that $\underline{x} = (x_1, ...x_n)$ and $\underline{\text{y}} = (\text{y}_1, ...\text{y}_n)$

$$\mathbb{Q}^n = \{(c_1, ...c_n) \,|\, c_i \in \mathbb{Q}\}$$

Where $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, also as $\mathbb{Q}$ is countable so all its n-tuples are also countable (which means $(c_1, ...c_n)$ is countable).

$\mathbb{Q}^n$ is a countable subset of $\mathbb{R}^n$ which is dense $\mathbb{R}^n$, so $\mathbb{R}^n$ is separable space.

The Complex Plane $\mathbb{C}$

In complex plane the numbers are in the form of $\{a + ib \,|\, a, b \in \mathbb{R}\}$, $i = \sqrt{-1}$

Or same as $\{(a, b) \,|\, a, b \in \mathbb{R}\}$

For $\mathbb{R}^2$ we can define another set $\mathbb{Q}^2 = \{(c_1, c_2) \,|\, c_1, c_2 \in \mathbb{Q}\} = \{c_1, ic_2 \,|\, c_1, c_2 \in \mathbb{Q}\}$

In previous example $\mathbb{R}$ was dense in n-dimension $\mathbb{R}^n$, here are only two complex numbers c_1, c_2 in $\mathbb{Q}^2$ so $\mathbb{Q}^2$ is also dense in $\mathbb{C}$ and also countable as $\{(c_1, c_2) \,|\, c_1, c_2 \in \mathbb{Q}\}$.

Now $\mathbb{C}$ has a subset $\mathbb{Q}^2$ which is dense and countable so, $\mathbb{C}$ is separable.

Discrete Metric Space:

In discrete metric space if elements are same then distance is 0 and if elements are different then distance is 1. There is no condition on set. Set can be any set.

Example 2:

In discrete metric space we have a condition on distance function which is if elements are same then distance is zero, and if elements are different then distance is equal 1. When we have a discrete metric space then the set X itself is dense and there is no limiting point, no other subset is dense in X. Now for separable space we need two conditions, 1: subset is dense, 2: countable. As in Discrete metric space the set is itself dense, so we need only to check that is countable or not, if it is countable then it is separable. Hence in Discrete metric space we only check that the set is countable or not, if it is countable then it is separable else it is not separable.

Module No. 22

Examples Separable Spaces:

Space l^p

Space mean “a set”, in this set elements are sequences, which may be real (called real space l^p) or complex numbers (called complex space l^p). Then we have define its metric, a metric means a distance function.

Now we have to show that a subset of l^p is dense and also countable then l^p is a separable space.

In previous examples we take $\mathbb{Q}$ as countable and then use it as generalize form. Here we also use $\mathbb{Q}$ as countable.

Space l^p

The space l^p with $1 \le p < +\infty$ is a separable.

To find a countable subset which is dense in l^p where l^p is a space consisting of sequences $x = \{\xi_i\}$, which are bounded sequences such that $\sum_{i=1}^{\infty} |\xi_i|^p < \infty$ is convergent.

The metric is $d(x,y)=\left(\sum_{i=1}^{\infty}|\xi_i|^p\right)^{\frac{1}{p}}$ *where* $x=(\xi_i)$ *and* $y=(\eta_i)$

x and y in d(x,y) are sequences. Now we find the countable subset of l^p which is dense in l^p.

Let M be the set of all sequences of the form $y=(\eta_1,\eta_2,.........\eta_n,0,0,.........)$

n-positive integer and η_i's are rational numbers.

$\eta_1,\eta_2,.........\eta_n$ are rational numbers and 0,0,………… are constant, so M is countable.

We need to probe that M is dense in l^p.

$$\bar{M}=l^p$$

Let $x=\{\xi_i\}\in l^p$ be arbitrary. We need to show that
$\exists \quad y\in M \quad$ *such that* $d(x,y)<\varepsilon$

Now $x=\{\xi_i\}\in l^p$

$$\Rightarrow \quad \sum_{i=1}^{\infty}|\xi_i|^p<\infty \text{ (convergent)}$$

$$\Rightarrow \quad \left(\sum_{j=1}^{n}|\xi_i|^p+\sum_{j=n+1}^{\infty}|\xi_i|^p\right)<\infty \text{ (Convergent)}$$

Less than infinity means sum is finite.

Then for every $\varepsilon>0$

(here Epsilon represent the small value) there is n(depend ε)

$$\Rightarrow \quad \sum_{j=n+1}^{\infty}|\xi_i|^p<\frac{\varepsilon^p}{2} \quad \text{……….(i,}$$

$$d(x,y)<\varepsilon$$

Now the rational numbers are dense in $\mathbb{R}$.

originally $x=(\xi_j)$ we have covert it into two parts

$(\xi_1,\xi_2,........,\xi_n)$ and $(\xi_{n+1},.............)$
overlall $(\xi_1,\xi_2,........,\xi_n,\xi_{n+1},.............)$

Now $y\in M$, $y=(\eta_1,\eta_2,.........\eta_n,0,0,.........)$

$$\sum_{j=1}^{n}\left|\xi_i-\eta_i\right|^p<\frac{\varepsilon^p}{2} \qquad \ldots\ldots\ldots\ldots\ldots(ii$$

Using both relations

$$\left[d(x,y)\right]^p=\sum_{j=1}^{\infty}\left|\xi_j-\eta_j\right|^p$$

$$=\sum_{j=1}^{n}\left|\xi_j-\eta_j\right|^p+\sum_{j=n+1}^{\infty}\left|\xi_j-\eta_j\right|^p$$

$$\sum_{j=1}^{n}\left|\xi_j-\eta_j\right|^p<\frac{\varepsilon^p}{2} \qquad \text{and}$$

$$\sum_{j=n+1}^{\infty}\left|\xi_j-\eta_j\right|^p<\frac{\varepsilon^p}{2}$$

$$\Rightarrow \qquad \left[d(x,y)\right]^p=\sum_{j=1}^{n}\left|\xi_j-\eta_j\right|^p+\sum_{j=n+1}^{\infty}\left|\xi_j-\eta_j\right|^p<\varepsilon^p$$

$$\Rightarrow \qquad d(x,y)<\varepsilon \qquad y\in M$$

We have found a limit point y which belongs to X.

In this module we have proven that *l* is separable.

Here we have defined a set M and using the properties of rational number we see that it was countable. Then we prove that it is dense, for this we take a sequence in l^p and proved that its limiting point is also in M. so, M along with limiting point y becomes whole l^p .

Module No. 23

Bounded Sequence:

Definition:

We call a nonempty subset $M\subset X$ a bounded set if its diameter

$\delta(M)=\sup_{x,y\in M} d(x,y)$ is finite.

Here we check all distance pairs for each point against all other points, lineup all those distances and take the supremum distance point we call the diameter of that set.

It means supremum of all distance is finite then the set is bounded.

A sequence (x_n) in X is bounded sequence if the corresponding point set is a bounded subset of X.

Hence bounded sequence means finite diameter and if diameter is infinite then sequence is unbounded.

Module No. 24

Sequences:

- ***Convergence of a sequence***
- ***Limits***

Sequence is a function whose domain is natural numbers.

Convergence of a sequence:

A sequence (x_n) in a metric space $X = (X, d)$ is said to converge or to be convergent if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0$$

x is called the limit of (x_n) and we write

$$\lim_{n\to\infty} x_n = x$$

or simply $x_n \to x$

Example 1:

$$x_n = \frac{1}{n} \quad ,$$

$$n \text{ varies } as \quad \{1, 2, 3, ...\}$$

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \to 0$$

Its domain is set of natural numbers.if n varies from 1 to ∞ then $\frac{1}{n}$ approaches to 0. Or its limit point x is 0.

Example 2:

$x_n = (-1)^n$ here x varies from 1 to n.

1 n-even not convergent

-1 n-odd convergent.

As this sequence is not converging at one value. It varies between 1 and -1 so it is not converging.

Example 3:

$$x_n = \begin{cases} 1 & \text{if n is a square} \\ 0 & \text{if otherwise} \end{cases}$$

Now $x_4 = 1$, $x_5 = 0$, $x_6 = 0$

$x_7 = 0$, $x_8 = 0$, $x_9 = 1$

....

Here $x_1 = 1$, $x_2 = 0$, $x_3 = 0$

$x_4 = 1$, $x_5 = 0$, $x_6 = 0$

$x_7 = 0$, $x_8 = 0$, $x_9 = 1$

....

....

....

are not convergent.

Convergence of a sequence: (Another Definition)

We say that (x_n) converges to x or has the limit x, if (x_n) is not convergent, it is said to be divergent.

$$(x_n) \qquad ; \qquad x_n \to x$$

$\varepsilon > 0$ being given, there is $N = N(\varepsilon)$ such that all (x_n) with n>Nwilllie in the ε -neighborhood $B(x;\varepsilon)$ of x. then we call it convergent.

Module No. 25

Here we relate the convergent sequence and bounded sequence.

Lemma:

Let $X = (X, d)$ be a metric space, then

a) A convergent sequence in X is bounded and its limit is unique.

b) If $x_n \to x$ and $y_n \to y$ in X then $d(x_n, y_n) \to d(x, y)$

a):

Given that sequence is convergent and we have to prove that it is also bounded.

Bounded means that its corresponding diameter is finite.

For convergent sequence mean for every $\varepsilon > 0$ there exist $N = N(\varepsilon)$ such that for all x_n with n>N lie in the ε - neighborhood $B(x;\varepsilon)$ of x .

As it is true for all ε so we choose

$$\varepsilon = 1, \quad \textit{then we will find there exist } N \textit{ such that}$$
$$d(x_n, x) < 1$$

We have values $x_1, x_2, \ldots\ldots\ldots x_N, x_{N+1}, \ldots\ldots\ldots$

which are entries of sequence.

If we choose n<N, say from this part of the sequence $x_1, x_2, \ldots\ldots\ldots x_N$ then $d(x_n, x)$ *is greater than* 1, and

if we choose n>N, say from this part of the sequence $x_{N+1}, \ldots\ldots\ldots$ then $d(x_n, x) < 1$,

$$d(x_n, x) < 1 \quad \forall n>N$$

Now we have calculated the distance of *x* from the point x_i where i=1,2……….N is

$$d(x_1, x), d(x_2, x), \ldots\ldots\ldots d(x_N, x)$$

We take the maximum of all these distances,max(………………),

let say this max distance is "*d*".

Now the distance before N is less than *d* and the value after N is less than 1,

$$d(x_n, x) < d + 1 \quad , \quad \forall \ n$$

In first part (a) we have to prove two things

i):sequence is convergent (i.e we have to prove that it is bounded) and

ii): converging value is unique.

2nd Part Uniqueness:

Let say that $x_n \to x$, $x_n \to z$

$d(x,z) \geq 0$ ……………..(i

If we take any two values from a set then the distance between them is always greater than 0.

For uniqueness we have to prove that x=z, in other word the distance between x and z is $d(x,z) = 0$

Using the 4th axioms of metric space that

$$d(x,z) \leq d(x,x_n) + d(x_n,z)$$

x_n converges to x and also z (our supposition),

$d(x,x_n) \to 0$

and $d(x_n,z) \to 0$

now $d(x,z) \leq 0$ ……………….(ii

From (i and (ii

$$d(x,z) = 0$$

$\Rightarrow$ x=z

Hence proved that it converges to a unique value.

b):

Let say that $x_n \to x$, $y_n \to y$

then we have to prove that $d(x_n, y_n) \to d(x,y)$

$$d(x_n, y_n) \leq d(x_n, y) + \mathrm{d(x, y)} + d(y, y_n) \quad , \qquad \text{triangular inequality}$$

$$d(x_n, y_n) - \mathrm{d(x, y)} \leq d(x_n, y) + d(y, y_n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{iii}$$

interchanging x_n and x and y_n and y,

$$d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)$$

$$d(x, y) - d(x_n, y_n) \leq d(x, x_n) + d(y_n, y)$$

Multiplying -1 on both sides of inequality

$$d(x_n, y_n) - d(x, y) \geq -(d(x, x_n) + d(y_n, y)) \ldots\ldots\ldots\ldots\ldots \text{iv} \quad \text{(CHECK SIGN)}$$

(i.e $|x| \leq a \quad \Rightarrow \quad -a \leq x \leq a$)

usingabove inequality from iii and iv

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, \mathrm{x}) + d(y_n, \mathrm{y}) \ldots\ldots\ldots\ldots \text{v}$$

As $x_n \to x$, $y_n \to y$ so, $d(x_n, x) \to 0$ *and* $d(y_n, y) \to 0$

$$\Rightarrow \quad \{d(x_n, y_n) - d(x, y)\} \to 0$$

$$\Rightarrow \quad d(x_n, y_n) \to d(x, y)$$

Hence proved.

Module No. 26

Cauchy Sequence:

Definition:

A sequence (x_n) in a metric space $X = (X, d)$ is said to be Cauchy (or fundamental) if

for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon$ for every m,n>N.

Equivalent notation $d(x_m, x_n) \to 0$ as $m, n \to \infty$

Example 1:

$$a_n = \frac{1}{n} \quad , \quad \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \subset (0,1]$$

The distance function is $d(x, y) = |x - y|$, is a Cauchy sequence because

Let say we have any m and n positive numbers, then

$$\left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} + \frac{1}{n}$$

as $\quad m \to \infty \quad , \quad n \to \infty \quad ,$

then $\quad \frac{1}{m} \to 0 \quad , \quad \frac{1}{n} \to 0$

so $\quad \left|\frac{1}{m} - \frac{1}{n}\right| \to 0 \quad$ as $\quad m \to \infty \quad , \quad n \to \infty$

this is the Cauchy sequence condition that

$$d(x_m, x_n) \to 0 \text{ as } m, n \to \infty$$

Completeness:

Definition:

The space X is said to be complete if every Cauchy sequence in X converges that is, has a limit which is an element of X.

Example 2:

$$a_n = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \subset (0,1]$$

is a Cauchy sequence as $a_n \to 0 \notin (0,1]$

Hence the sequence a_n in space$X$is converging to 0 but this does not belong to that (0,1], the function define on space is

$$d(x, y) = |x, y| \text{ is Cauchy.}$$

$\Rightarrow$ this space (0,1] is not complete.

For every Cauchy sequence, it should converge to element of that space; if it converges to space then we say that it is complete space.

Module No. 27

Here we relate the convergent sequence and bounded sequence.

Theorem Convergent Sequence:

Theorem:

Every convergent sequence in a metric space is a Cauchy sequence.

Proof:

Let $\{x_n\}$ be a convergent sequence such that $x_n \to x$ for every $\varepsilon > 0$ there exist $N = N(\varepsilon)$ *such that* $d(x_n, x) < \frac{\varepsilon}{2}$ $\forall$ n>N

Now we have to prove that $\{x_n\}$ is a Cauchy sequence, for this we have to prove that

$$d(x_m, x_n) < \varepsilon \quad ; \qquad m, n > N$$

We first choose that m>N then by triangular inequality,

$$d(x_m, x_n) \le d(x_m, x) + d(x_n, x) \quad ; \qquad m, n > N$$

as $\qquad d(x_m, x) < \frac{\varepsilon}{2} \quad , \quad d(x_n, x) < \frac{\varepsilon}{2}$

$\Rightarrow \qquad d(x_m, x_n) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad ; \qquad m, n > N$

$\Rightarrow \qquad d(x_m, x_n) < \varepsilon \quad ; \qquad m, n > N$

That we have to prove, hence $\{x_n\}$ is a Cauchy sequence.

Converse:

Now we check that every Cauchy sequence (x_n) in that space is convergent.

The converse is not true.

"every Cauchy sequence (x_n) in that space is not convergent".

Example 1:

The counter example is

$$a_n = \left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$$

This is a Cauchy sequence in $(0,1]$ but it is not convergent in $(0,1]$.

Example 2:

The metric space $\mathbb{Q}$, $d(x, y) = |x - y|$, This metric space is not complete, we need at least one Cauchy sequence which is not converging in this space.

So, we have a sequence $\{x_n\}$ which is

$$x_n = \left(1+\frac{1}{n}\right)^n ; \qquad n = 1, 2, 3, ...$$

This sequence is Cauchy sequence in $\mathbb{Q}$ and this sequence is converging to 'e' in $\mathbb{R}$, $e \notin \mathbb{Q}$, where e is an irrational number and does not belong to $\mathbb{Q}$. So this sequence is such that it is converging to irrational number.

This means that $\mathbb{Q}$ is not complete metric space.

Module No. 28

Here we relate the convergent sequence and bounded sequence.

Theorem (Closure, Closed Set):

Closure is a collection of limit points and the set itself.

Limit point is such point that if we draw an open ball around it then we can find another point other than that point which belongs to that set.

A set is called a ***closed set*** if all the limits points are present in that set.

Theorem:

Let M be a nonempty subset of a metric space $d(X,d)$ *and* $\bar{M}$ its closure as defined before then,

a): $x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that $x_n \to x$.

b): M is closed if and only if the situation $x_n \in M$, $x_n \to x$ implies that $x \in M$.

a): Proof:

$$x \in \bar{M} \Leftrightarrow \text{there is a sequence } (x_n) \text{ in M such that } x_n \to x$$

$\bar{M}$ is a collection of M and its limit points.Now there are two option, i): x belong to M, $x \in \overline{M}$

ii): x does not belong to M, $x \notin M$

i) $x \in \overline{M}$

now if x does not belong to M then x is a limit point of M.

and if $x \in M$ then $x_1 = x,\ x_2 = x,\ x_3 = x, \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

or $(x_1, x_2, \ldots\ldots\ldots) = (x_n) = (x_1 x_2 \ldots\ldots\ldots),$

hence $x_n \to x \in M$

ii) $x \notin M$

For every n=1,2,3,……………… the ball $B\left(x; \frac{1}{n}\right)$, here ε is $\frac{1}{n}$.

Containing an $x_n \in M$, other than x.

Now as

$$n \to \infty$$

$$\Rightarrow \quad \frac{1}{n} \to 0$$

$$\Rightarrow \quad x_n \to x.$$

Hence x_n converges to x.

Conversely,

There is a sequence $\{x_n\}$ in M such that $x \in \overline{M}$, so, we have a sequence $x_n \to x \quad and \ (x_n) \ in \ M..$

Here we have two cases

i) $x \in M$ or

ii) every neighborhood of x contains points $x_n \neq x$

this implies that x is a limit point. i.e $x \in \overline{M}$

b):

M is closed, if and only if the situation $x_n \in M, \quad x_n \to x$.

$$\Rightarrow \quad x \in M$$

M is closed if and only if $\bar{M} = M$,

Now we have to prove that $\bar{M} = M$ for this we prove that $\bar{M} \subseteq M$, $M \subseteq \bar{M}$

i): $\bar{M} \subseteq M$

by definition M contains M and its limit point so this condition is fulfilled.

ii): $M \subseteq \bar{M}$

Now we prove that $M \subseteq \bar{M}$

Let $x \in \bar{M}$, we will show that $x \in M$.

Now if we take $x$belongs to $\bar{M}$ then from above “a” part of this theorem, we have a sequence x_n in M such that $x_n \to x$ this implies $x \in M$.

That means $M \subseteq \bar{M}$.

Hence $\bar{M} = M$

Module No. 29

Theorem (Complete Subspace):

Theorem:

A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X.

As this condition is if and only if so vice versa.From previous theorem we have

Theorem:

Let M be a nonempty subset of a metric space $d(X,d)$ *and* $\bar{M}$ its closure as defined before then,

a): $x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that $x_n \to x$.

b): M is closed if and only if the situation $x_n \in M$, $x_n \to x$ implies that $x \in M$.

Proof:

Let M is subspace of X over d is then (X,d) complete.

$$M \subset (X,d),$$

M is complete if and only if M is closed, and M is closed if and only if

$$M = \bar{M}.$$

Now we can say that

$$M \subset (X,d) \Leftrightarrow M = \bar{M}.$$

Suppose M is complete and we need to show that $M = \bar{M}$.

Now by definition $M \subseteq \bar{M}$. Now we need to prove that $\bar{M} \subseteq M$ (to be proved).

"Let M be a nonempty subspace of a metric space $d(X,d)$ *and* $\bar{M}$ its closure as defined before then,

From the part "a" of previous theorem

a):

$x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that $x_n \to x$.

Now $x \in \bar{M}$

As M is a subspace of a complete metric space $d(X,d)$ *and* x_n is also in X *so,*

$\Rightarrow$ there is a sequence (x_n) in X such that $x_n \to x$.

Since every convergent sequence in a metric space is Cauchy, then (x_n) is Cauchy.

Our supposition is that M is complete. So, (x_n) converges in M

$$\Rightarrow \quad x_n \to x \in M$$

$$\Rightarrow \quad \bar{M} \subseteq M$$

we start from $x \in \bar{M}$ and obtained $x \in M$

$$\Rightarrow \quad M = \bar{M}$$

Hence M is closed.

Conversely:

M is closed $\Rightarrow$ $M = \bar{M}$

and we need to show that M is complete.

For this we need to show that every Cauchy sequence in *M* converges in

$$M,\ x \in M\ .$$

Let (x_n) be a Cauchy sequence in M such that $x_n \to x$,

By the previous theorem $x \in \bar{M}$

but $\bar{M} = M \Rightarrow x \in M$

Since (x_n) is an arbitrary sequence,

$\Rightarrow$ true for all Cauchy sequences in *M*,

Hence proved

MTH 641

Functional Analysis

MODULE NO. 29 To 63

(MID TERM SYLLABUS)

THESE ARE JUST SHORT HINT FOR THE PREPARATION OF MTH 641

Don't look for someone who can solve your problems,
Instead go and stand in front of the mirror,
Look straight into your eyes,
And you will see the best person who can solve your problems!
Always trust yourself.

A gift from Unknown to Juniors VU Mathematics Students

Module No. 29

Theorem (Complete Subspace):

Theorem:

A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X.

As this condition is if and only if so vice versa.From previous theorem we have

Theorem:

Let M be a nonempty subset of a metric space $d(X,d)$ *and* $\bar{M}$ its closure as defined before then,

a): $x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that $x_n \to x$.

b): M is closed if and only if the situation $x_n \in M,\ \ x_n \to x$ implies that $x \in M$.

Proof:

Let M is subspace of X over d is then (X,d) complete.

$$M \subset (X,d),$$

M is complete if and only if M is closed, and M is closed if and only if

$$M = \bar{M}.$$

Now we can say that

$$M \subset (X,d) \Leftrightarrow M = \bar{M}.$$

Suppose M is complete and we need to show that $M = \bar{M}$.

Now by definition $M \subseteq \bar{M}$. Now we need to prove that $\bar{M} \subseteq M$ (to be proved).

"Let M be a nonempty subspace of a metric space $d(X,d)$ *and* $\bar{M}$ its closure as defined before then,

From the part "a" of previous theorem

a):

$x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that $x_n \to x$.

Now $x \in \bar{M}$

As M is a subspace of a complete metric space $d(X,d)$ *and* x_n is also in *X so,*

$\Rightarrow$ there is a sequence (x_n) in X such that $x_n \to x$.

Since every convergent sequence in a metric space is Cauchy, then (x_n) is Cauchy.

Our supposition is that M is complete. So, (x_n) converges in M

$\Rightarrow$ $x_n \to x \in M$

$\Rightarrow$ $\bar{M} \subseteq M$

we start from $x \in \bar{M}$ and obtained $x \in M$

$\Rightarrow$ $M = \bar{M}$

Hence M is closed.

Conversely:

M is closed

$\Rightarrow$ $M = \bar{M}$

and we need to show that M is complete.

For this we need to show that every Cauchy sequence in *M* converges in

$M,\ x \in M$.

Let (x_n) be a Cauchy sequence in M such that $x_n \to x$,

By the previous theorem $x \in \bar{M}$

but $\bar{M} = M$ $\Rightarrow$ $x \in M$

Since (x_n) is an arbitrary sequence,

$\Rightarrow$ true for all Cauchy sequences in *M*,

Hence proved

Module No. 30

Theorem (Continuous Mapping):

Theorem:

A mapping $T:X\to Y$ of a metric space (X, d) into a metric space $(Y,\tilde{d})$ is continuous at a point $x_o\in X$ if and only if $x_n\to x_o$ implies $Tx_n\to Tx_o$.

Proof:

Suppose T is continuous, we will prove that if $x_n\to x_o$ implies $Tx_n\to Tx_o$.

T is continuous means $T:X\to Y$

a given $\varepsilon>0$ there exist $\delta>0$ such that

$$d(x,x_o)<\delta \quad \tilde{d}(Tx,\ Tx_o)<\varepsilon$$

So, let $x_n\to x_o$ there exist a $\mathbb{N}$ such that for all n> $\mathbb{N}$ we have

$$d(x_n,\ x_o)<\delta$$

This is δ of convergence.

$$\tilde{d}(Tx,\ Tx_o)<\varepsilon \quad , \quad n>\mathbb{N}$$

By definition $\quad Tx_n\to Tx_o$

Converse:

Let $x_n\to x_o$ implies $Tx_n\to Tx_o$ for all x_o .

We have to show that T is continuous by contradiction.

We suppose that it is not true then there is an $\varepsilon>0$ such that for every $\delta>0$ there is some $x\neq x_o$ such that

$$d(x,x_o)<\delta \quad \Rightarrow \quad \tilde{d}(Tx,Tx_o)\geq\varepsilon$$

In particular $\quad \delta=\dfrac{1}{n} \qquad d(x,x_o)<\dfrac{1}{n}$

$$\Rightarrow \qquad x_n\to x_o$$

$$\Rightarrow \qquad Tx\ not\to Tx_o$$

$$\Rightarrow \qquad \tilde{d}(Tx,Tx_o)\geq\varepsilon$$

Module No. 31

Exmaples (Completeness):

- $\mathbb{R}$

We will show that $\mathbb{R}$ and $\mathbb{C}$ are completes. In this module we show only that $\mathbb{R}$ is a complete metric space which means every sequence in $\mathbb{R}$ is convergent in $\mathbb{R}$ and every Cauchy sequence is convergent.

Lemma a:

Every Cauchy sequence in a metric space is bounded.

This is for every metric space.

Lemma b:

If a Cauchy sequence has a subsequence that converges to $\bar{x}$, then the sequence converges to $\bar{x}$.

Proposition:

Every sequence of real numbers has a monotone subsequence.

Proof:

Suppose the sequence $\{x_n\}$ has no monotone increasing subsequence, we will show that it has a monotone decreasing sequence. The sequence $\{x_n\}$ must have a first term, say x_{n_1} such that all subsequent terms are smaller

$n > n_1$ means that n comes after n_1, $\Rightarrow x_n < x_{n_1}$.

Otherwise, $\{x_n\}$ would have a monotone increasing subsequence.

Similarly, the remaining sequence $\{x_{n_2}, x_{n_3} ....\}$ it must have some first term.

Let first term of remaining sequence is x_{n_2} , Now this x_{n_2} is less than x_{n_1} , $x_{n_2} < x_{n_1}$.

Now we take the remaining sequence $\{x_{n_3} ....\}$, whose first term is x_{n_3} , now this $x_{n_3} < x_{n_2}$.

Hence this process will continue $x_{n_1} > x_{n_2} > x_{n_3 ....}$,

and is a monotonic decreasing subsequence.

We have proved that every sequence of Real numbers has a monotone subsequence.

Now using lemma a, b and proposition we have a theorem.

Theorem:

$\mathbb{R}$ is a completer metric space, i.e., every Cauchy sequence of real numbers converges.

Proof:

Let $\{x_n\}$ be a Cauchy sequence.

Remark a implies that $\{x_n\}$ is bounded. Now if the given Cauchy sequence is bounded then its subsequence is also bounded.

Every subsequence of $\{x_n\}$ is bounded.

Also $\{x_n\}$ has a monotone subsequence.Now $\{x_n\}$ is monotone as well as bounded.

Monotone Convergence Theorem:

If a sequence $\{x_n\}$ is monotone and bounded this implies that it is convergent.

This implies that subsequence is convergent. Now using remarks 2 if we have a Cauchy sequence has a subsequence is convergent than the original sequence will also convergent. $\{x_n\}$ is convergent. As this general sequence $\{x_n\}$ from $\mathbb{R}$ so, every Cauchy sequence from $\mathbb{R}$ is convergent which means that $\mathbb{R}$ is complete.

Module No. 32

Exmaples (Completeness):

- $\mathbb{R}^n$

Here we prove that $\mathbb{R}^n$ is complete

Example:

The Euclidean space $\mathbb{R}^n$ is complete.

Proof:

Let $\mathbb{R}^n$, the elements of $\mathbb{R}^n$ are n-tuples say

$$x = (a_1, a_2, \ldots\ldots\ldots. a_n) \quad ; \quad a_i, b_i \in \mathbb{R}$$
$$y = (b_1, b_2, \ldots\ldots\ldots. b_n)$$

The distance function in $\mathbb{R}^n$ is

$$d(x,y) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \ldots\ldots + (a_n - b_n)^2}$$

Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}^n$

$$x_m = (a_1^{(m)}, a_2^{(m)}, ...a_n^{(m)})$$

(i.e .

$$x_1 = (a_1^{(1)}, a_2^{(1)}, ...a_n^{(1)})$$
$$x_1 = (a_1^{(2)}, a_2^{(2)}, ...a_n^{(2)})$$
$$. \qquad)$$
$$.$$
$$.$$
$$x_r = (a_1^{(r)}, a_2^{(r)}, ...a_n^{(r)})$$

The distance function is

$$d(x_m, x_r) = \sqrt{(a_1^{(m)} - a_1^{(r)})^2 + (a_2^{(m)} - a_2^{(r)})^2 + ...+ (a_n^{(m)} - a_n^{(r)})^2} < \varepsilon \quad , \quad \forall m, r > N$$

Taking power two, we have

$$(a_1^{(m)} - a_1^{(r)})^2 + (a_2^{(m)} - a_2^{(r)})^2 + ...+ (a_n^{(m)} - a_n^{(r)})^2 < \varepsilon^2$$

$$(a_j^{(m)} - a_j^{(r)})^2 < \varepsilon^2,$$
$$\left|a_j^{(m)} - a_j^{(r)}\right| < \varepsilon, \quad \forall m, r > N, \quad j = 1, 2, ...n$$

For a fixed j $(a_j^{(1)} + a_j^{(2)} + ...)$ is a Cauchy sequence, this implies it is converging in $\mathbb{R}$ because $\mathbb{R}$ is a complete metric space.

$$\Rightarrow \qquad a_j^{(m)} \to a_j^{(r)}, \quad m \to \infty, \quad a_j \in \mathbb{R}, \text{ j=1,2,......,N}$$

$$a_1^{(m)} \to a_1$$
$$a_2^{(m)} \to a_2$$
$$.$$
$$.$$
$$a_n^{(m)} \to a_n$$

All these values $a_1, a_2, ...a_n$ called x, As $x = (a_1, a_2, ...a_n) \in \mathbb{R}^n$

$$\Rightarrow \qquad d(x_m, x) \le \varepsilon, \qquad r \to \infty, \quad x_m \to x$$

$\Rightarrow$ x is a limit of $\langle x_m \rangle$,

$\Rightarrow$ $\langle x_m \rangle$ was general element

$\Rightarrow$ $\mathbb{R}^n$ is completer

Module No. 33

Exmaples (Completeness):

- $\mathbb{C}[a,b]$

Here we prove that $\mathbb{C}[a,b]$ is complete metric space

Example:

The function space $\mathbb{C}[a,b]$ is complete; here $[a,b]$ is any given closed interval on $\mathbb{R}$.

Let (x_m) be any Cauchy sequence in $\mathbb{C}[a,b]$.

The metric space in $\mathbb{C}[a,b]$ is

$$d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)| \ , \quad \text{where [a,b]=J}$$

There is an N such that for all m,n>N

$$d(x_m, x_n) = \max_{t\in J} |x_m(t) - x_n(t)| < \varepsilon$$

Hence for any fixed $t = t_o \in J$

$$|x_m(t_o) - x_n(t_o)| < \varepsilon$$

$\Rightarrow$ $x_1(t_o), x_2(t_o),$............ is a Cauchy sequence of real numbers and $\mathbb{R}$ is complete.

$\Rightarrow$ sequence converges $x_m(t_o) \to x(t_o)$ as $m \to \infty$

In this way to each $t \in J$, a unique real number x(t). This definespointwise function on J.

Now we well show that $x(t) \in \mathbb{C}[a,b]$ *and* $x_m \to x$

$$\max_{t\in J} |x_m(t) - x(t)| \le \varepsilon$$

We are comparing with $\max_{t\in J} |x_m(t) - x_n(t)| < \varepsilon,$ *as* $n \to \infty$

$\Rightarrow$ for every $t \in J$ $|x_m(t) - x(t)| \le \varepsilon$

$\Rightarrow$ $x_m(t)$ converges to x(t) uniform;

If a sequence (x_m) of continuous function on [a,b] converges on [a,b] and the convergence is uniform on [a,b], then the limit function x is continuous on [a,b]

$\Rightarrow$ $x(t)$ is continuous on [a,b]

$\Rightarrow$ $x(t) \in \mathbb{C}[a,b]$.

Module No. 34

Exmaples (Completeness):

- l^{∞}

Here we prove that l^{∞} is complete metric space

Example:

The function space l^{∞} is complete; here $[a,b]$ is any given closed interval on $\mathbb{R}$.

Proof:

Let (x_m) be any Cauchy sequence in l^{∞} such that

In l^{∞} the elements are of the form

$$x = (a_1, a_2, ...) \quad \Rightarrow \quad |a_j| < c_x$$

$$y = (b_1, b_2, ...), \quad \Rightarrow \quad |b_j| < c_y$$

The distance or metric function is

$$d(x,y) = \sup_{j \in \mathbb{N}} |a_j - b_j|$$

Here $\quad x_m = (a_1^{(m)}, a_2^{(m)}, ...),\quad$ as

$$x_1 = (a_1^{(1)}, a_2^{(1)}, ...),$$
$$x_2 = (a_1^{(2)}, a_2^{(2)}, ...) \quad \text{so on}$$

For any q>0 , there exist $\mathbb{N}$ such that for all m,n> $\mathbb{N}$.

$$d(x_m, y_n) = \sup_{j \in \mathbb{N}} |a_j^{(m)} - b_j^{(n)}|$$

So, if $\sup < \varepsilon$ for a fixed j

$$|a_j^{(m)} - a_j^{(n)}| < \varepsilon \quad , \quad m,n \geq \mathbb{N}$$

$\Rightarrow$ for every fixed j, the sequence $(a_j^{(1)}, a_j^{(2)}, ...)$ is a Cauchy sequence of real numbers $\mathbb{R}$.

Since $\mathbb{R}$ is complete, $a_j^{(m)}$ is convergent in $\mathbb{R}$.

$$a_j^{(m)} \to a_j \in \mathbb{R} \qquad as \quad m \to \infty \quad for\ j = 1, 2, ...$$

For these infinite limits $a_1, a_2, ...$ such that $a_1^{(m)} \to a_1, \quad a_2^{(m)} \to a_2, ...$

We define $\quad x = (a_1, a_2, ...) \in \mathbb{R}$

We need to prove $\quad x = (a_1, a_2, ...) \in l^\infty$

$$\left|a_j^{(m)} - a_j^{(n)}\right| < \varepsilon$$

$\Rightarrow \quad \left|a_j^{(m)} - a_j\right| < \varepsilon \quad$ as $n \to \infty$. then $x_m \to x$

From above inequality,

$$d(x, y) = \sup\left|a_j^{(m)} - a_j\right| < \varepsilon$$

Which means $\quad x_m \to x$

Since $\quad x_m = (a_j^{(m)}) \in l^\infty$

$$\left|a_j^{(m)}\right| < k_m \quad \textit{for all } j$$

$$\begin{aligned} \left|a_j\right| &= \left|a_j - a_j^{(m)} + a_j^{(m)}\right| \\ &\le \left|a_j - a_j^{(m)}\right| + \left|a_j^{(m)}\right| \\ &< \varepsilon + k_m \end{aligned}$$

$\Rightarrow \quad a_j$ is bounded , $\quad x = \left|a_j\right| \in l^\infty$

MODULE NO. 35

EXMAPLES (COMPLETION OF METRIC SPACES):

- Space $\mathbb{Q}$
- Space of Polynomials
- Isometric mappings/spaces

here we prove that l^∞ is complete metric space

Isometric Mappings:

Let $\quad X = (X, d)$ and $\tilde{X} = (\tilde{X}, d)$ be metric spaces.

A mapping $T : X \to \tilde{X}$ is said to be isometric or isometry if T preserve distance.

Preseve distance mean after applying the mapping the distance is preserve, i.e. for all $x, y \in X$

$$\tilde{d}(T_x, T_y) = d(x, y)$$

Isometric Spaces:

The space X is said to be isometric with space $\tilde{X}$ if there exist a bijectiveisometry of X onto $\tilde{X}$.

X and $\tilde{X}$ are then called isometric spaces.

Theorem(Completion)

For a metric space $X=(X,d)$ there exists a complete metric space $\hat{X}=(\hat{X},d)$ which has a subspace W that is isometric with X and is dense in $\hat{X}$.

This space $\hat{X}$ is unique except for isometries, that is if $\tilde{X}$ is any complete metric space having a dense subspace $\tilde{W}$ isometric with X, then $\tilde{X}$ and $\hat{X}$ are isometric.

MODULE NO. 36

VECTOR SPACE

Definition:

A vector space (or linear space) over a field K is a nonempty set X of elements x,y,……………….(called vectors) together with two algebraic operations.

These operations are called vector addition and multiplication of vectors by scalars, that is, by elements of K.

Vector Addition associates with every ordered pair (x,y) of vectors a vector x+y, called the sum of x and y, in such a way that the following properties hold

Vector addition is commutative and associative.

There exists a vector 0, called the zero vector, and for every vector x there exists a vector –x, such that for all vectors.

Vector Space

$$x+0=x$$

$$x+(-x)=0$$

Multiplication by scalar associates with every vector x and scalar α a vector αx (also written $x\alpha$), called the product of α and x, in such a way that for all vectors x, y and scalar α, β we have

$$\alpha(\beta x)=(\alpha\beta)x \qquad \text{or} \qquad 1x=x$$

and the distributive laws hold.

MODULE NO. 37

EXAMPLES(VECTOR SPACE)

- ***Space*** $\mathbb{R}^n$
- ***Space*** $\mathbb{C}^n$
- ***Space*** $\mathbb{C}[a,b]$
- ***Space*** l^2

1. *Space* $\mathbb{R}^n$

$$x=(\xi_1,...........,\xi_n)\ ,\qquad \xi_i\in\mathbb{R}$$

$$y=(\eta_1,...........,\eta_n)\ ,\qquad \eta_i\in\mathbb{R}$$

Addition:

$$\boldsymbol{x+y}=(\xi_1+\eta_1,...........,\xi_n+\eta_n)$$

scalar Multiplication:

let α be a scalar then

$$\alpha x=(\alpha\xi_1,...........,\alpha\xi_n)$$

Now addition and scalar multiplication in $\mathbb{R}^n$ is a vector space.

2. *Space* $\mathbb{C}^n$

Addition:

Let $$x=(\xi_1,...........,\xi_n)\ ,\qquad \xi_i\in\mathbb{C}$$

$$y=(\eta_1,...........,\eta_n)\ ,\qquad \eta_i\in\mathbb{C}$$

Scalar Multiplication:

addition and scalar multiplication is same as in $\mathbb{R}^n$, so $\mathbb{C}^n$ is a vector space.

3. *Space* $\mathbb{C}[a,b]$

Let $x\in\mathbb{C}[a,b]$ *and* $y\in\mathbb{C}[a,b]$

where x and y are fucntions and operating on t

Addition:

$$(x+y)(t)=x(t)+y(t)$$

Scalar Multiplication:

$$(\alpha x)(t) = \alpha x(t)$$

So under addition and scalar multiplication $\mathbb{C}[a,b]$ is vector space over a field $\mathbb{R}$ or $\mathbb{C}$.

4. Space l^2:

In this space we have sequences,if $x \in l^2$ then x is a sequence, say

$$x = (\xi_1, \ldots\ldots\ldots\ldots, \xi_n) \; , \qquad x \in l^2$$

and $$y = (\eta_1, \ldots\ldots\ldots\ldots, \eta_n) \; , \qquad y \in l^2$$

Addition:

$$x + y = (\xi_1 + \eta_1, \ldots\ldots\ldots\ldots, \xi_n + \eta_n)$$

Scalar Multiplication:

$$\alpha x = (\alpha\xi_1, \ldots\ldots\ldots\ldots, \alpha\xi_n)$$

So under addition and scalar multiplication the space l^2 is vector space over a field $\mathbb{R}$ or $\mathbb{C}$

MODULE NO. 38

VECTOR SPACE

- ***Subspace***
- ***Basis of a Vector Space***

Subspace:

A subspace of a vector space X is a nonempty subset Y of X such that addition and scalar multiplication are closed in Y.

Hence T is itself a vector space, the two algebraic operations being those induced from X.

Two Types of subspaces

- Improper Subspace: If the span of a subspace is equal to that vector space ;
- Proper Subspace: If the span of a subspace is not equal to that vector space

Linear Combination

A linear combination of vectors $x_1, \ldots\ldots\ldots, x_n$ of a vector space X is an axpression of the form

$a_1x_1 + \ldots\ldots\ldots\ldots a_m x_m$ where the coefficients $a_1, \ldots\ldots\ldots\ldots, a_m$ are any scalars.

Span of a Set:

For any nonempty subset $M \subset X$ the set of all linear combinations of vectors of M is called the span of M.

Written as "span M".

Obviously, this is a subspace Y of X, and we say that Y is spanned or generated by M.

Linear Independence:

If two vectors have same direction and different in magnitude then on vector is multiple of other which means that one is dependent to other.

If two vectors have not same direction then one vector is independent to other.

Mathematically:

linearly independent.

$$c_1x_1 + c_2x_2 + ... + c_mx_m = 0$$

if and only if all constant are zero

$$c_1 = c_2 = ... = c_m = 0$$

We call $x_1, x_2, ..., x_m$ linearly independent.

linearly dependent.

If vectors are dependent then their coefficients are not equal to 0 as

let

$$x_1 = 2x_2$$

$$\Rightarrow \quad x_1 - 2x_2 = 0$$

Here coefficient $1 \neq 2 \neq 0$, so x_1 is dependent of x_2 .

Basis of a Vector Space:

As span of M is also a subspace, if the subspace (collection of vectors) is improper subspace(means span of M is equal to that vector space) and linearly independent(coefficients are equal to zero) then that particular subspace is a Basis of a Vector Space.

So, for basis the subspace have to improper subspace and linear independent.

Module No. 39

Vector Space

Dimension (definition):

The number of elements in subspace of a basis is called dimension of that vector space.

- ***Dimension***
  - i. ***Finite dimensional vector space***
  - ii. ***Infinite dimensional vector space***

Examples:

In $\mathbb{R}^n$ space

Elements of basis of $\mathbb{R}^n$ are $e_1, e_2, ..., e_n,$

$$e_1 = (1, 0................, 0)$$
$$e_2 = (0, 1................, 0)$$
$$.$$
$$.$$
$$e_n = (0, 0................, 1)$$

Sometimes it is called Canononical basis of $\mathbb{R}^n$ basis $\mathbb{R}^n$.

Similarly in $\mathbb{C}^n$ space n-dimension

$C[a,b]$ is infinite dimension vector space because there is no finite set which can span the set of function.

In l^2 space, there are sequences, this is also infinite dimensional vector space.

Result :

Every nonempty vector space $X \neq \{0\}$ has a basis.

Theorem:

Let X be an n dimensional vector space. Then any proper subspace Y of X has dimension less than n.

Proof:

If n=0 this implies X={0}

There is no proper subspace. Hence we can’t continue.

If dimension of Y is zero.

Dim Y=0

and $X \neq Y$ Y={0}

{Y is proper subspace of X}

dim Y < dim X

suppose dim Y=n

$\Rightarrow$ Y would have a basis of n elements.

$\Rightarrow$ that basis would also be a basis for X, as element in basis are same, they span and linearly independent.

dim X=n when basis are same then X=Y

but it is contradict to our supposition as we suppose that Y is a proper subset of Xi.e $Y \subset X$ which means X and Y are not equal.

$\Rightarrow$ any linearly independent set of vectors in Y must have less elements then n.

$\Rightarrow$ dim Y< n

That we have to prove.

Module No. 40

Normed Space, Banach Space

- ***Norm***
- ***Normed Space***
- ***Banach Space***

Norm (definition):

A norm on a (real or complex) vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by $\|x\|$.

(This like the notation of mod but it has two vertical lines on left and right side.)

It has following properties:

i): $\|x\| \geq 0$ (N1)

ii): $\|x\| = 0 \Leftrightarrow x = 0$ (N2)

Norm is equal to zero if and only if x=0. Length is always positive or zero but not –ve.

iii): $\|\alpha x\| = |\alpha|\|x\|$ (N3)

if we multiply the length of norm with α (any number) then it will increase the length of Norm α times.

iv): $\|x+y\| \le \|x\| + \|y\|$ (N4) triangular inequality

if x and y are two vectors then their sum of Norms is equal to individual sum of their norm.

Norm metric:

A norm on X defines a metric d on X which is given by

$$d(x,y) = \|x-y\| \qquad \textit{where } x,y \in X$$

and is called the metric induced by the norm as this metric depend on norm so we call it metric induced by norm.

from the property $\|x+y\| \le \|x\| + \|y\|$

we can write $\big|\|y\| - \|x\|\big| \le \|y-x\|$

The norm is real valued function so it is continuous function. Continuous function mean if we define norm on x then it will give us the value of norm x as

$$x \to \|x\|$$

and this mapping is continuous and is mapped $(X, \|.\|) \to \mathbb{R}$.

Norm is always a continuous function.

Norm Space:

A normed space X is a vector space with a norm defined on it.

A normed space is denoted by $(X, \|.\|)$ or simply by X.

Banach Space:

A Banach space is a complete normed space, (Complete in the metric defined by the norm).

Module No. 41

Examples (Normed Space)

- ***Euclidean Space*** $\mathbb{R}^n$
- ***Unitary Space*** $\mathbb{C}^n$
- ***Space*** l^p
- ***Space*** l^∞
- ***Space*** $\mathbb{C}[a,b]$

Euclidean Space $\mathbb{R}^n$

This is a metric space and elements in $\mathbb{R}^n$ is in n-tuples form,

$$x=(\xi_1,\xi_2,..........,\xi_n) \qquad \text{where } \xi_i\in\mathbb{R}\ , \quad x\in X$$

$$\|x\|=\sqrt{|\xi_1|^2+........+|\xi_n|^2}$$

$$=\left(\sum_{i=1}^{n}|\xi_i|^2\right)^{\frac{1}{2}}$$

$$y=(\eta_1,\eta_2,..........,\eta_n) \qquad \text{where } \eta_i\in\mathbb{R}$$

The distance function $\qquad d(x,y)=\|x-y\|$

$$d(x,y)=\sqrt{|\xi_1-\eta_1|^2+........+|\xi_n-\eta_n|^2}$$

Unitary Space $\mathbb{C}^n$

This is a metric space and elements in $\mathbb{C}^n$ is in n-tuples form,

$$x=(\xi_1,\xi_2,..........,\xi_n) \qquad \text{where } \xi_i\in\mathbb{C}\ , \quad x\in X$$

$$\|x\|=\sqrt{|\xi_1|^2+........+|\xi_n|^2}$$

$$=\left(\sum_{i=1}^{n}|\xi_i|^2\right)^{\frac{1}{2}}$$

$$y=(\eta_1,\eta_2,..........,\eta_n) \qquad \text{where } \eta_i\in\mathbb{C}$$

The distance function

$$d(x,y)=\|x-y\|$$

$$=\sqrt{|\xi_1-\eta_1|^2+........+|\xi_n-\eta_n|^2}$$

Space l^p

$$x = (\xi_1, \xi_2, ...) \ ,$$

$$y = (\eta_1, \eta_2, ...)$$

$$\|x\| = \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{\frac{1}{p}}$$

The distance function $$d(x, y) = \|x - y\|$$

$$= \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p \right)^{\frac{1}{p}}$$

Space l^∞

$$x \in l^\infty$$

The metric is given by $$\|x\| = \sup_j |\xi_j|$$

Space $\mathbb{C}[a,b]$:

This is a space of all real valued continuous functions defined on closed interval [a,b]

The norm of the function is $\|x\| = \max_{t \in J} |x(t)|$, with this metric space it is a norm space.

Module No. 42

Unit Sphere

- ***Unit Sphere***

Unit Sphere

The sphere with center 0 and radius 1, S(0;1), this we define in $\mathbb{R}^2$, but in any metric space

Those points from x whose norm is 1. $\{x \in X \mid \|x\| = 1\}$,

In a normed space X is called the unit sphere. In norm space the collection of all those points which are equal to 1 is called a Unit Sphere.

Let $\|x\|$ be a norm, and space is $\mathbb{R}^2$, the element in $\mathbb{R}^2$ are $x = (\xi_1, \xi_2)$

Example:

(i.e x=(2,-3), $\|x\| = |2| + |-3| = 2 + 3 = 5$)

$$\|x\| = |\xi_1| + |\xi_2|$$

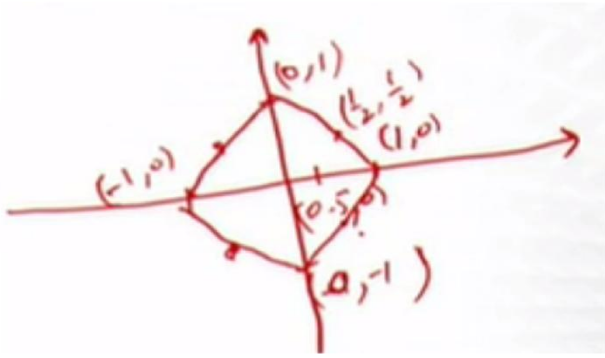
Norm of (1,0) is 1, and similarly norm of point (0,1) is also 1.

Similarly for Norm of (-1,0) is 1, and also norm of point (0,-1) is also 1.

This norm is according to function $\|x\| = |\xi_1| + |\xi_2|$,

for x=(1,0)

$$\|(1,0)\| = 1 + 0 = 1$$



Another Example.

The norm is defined as $\|x\| = |\xi_1^2 + \xi_2^2|^{1/2}$ similar to equation of circle.

In unit sphere we have the condition that norm of x is 1, $\|x\| = 1$

$$1 = \left(\xi_1^2 + \xi_2^2\right)^{1/2}$$

$$1 = \xi_1^2 + \xi_2^2$$

Another Example.

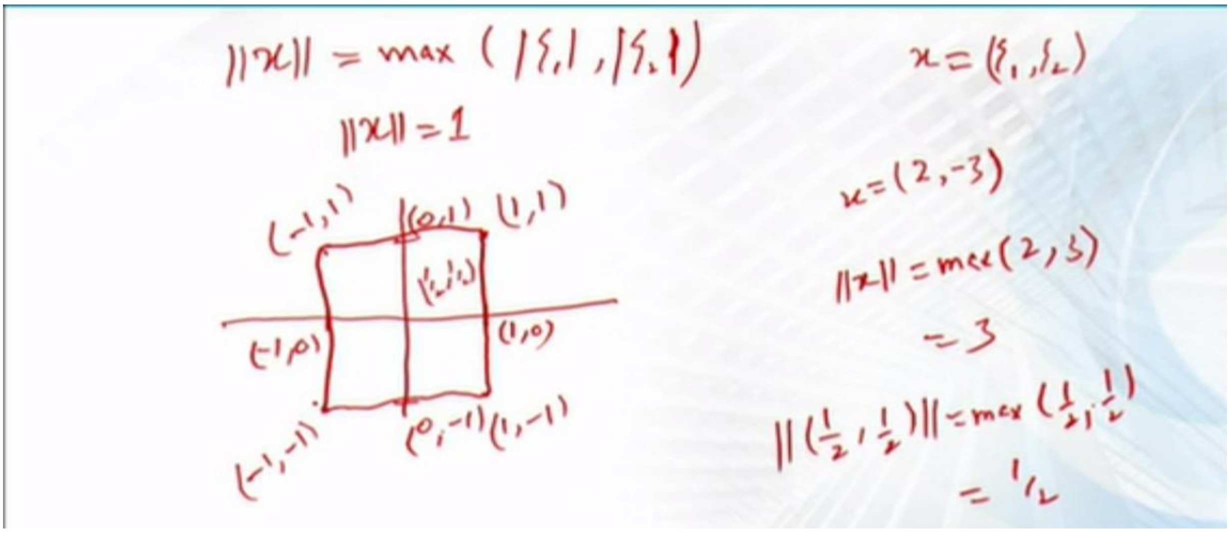
The norm is defined as $\|x\| = \max\left(|\xi_1|, |\xi_2|\right)$ similar to equation of circle.

Suppose $x \in \mathbb{R}^2$, such that $x = (\xi_1, \xi_2)$,

Let say $x = (2, -3)$

According to given condition,

$$\|x\| = \max(|2|, |-3|) = \max(2, 3) = 3$$



Here the sphere is a square.

We have discussed only $\mathbb{R}^2$ norm space and also its sketches, but it can be $\mathbb{R}^n$, $\mathbb{C}^n$ or any other space like space of functions C[a,b].

When we defined different norm then the shape of the unit sphere is depends on the norm define.

Module No. 43

Normed Spaces

- ***Subspace***

Subspace (definition)

A subspace Y of a normed space X is a subspace of X considered as a vector space, with the norm obtained by restricting the norm on X to the subset Y.

This norm on Y is said to be induced by the norm on X.

If Y is closed in X, then Y is called a closed subspace of X.

Subspace l^p :

A subspace Y of a Banach space X is a subspace of X considered as a normed space.

Hence we do not require Y to be complete.

Theorem :

A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.

Convergence in Normed Spaces.

The metric function is $d(x,y) = \|x - y\|$

For convergence we define as

i): A sequence (x_n) in a normed space X is convergent if X contains an x such that

$$\lim_{n\to\infty}\|x_n - x\| = 0$$

$$x_n \to x \ , \qquad x \ \lim it \ of \ (x_n)$$

Now this definition define for Cauchy sequence

ii): A sequence (x_n) in a normed space X is a Cauchy sequence if for every $\varepsilon > 0$ there is an N such that

$$\|x_m - x_n\| < \varepsilon \quad for \ all \ m,n > N$$

MODULE NO. 44

NORMED SPACES

- ***Convergence of Infinite Series***
- ***Basis in Normed Spaces***
- ***Completion in Normed Spaces (Theorem***

Convergence of Infinite Series

A sequence (x_k) is associate with a sequence of partial sum s_n .

$$s_n = x_1 + x_2 + \ldots\ldots\ldots\ldots\ldots x_n \qquad \text{where n=1,..........,}$$

If s_n convergent, $s_n \to s$, then

$$\sum_{i=1}^{\infty} x_i = x_1 + x_2 + \ldots\ldots\ldots\ldots \text{is also convergent.}$$

if $\|s_n - s\| \to 0$ then $s_n \to s$.

If we have following series

$$\|x_1\| + \|x_2\| + \ldots\ldots\ldots\ldots \text{converges,}$$

$$\Rightarrow \qquad \sum_{i=1}^{\infty} x_i \quad \text{absolutely convergent.}$$

So , we have transform the convergence and absolutely convergence in term of norm.

Basis:

In a normed space X is a Cauchy sequence if for every $\varepsilon > 0$ there is an N such that

Elements of basis of $\mathbb{R}^n$ are $e_1, e_2, ..., e_n$, such that

$$\begin{aligned} e_1 &= (1, 0................, 0) \\ e_2 &= (0, 1................, 0) \\ &. \\ &. \\ e_1 &= (0, 0................, 1) \end{aligned}$$

Sometimes it is called Canononical basis of $\mathbb{R}^n$.

Elements are spanning and are linearly independent.

Any element $x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$ in the form of norm is

$$\|x - \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n\| \to 0$$

and if this condition is hold then we say that it is a basis in the norm space.

Theorem Completion:

Let $X = (x, \|.\|)$ be a normed space then there is a Banach space $\hat{X}$ and an isometry A from X onto a subspace W of $\hat{X}$ which is dense in $\hat{X}$.

The space $\hat{X}$ is unique, except for isometries.

Module No. 45

Fininte Dimensional Normed Spaces

- ***Lemma (Linear Combination)***

Lemma

Let $\{x_1, ..., x_n\}$ be a linearly independent set of vectors in a normed space X (of any dimension).

Then there is a number c>0 such that for every choice of scalars $\alpha_1, ..., \alpha_n$ we have

$$\|\alpha_1 x_1 + ... + \alpha_n x_n\| \geq c\left(|\alpha_1| + ... + |\alpha_n|\right)$$

Proof:

$$S=|\alpha_1|+............+|\alpha_n|=\left(|\alpha_1|+............+|\alpha_n|\right)$$

$$\|\alpha_1 x_1+.........+\alpha_n x_n\|\geq c\left(|\alpha_1|+............+|\alpha_n|\right),\quad \text{where } c>0$$

Now we have two cases:

i): If S=0

It means $|\alpha_i|=0 \quad \Rightarrow \quad \alpha_i=0 \;\; for \;\; all \;\; i=1,.......,n$

ii): If S>0

$$\|\alpha_1 x_1+.........+\alpha_n x_n\|\geq cS \qquad \text{as S>0 so we can divide it}$$

$$\frac{\|\alpha_1 x_1+.........+\alpha_n x_n\|}{S}\geq c$$

$$\left\|\frac{\alpha_1 x_1}{S}+.........+\frac{\alpha_n x_n}{S}\right\|\geq c$$

$$\|\beta_1 x_1+.........+\beta_n x_n\|\geq c$$

If we define $\beta_i=\frac{\gamma_i}{S}$ then from S we have

$$\frac{|\alpha_1|+............+|\alpha_n|}{S}=1$$

$$\frac{|\alpha_1|}{S}+............+\frac{|\alpha_n|}{S}=1$$

$$\sum_{i=1}^{n}|\beta_i|=1$$

To prove $\|\beta_1 x_1+.........+\beta_n x_n\|\geq c$ We have to prove $\sum_{i=1}^{n}|\beta|_i=1$

We do this by contradiction.

Suppose it is false that $\|\beta_1 x_1+.........+\beta_n x_n\|\geq c$

So we can find a sequence $\langle y_m\rangle$ of vectors $y_m=\beta_1^{(m)}x_1+......+\beta_n^{(m)}x_n$ such that

$$\|y_m\|\to 0 \qquad as \quad m\to\infty$$

as we suppose that $\|\beta_1 x_1+.........+\beta_n x_n\|\leq c$

so we will find values smaller than c.

$$\sum_{j=1}^{n}\left|\beta_j^{(m)}\right| = 1 \qquad \Rightarrow \qquad \left|\beta_j^{(m)}\right| \le 1$$

Thus for each fixed $\left\langle \beta_j^{(m)} \right\rangle = \left(\beta_j^{(1)} + \beta_n^{(2)} ....\right)$ is bounded.

By Bolzano-Weisrtren theorem has a convergent subspace.

For all j=1,2,…………,n

$\Rightarrow \quad \left\langle \beta_1^{(m)} \right\rangle$ has converged subsequence say $\gamma_1^{(m)}$ converges to β_1

$$y_m = \beta_1^{(m)} x_1 + ... + \beta_n^{(m)} x_n$$
$$y_{m,1} = \gamma_1^{(m)} x_1 + ... + \beta_n^{(m)} x_n$$

$$\beta_2^{(m)} \to \gamma_2^{(m)} \to \beta_2$$

This is also true for

$$y_{m,2} = \gamma_2^{(m)} x + \gamma_2^{(m)} x + , ..., + \beta_n^{(m)} x_n$$

..

..

$$y_{m,n} = \sum_{j=1}^{n} \gamma_j^{(m)} x_j \quad \text{for all} \quad \sum_{j=1}^{n}\left|\gamma_j^{(m)}\right| = 1.$$

$$\gamma_j^{(m)} \to \beta_j \qquad \text{as} \qquad m \to \infty$$

$$y_{m,n} \to y = \sum_{j=1}^{n} \beta_j x_j \quad \text{with} \sum \beta_j = 1 \quad \Rightarrow \quad \text{all } \beta_j \ne 0$$

Using the linearly independence condition $\{x_1, ..., x_n\}$ are linearly independent.

This implies $\beta_1 x_1 + ... + \beta_n x_n \ne 0 \qquad \Rightarrow \qquad y \ne 0$

Now $y_{m,n} \to y \qquad \left\|y_{m,n}\right\| \to \|y\|$ where $\|.\|$ is continuous

Hence $\|y_m\| \to 0$ and $y_{m,n}$ is a subsequence of y_m but we have supposed that $y \ne 0$

$$\|y_{m,n}\| \to 0 = \|y\| \to y = 0 \qquad \text{N2 proved}$$

Hence proved

Module No. 46

Normed Spaces

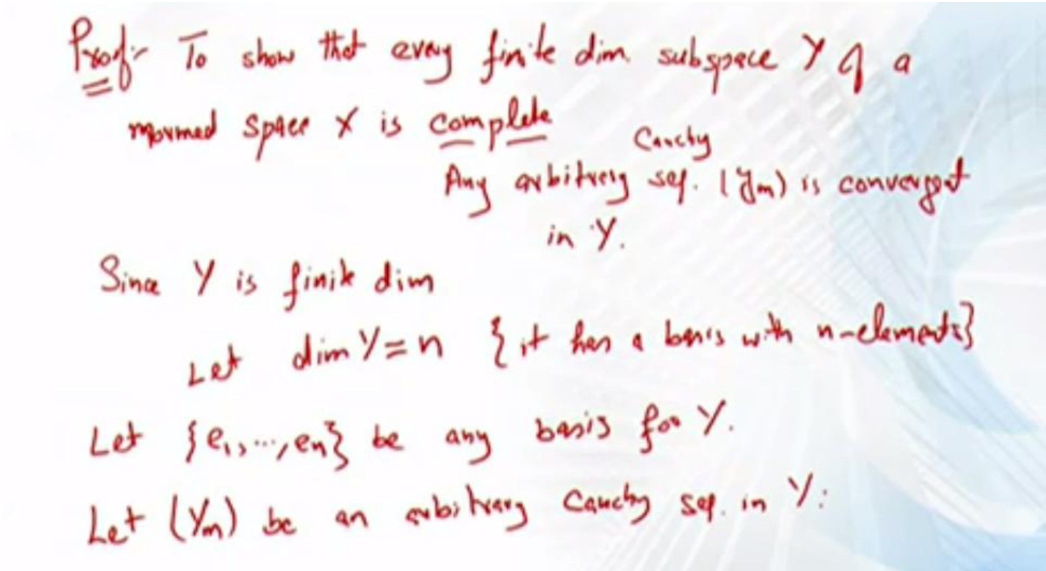
- ***Theorem (Completeness)***

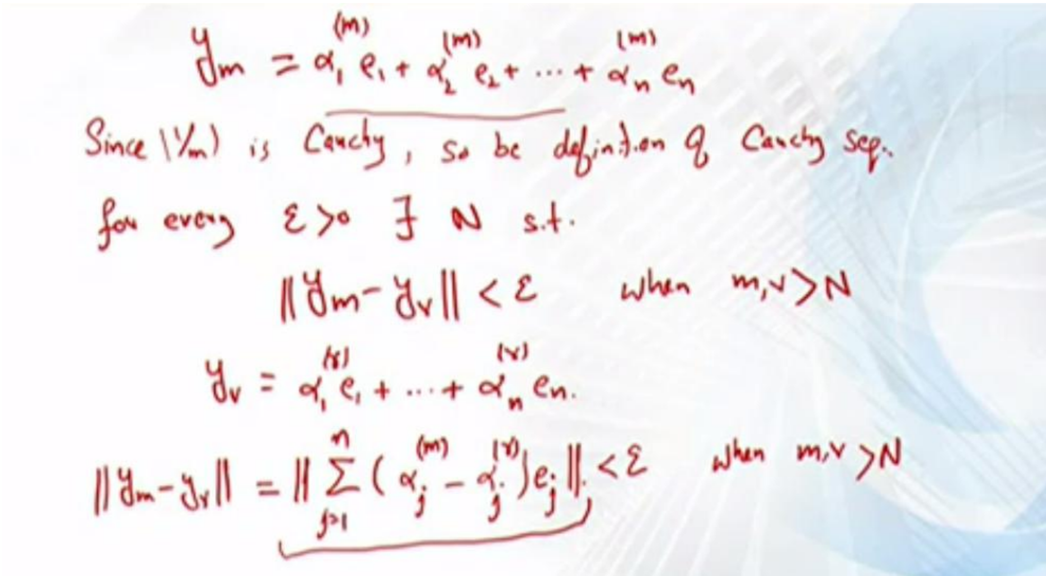
Theorem

Every finite dimensional subspace Y of a normed space X is complete. In particular, every finite dimensional normed space is complete.

Proof:

Prove it yourself:





$$\varepsilon > \left\| \sum_{j=1}^{n} (\alpha_j^{(m)} - \alpha_j^{(r)}) e_j \right\| \geq c \sum_{j=1}^{n} |\alpha_j^{(m)} - \alpha_j^{(r)}| \quad \{\because \text{ by Lemma } 45$$

$$\Rightarrow \quad c \sum_{j=1}^{n} |\alpha_j^{(m)} - \alpha_j^{(r)}| < \varepsilon \qquad c > 0$$

$$\Rightarrow \quad \sum_{j=1}^{n} |\alpha_j^{(m)} - \alpha_j^{(r)}| < \frac{\varepsilon}{c} \qquad m, r > N$$

For fixed j $\quad |\alpha_j^{(m)} - \alpha_j^{(r)}| < \frac{\varepsilon}{c} \Rightarrow$ A Cauchy seq. of $\mathbb{R}$ or $\mathbb{C}$.

Hence it is convergent. Let α_j denote the limit of each seq.. So for these 'n' sequences, let $\alpha_1, \ldots, \alpha_n$ be the limit.

Set $\quad y = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$

Also

$$\|y_m - y\| = \left\| \sum_{j=1}^{n} (\alpha_j^{(m)} - \alpha_j) e_j \right\|$$

$$\leq \sum_{j=1}^{n} |\alpha_j^{(m)} - \alpha_j| \, \|e_j\|$$

$\Rightarrow \alpha_j^{(m)} \longrightarrow \alpha_j$ as $m \to \infty$

$\|y_m - y\| \to 0$

$y_m \to y \in Y$

(y_m) is convergent to $y \in Y$

since (y_m) is an arbitrary, $\Rightarrow$

Y is complete.

Module No. 47

Normed Spaces

- ***Theorem (Closedness)***

As we have already proved that every finite dimensional subspace is complete and we also know that a subspace is complete if and only if it is closed.

Theorem

Every finite dimensional subspace Y of a normed space X is closed in X. This result is true for finite dimensional subspace but for infinite space it is not true.

Infinite dimensional subspaces are like C[0,1], l^2 are infinite dimensional normed space which are not closed space. We use dense, limit points to prove this.

Module No. 48

Normed Spaces

- ***Theorem (Equivalent Norms)***

Definition

A norm $\|.\|$ on a vector space X is said to be equivalent to a norm $\|.\|_o$ on X if there are positive numbers a and b such that for all $x \in X$ we have

$$a\|x\|_o \leq \|x\| \leq \|x\|_o b$$

This property should hold for every element x of vector space X.($a\|x\|_o$ read a times x not norm).

If we prove about condition then we say that these two norms are equivalent.

Equivalent norms on X define the same topology for X.

Theorem (Equivalent norms)

One finite dimensional vector space X, any norm $\|.\|$ is equivalent to any other norm $\|.\|_o$

Proof:

Proof: $\|\cdot\| \cong \|\cdot\|_0$

$\forall\ x \in X,\ \exists\ a, b$ s.t.

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0$$

Let $\dim X = n$, $\{e_1, \cdots, e_n\}$ be any basis of X.

Then every $x \in X$ has a unique representation

$$x = \alpha_1 e_1 + \cdots + \alpha_n e_n \quad \text{——— (1)}$$

Now by Lemma (45) $\exists\ c > 0$ s.t.

$$\|x\| \geq c(|\alpha_1| + \cdots + |\alpha_n|) = c\sum_{j=1}^{n}|\alpha_j| \quad \text{— (2)}$$

by applying $\|\cdot\|_0$ we get by (1)

$$\|x\|_0 = \|\alpha_1 e_1 + \cdots + \alpha_n e_n\|_0$$

$$\leq \sum_{j=1}^{n}|\alpha_j|\|e_j\|$$

Let

$$K = \max_{j=1,\cdots,n}\|e_j\|$$

$$\leq K\sum_{j=1}^{n}|\alpha_j| \Rightarrow \frac{\|x\|_0}{K} \leq \sum_{j=1}^{n}|\alpha_j|$$

$$\|x\| \geq c\sum_{j=1}^{n}|\alpha_j| \geq c\frac{\|x\|_0}{K}$$

$$\Rightarrow \|x\| \geq \frac{c}{K}\|x\|_0 \Rightarrow \|x\| \geq a\|x\|_0, \quad a = \frac{c}{K}$$

by interchanging both norms $\Rightarrow \|x\|_0 \geq \frac{1}{b}\|x\|$

$$\Rightarrow b\|x\|_0 \geq \|x\|$$

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0 \quad \text{required}$$

Module No. 49

Compactness and finite Dimension

- ***Lemma (Compactness)***

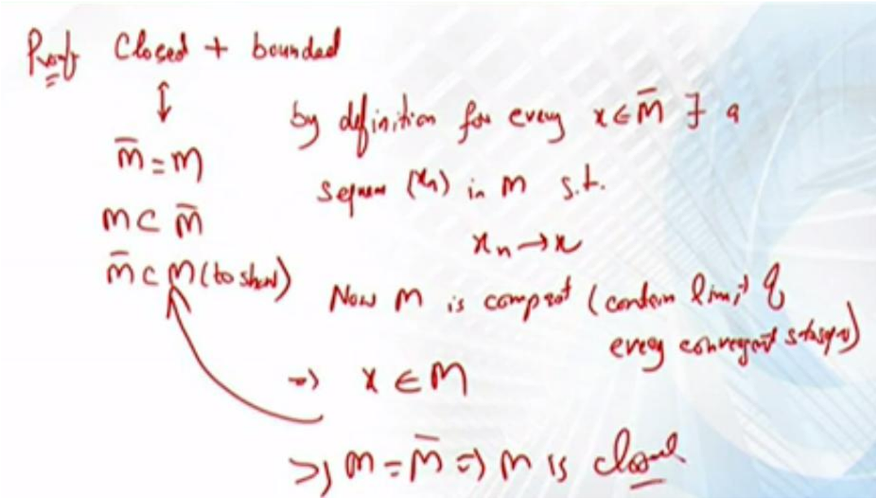
Definition

A metric space X is said to be compact if every sequence in X has a convergent subsequence. A subset M of X is said to be compact if M is compact considered as a subspace of X, that is if every sequence in M has a convergent subsequence whose limit is an element of M.

Lemma (Compactness)

A compact subset M of a metric space is closed and bounded.

For close of M we show that $\bar{M} = M$. Now we have to prove closed and bounded



Proof Closed + bounded

$\updownarrow$

$\bar{M} = M$

$M \subset \bar{M}$

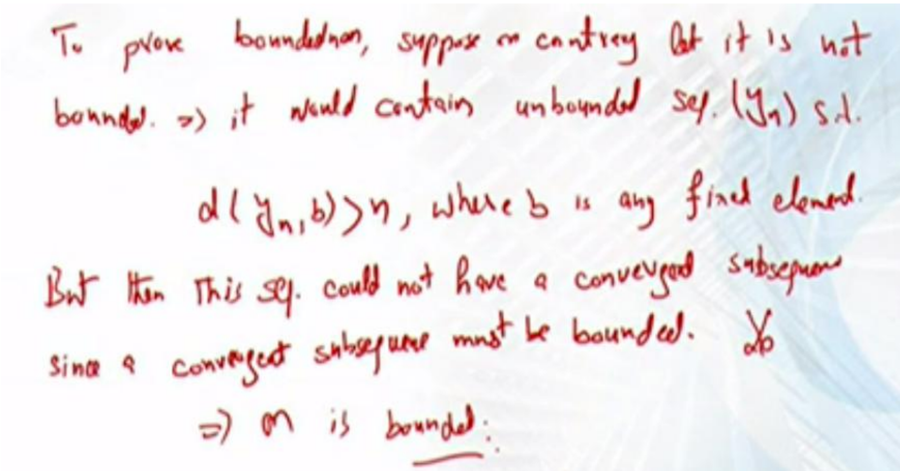
$\bar{M} \subset M$ (to show)

By definition for every $x \in \bar{M}$ $\exists$ a sequence (x_n) in M s.t. $x_n \to x$

Now M is compact (contain limit & every convergent subseq)

$\Rightarrow x \in M$

$\Rightarrow M = \bar{M} \Rightarrow$ M is closed



To prove boundedness, suppose on contrary that it is not bounded. $\Rightarrow$ it would contain unbounded seq. (y_n) s.t.

$d(y_n, b) > n$, where b is any fixed element.

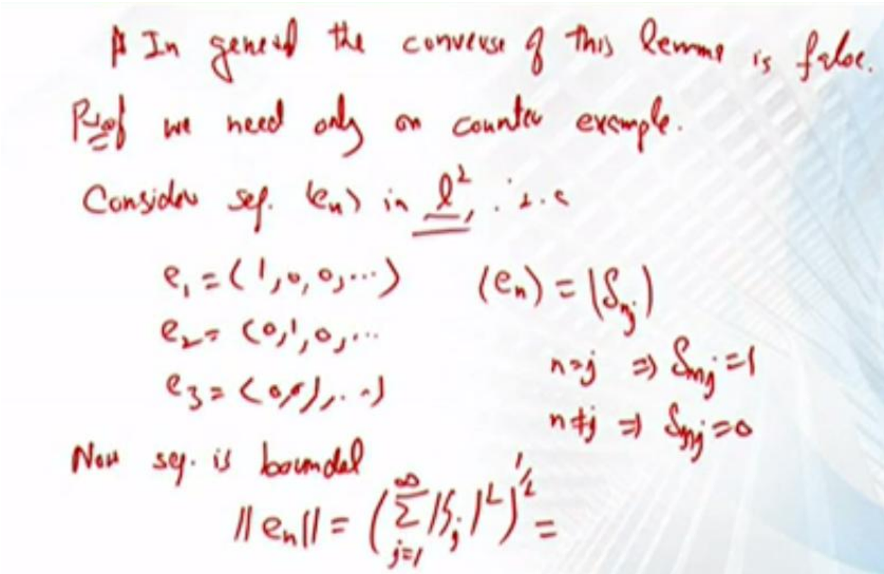
But then this seq. could not have a convergent subsequence since a convergent subsequence must be bounded. ※

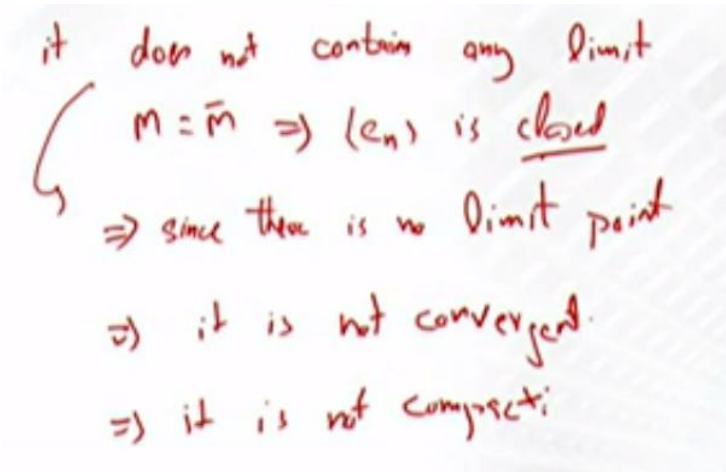
$\Rightarrow$ M is bounded.

Conversely

In general the converse of this lemma is false.

Proof


In general the converse of this lemma is false.
Proof we need only on counter example.
Consider seq. (e_n) in ℓ^2, i.e
$e_1 = (1, 0, 0, \dots)$
$e_2 = (0, 1, 0, \dots$
$e_3 = (0, 0, 1, \dots)$
$(e_n) = (\delta_{nj})$
$n = j \Rightarrow \delta_{nj} = 1$
$n \neq j \Rightarrow \delta_{nj} = 0$
Now seq. is bounded
$\|e_n\| = \left(\sum_{j=1}^{\infty} |\delta_j|^2\right)^{1/2} =$


it does not contain any limit
$M = \bar{M} \Rightarrow (e_n)$ is closed
$\Rightarrow$ since there is no limit point
$\Rightarrow$ it is not convergent.
$\Rightarrow$ it is not compact.

The above example is closed and bounded but not compact so the converse is false that a closed and bounded metric space is not compact.

Module No. 50

Theorem (Compactness)

- ***Lemma (Compactness)***

In case of finite dimensional subset M is a compact set if and only if it is closed and bounded. Here we prove both directions.

Theorem (Compactness)

In a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded.

Proof:

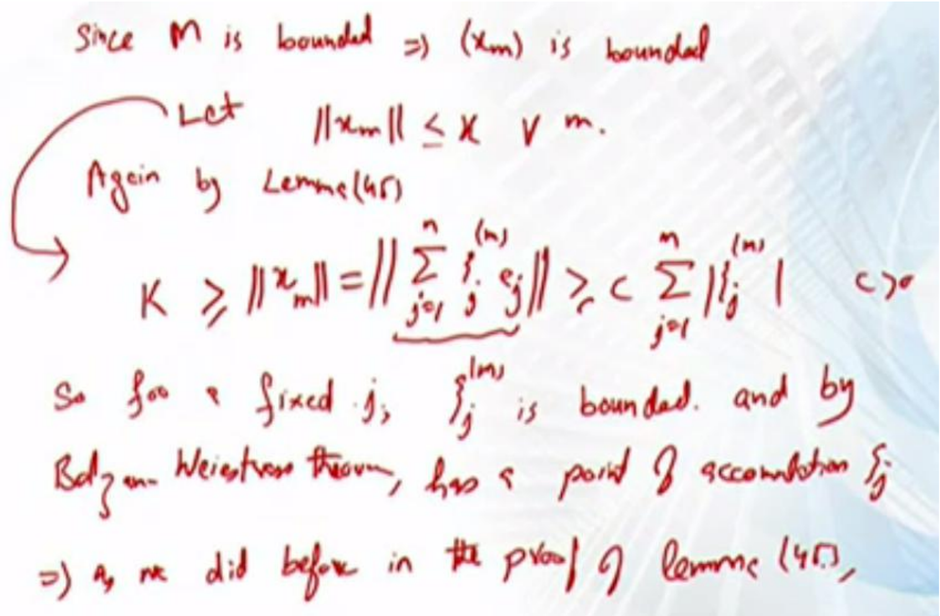
We have to prove that compact implies closed and bounded. This we have proved already. Now we prove the converse only. We have to prove only compact (for finite dimensional only).

Let M be closed and bounded, we need to show that M is compact (i.e. every sequence in M has a subseq which converges in M).

Let it is finite dimension so, say n, as $\dim X = n$ and $\{e_1 + \dots\dots\dots\dots + e_n\}$ be a basis for X

Let $\langle x_m \rangle$ be any sequence in M.

$$\Rightarrow \quad x_m = \xi_1^{(m)} e_1 + \dots\dots + \xi_n^{(m)} e_n$$



Since M is bounded ⇒ (x_m) is bounded

Let $\|x_m\| \le K \ \forall \ m$.

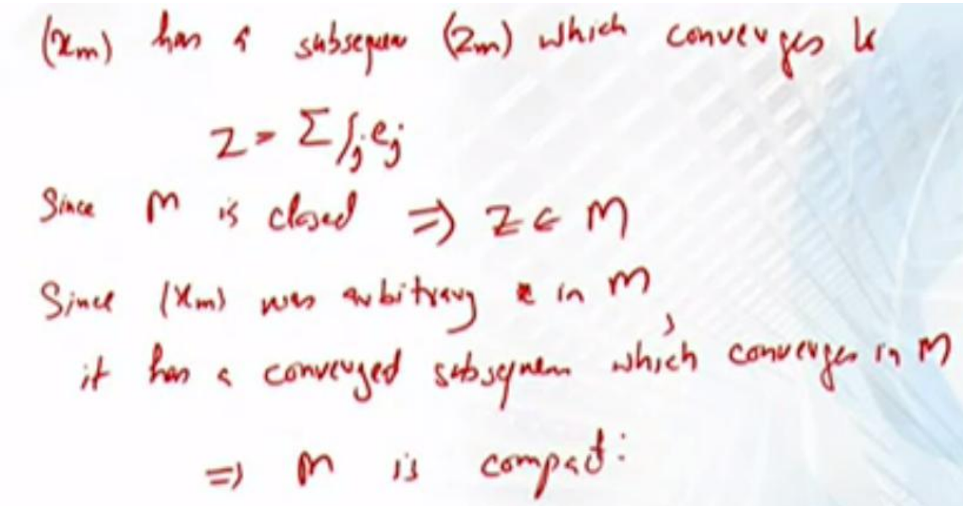
Again by Lemma (45)

$$K \ge \|x_m\| = \left\| \sum_{j=1}^{n} \xi_j^{(m)} e_j \right\| \ge c \sum_{j=1}^{n} |\xi_j^{(m)}| \qquad c>0$$

So for a fixed j, $\xi_j^{(m)}$ is bounded and by Bolzano Weierstrass theorem, has a point of accumulation ξ_j

⇒ As we did before in the proof of lemma (45),

Lemma 45 lecture,



(x_m) has a subsequence (z_m) which converges to

$$z = \sum \xi_j e_j$$

Since M is closed ⇒ $z \in M$

Since (x_m) was arbitrary seq in M, it has a converged subsequence which converges in M

⇒ M is compact.

Module No. 51

Compactness and Finite Dimension

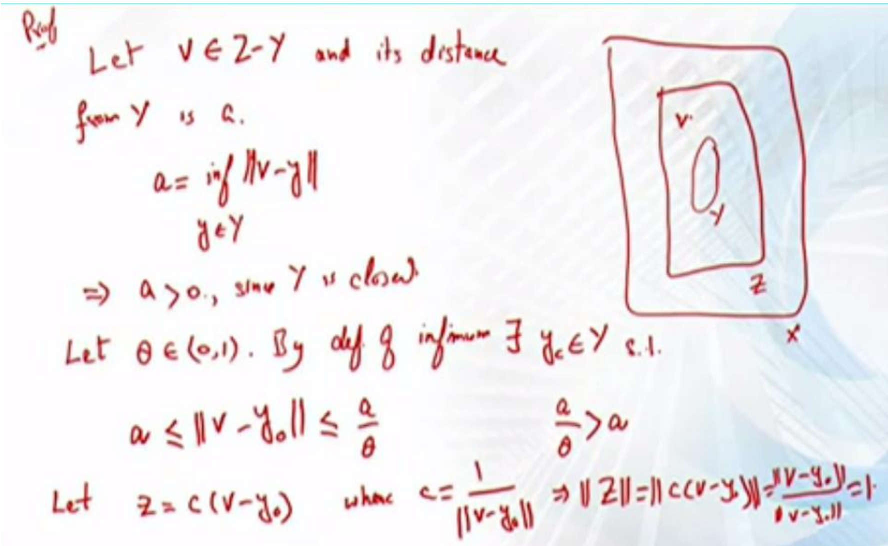
- ***F. Riesz's Lemma***

F. Riesz's Lemma

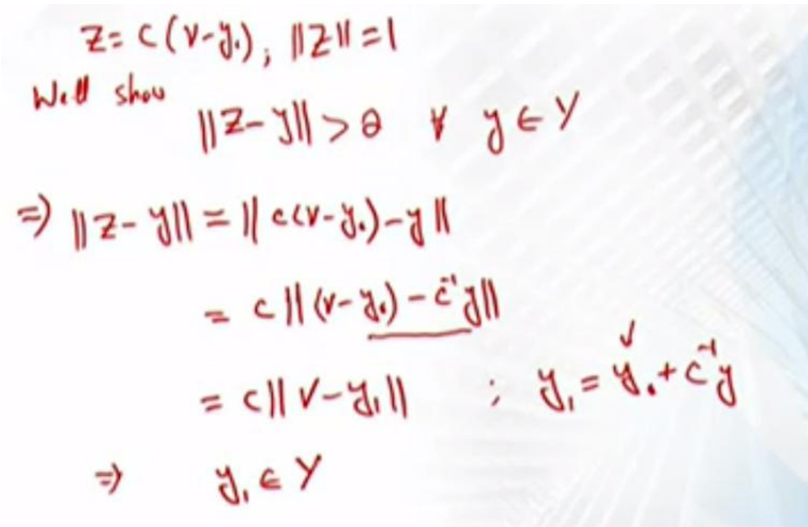
Let Y and Z be subspaces of a normed space X (of any dimension), and suppose that Y is closed and is a proper subset of Z, then for every real number θ in the interval (0,1) there is a $z \in Z$ such that

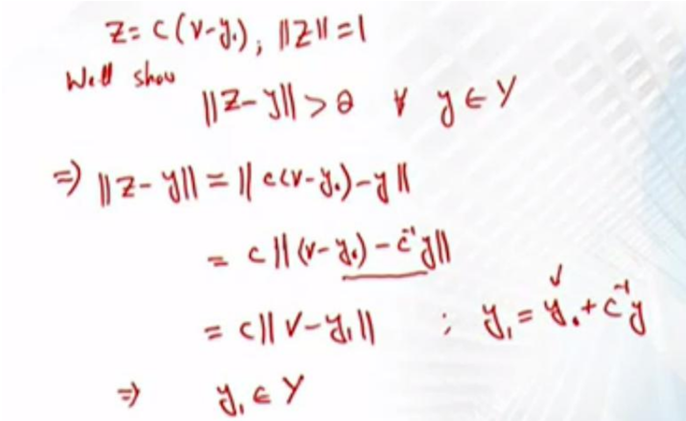
$$\|z\| = 1$$
$$\|z - y\| \geq \theta \quad for\ all\ y \in Y$$

First part $\|z\| = 1$ we prove as



Second part: $\|z - y\| \geq \theta \quad for\ all\ y \in Y$





$z = c(v - y_0)$; $\|z\| = 1$

We'll show $\|z - y\| > \theta \quad \forall\, y \in Y$

$\Rightarrow \|z - y\| = \|c(v - y_0) - y\|$

$= c\|(v - y_0) - c^{-1}y\|$

$= c\|v - y_1\| \quad ; \; y_1 = y_0 + c^{-1}y$

$\Rightarrow \quad y_1 \in Y$

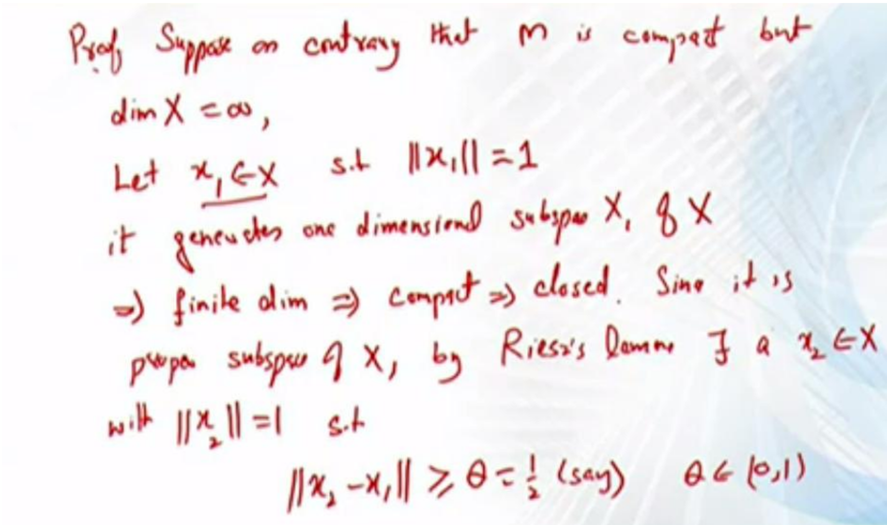
Module No. 52

Finite Dimension

- ***Theorem (Finite Dimension)***

Theorem

If a normed space X has the property that the closed unit ball $M = \{x \mid \|x\| \le 1\}$ is compact, then X is finite dimensional.



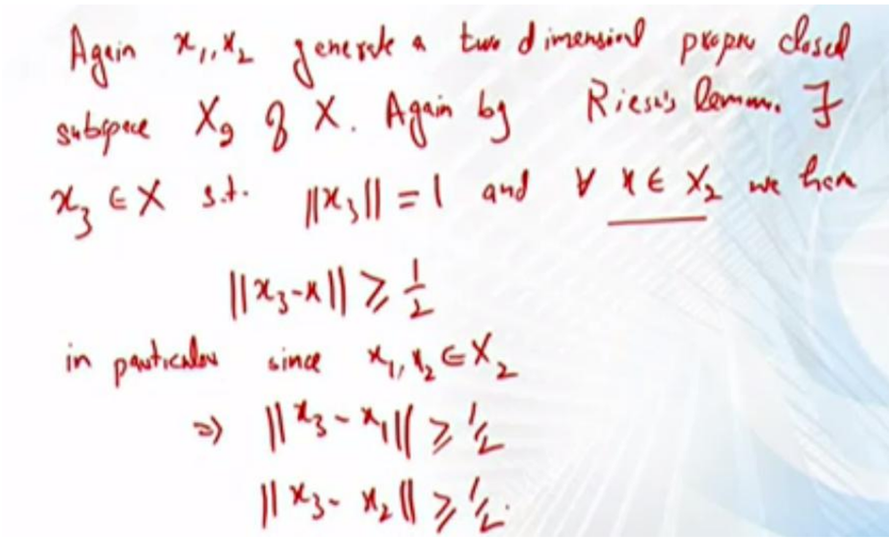
Proof Suppose on contrary that M is compact but $\dim X = \infty$,

Let $x_1 \in X$ s.t. $\|x_1\| = 1$

it generates one dimensional subspace X_1 of X

$\Rightarrow$ finite dim $\Rightarrow$ compact $\Rightarrow$ closed. Since it is proper subspace of X_1 by Riesz's Lemma $\exists$ a $x_2 \in X$ with $\|x_2\| = 1$ s.t.

$\|x_2 - x_1\| \geq \theta = \frac{1}{2}$ (say) $\quad \theta \in (0,1)$



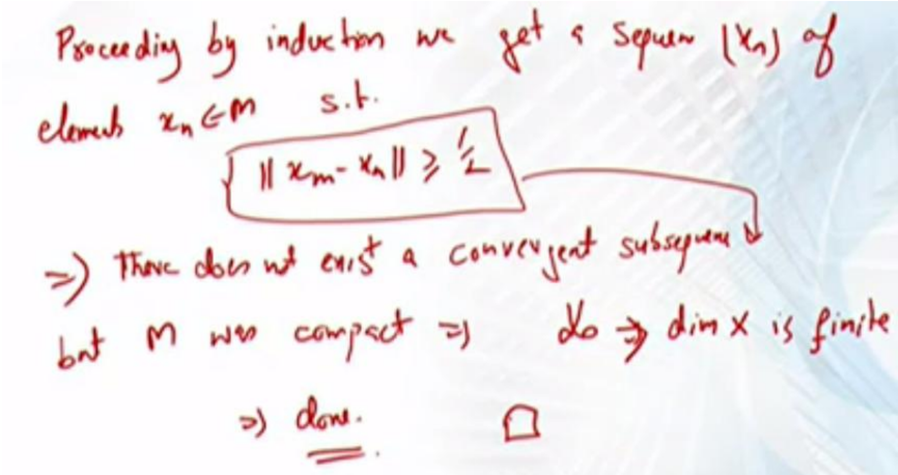
Again x_1, x_2 generate a two dimensional proper closed subspace X_2 of X. Again by Riesz's Lemma $\exists$ $x_3 \in X$ s.t. $\|x_3\| = 1$ and $\forall\, x \in X_2$ we have

$\|x_3 - x\| \geq \frac{1}{2}$

in particular since $x_1, x_2 \in X_2$

$\Rightarrow \|x_3 - x_1\| \geq \frac{1}{2}$

$\|x_3 - x_2\| \geq \frac{1}{2}$.



Module No. 53

Compactness and Finite Dimension

- ***Theorem (Continuous Mapping)***
- ***Corollary (Maximum and minimum)***

Theorem

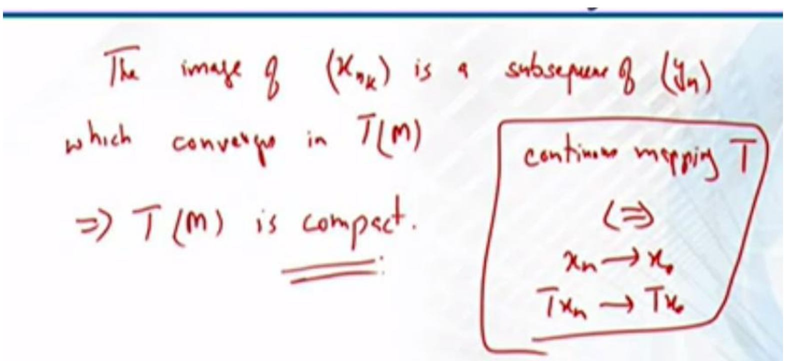
Let X and Y be metric spaces and $T: X \rightarrow Y$ be a continuous mapping.

Then the image of a compact subset M of X under T is compact.

Proof:

By definition of compactness we need to show that every sequence $\langle y_n \rangle$ in the image $T(M) \subset Y$ continuous a subsequence which converges in T(M).

Now since $y_n \in T(M)$, we have x_n such that $y_n = Tx_n$, for some $x_n \in M$. since M is compact, (x_n) contains subsequence $\langle x_{n_k} \rangle$ which converges in M.



Corollary (maximum and minimum)

A continuous mapping T of a compact subset M of a metric space X into R assumes a maximum and a minimum at some points of M.

$$T: M \to R$$

$\Rightarrow$ $T(M) \subset \mathbb{R}$

$T(M)$, $\left.\begin{array}{l} M \text{ - compact} \\ T - \text{continuous} \end{array}\right]$ by previous result

$\Rightarrow$ T(M) is compact.

which means it is closed and bounded because compactness implies close and bounded.

$$\Rightarrow \quad \inf T(M) \in T(M), \quad and \quad \sup T(M) \in T(M)$$

Inverse image of these two points consist of points of M at which Tx is minimum or maximum respectively. And that we have to prove.

Module No. 54

Functional Analysis

- ***Linear Operators***

In functional analysis if we define a metric on a set then it is a metric space and if we define a norm on a vector then it is called a norm space. In mapping if we take a and b as norms then we define a linear operator on the mapping and it should satisfied the certain properties.

Operator

In the case of vector spaces and, in particular, normed spaced, a mapping is called an operator.

Linear Operator

A linear operator T is an operator such that

i): the domain $\mathcal{D}$(T) of T is a vector space and the range R(T) lies in a vector space over the same field.

ii): for all $x, y \in D(T)$ and scalar α

$$T(x+y)=Tx+Ty \quad \text{also} \quad T(\alpha x) = \alpha Tx$$

By combining above two equations

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty \quad where\ \alpha\ and\ \beta\ are\ both\ scalar$$

$T(x) \simeq Tx$ is same.

Some more notations.

$\mathcal{D}$(T) domain of T

$\mathcal{R}$(T) range of T

$\mathcal{N}$(T) denotes the null space of T.

Null space are those element from the domain of T such that on which we operate gives the answer zero. $x \in D(T)$ such that Tx=0

Also null space of T is similar to kernel of T.

Let $D(T) \subset X$ *and* $R(T) \subset Y,\ X, Y$ *vector space.*

(vector spaces can be real and complex spaces).

Then T is an operator from $\mathcal{D}$(T) onto $\mathcal{R}$(T), the notation is

$T : D(T) \to R(T)$, D(T) covers all range so it is onto.

Or $\mathcal{D}$(T) into y $T : D(T) \to Y$ $R(T) \subset Y$

if $\mathcal{D}$(T) is the whole space X, then we write $T : X \to Y$

moreoverif we take $\alpha = 0 \Rightarrow$ T0=0.

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty \quad \text{where } \alpha \text{ and } \beta \text{ are both scalar}$$

T is a homomorphism when it is a linear operator.

$T : X \to Y$, wherewe have two kind of vector space, one vector space is X and other vector space is Y. we apply operations on X and also operation on Y.These operation may or may same on both vector spaces.

Module No. 55

Linear Operators

- ***Examples.***

Operator is a mapping whose domain and range is a vector space. It is subset of vector space. Below are different linear operators.

Identity Operator

Identity mean it operate on the same vector space. $I_x : X \to X$

$\Rightarrow$ $I_x(x) = x \quad \forall\ x \in X$

$\Rightarrow$ $I_x(\alpha x + \beta y)$ we have to prove

Zero Operator:

$$O : X \to Y \text{ such that } Ox = 0 \quad \forall \; x \in X$$

here the 0 on right side is belong to vector space Y.

Differentiation:

Let *X* be a vector space of all polynomials on [a,b]. A set of polynomial in denoted by *x(t)*

$$Tx(t) = x'(t) \qquad \forall \; x(t) \in X$$

When we apply T on polynomial x(t) then $x'(t)$ is also a polynomial. So this operatorT maps X onto itself. There is no polynomial whose derivative we can't find.

Integration:

Linear operator T for C[a,b] into itself can be defined by

$$Tx(t) = \int_a^t x(\tau)d\tau$$

taa τ is just a variable and C[a, b] is collection of all continuous function on a and b.

Multiplication by t:

Let C[a, b] be a collection of continuous functions defined on a and b.

$$Tx(t) = tx(t)$$

This operator plays an important role in quantum theory of physics.

Elementary vector algebra:

Here we have different types of maps we have

$$T_1 : \mathbb{R}^3 \to \mathbb{R}^3 \text{ cross product of two vectors is also a vector.}$$

For cross vector we need two vectors. Then each element is also a vector.

$$T_1 = \underline{a} \times \underline{x}$$

Similarly for dot product:

Dot product of two vector is a scalar, so the map on real numbers $\mathbb{R}$ as

$$T_2 : \mathbb{R}^3 \to \mathbb{R}$$

$$T_2(x) = \underline{a}.\underline{x} = a_1x_1 + a_2x_2 + a_3x_3 \in \mathbb{R} \qquad \text{where } x \in \mathbb{R}^3$$

For different map we fix a.

Matrices:

We denote matrix by capital letter say A. whose elements are in rows and column.

$$A = (\alpha_{jk})$$

Let with r rows and n column we define a linear operator which is

$$T : \mathbb{R}^n \to \mathbb{R}^r$$

Where $\mathbb{R}^n = \{(x_1, \ldots\ldots, x_n) \mid x_i \in \mathbb{R}\}$, in column form so that we use matrices multiplication

$$\begin{bmatrix} x_1 \\ . \\ . \\ x_n \end{bmatrix}$$

such as say

$$\underset{r\times 1}{\begin{bmatrix} y_1 \\ . \\ . \\ y_n \end{bmatrix}} = \underset{r\times n}{\begin{bmatrix} \alpha_{11} & . & . & . & \alpha_{1n} \\ . & & & & \\ . & & & & \\ \alpha_{r1} & . & . & . & \alpha_{rn} \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ . \\ . \\ x_n \end{bmatrix}}$$

For matrix multiplication number of first matrix column is equal to number of rows of second column.rxn is a fix matrix

To check the linear condition we use

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

Matrix multiplication satisfied this condition, hence this operator is a linear operator.

Module No. 56

Linear Operators

- ***Theorem (Range and Null space)***

Null space is the collection of those elements from the domain on which we apply the operator and the answer is zero.

Theorem

Let *T* be a linear operator. Then:

- The range *R(T)* is a vector space. (domain is also a vector space as discussed)
- If $\dim D(T) = n < \infty$, *then* $\dim R(T) \le n$ (dimension of domain vector space is finite then range is equal or less than the dimension of domain or equal.
- The null space *N(T)* is a vector space.

The first two results are about range and third result is about null space.

Proof: (a) R(T) is a vector space.

$$y_1, y_2 \in R(T)$$
$$\Rightarrow \quad \alpha y_1 + \beta y_2 \quad \in \mathrm{R(T)}, \quad \text{where } \alpha, \beta \text{ are scalar}$$

Since

$$y_1, y_2 \in R(T) \quad \text{and} \quad x_1, x_2 \in D(T)$$
$$T : D(T) \to Y$$
$$y_1 \in Tx_1 \quad , \quad y_2 \in Tx_2$$

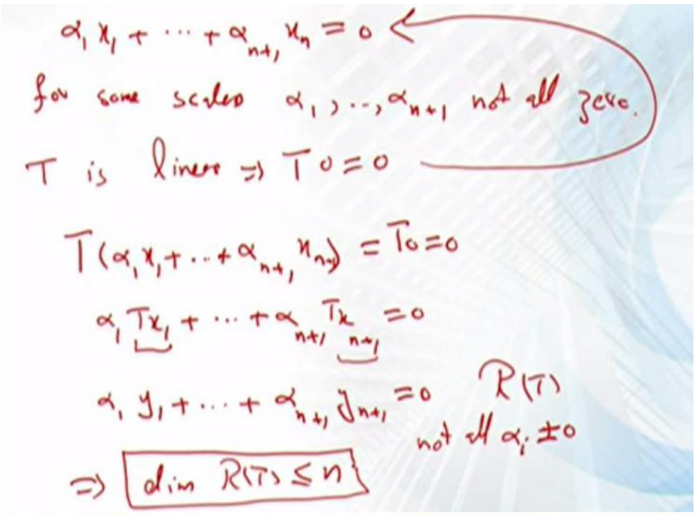
Also domain of T "*D(T")* is a vector space so, $\alpha x_1 + \beta x_2 \quad \in D(T)$ this is by definition of vector space.Since T is linear

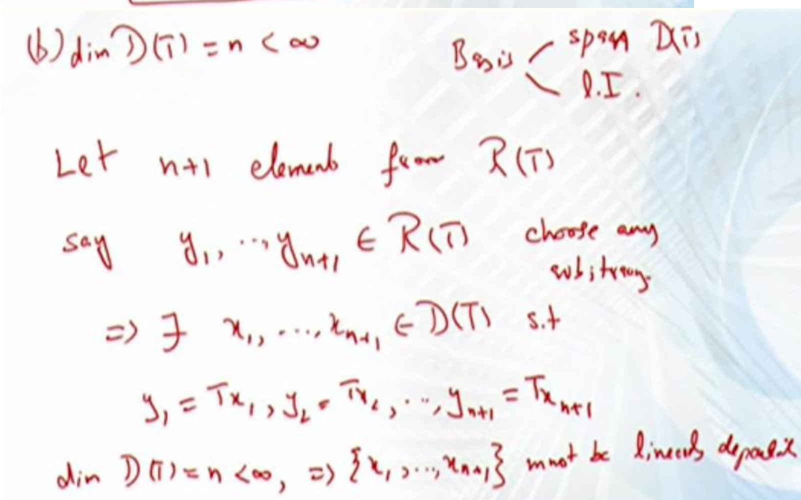
$$T(\alpha x_1 + \beta x_2) = \alpha\ Tx_1 + \beta Tx_2 = \alpha\ y_1 + \beta y_2 \in R(T)$$

Here $\alpha x_1 + \beta x_2$ is domain and gives $\alpha\ y_1 + \beta y_2$ range of T.Hence *R(T)* is a vector space.

Part (b):

Basis should span *D(T)*and it should linearly independent. More one than condition is if n element linearly independent then the elements other than n will be linearly dependent.





Linear operators preserve linearly dependence.

Part (c):

$$x_1, x_2 \in N(T)$$
$$Tx_1 = Tx_2 = 0$$

To prove it a vector space, we have to prove $\alpha x_1 + \beta x_2 \in N(T)$

$$T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2 = \alpha \times 0 + \beta \times 0 = 0$$

$\Rightarrow \quad \alpha x_1 + \beta x_2 \in N(T)$

$\Rightarrow \quad$ N(T) is a vector space (proved)

MODULE NO. 57

LINEAR OPERATORS

- ***Inverse Operators***

Operator is a mapping whose domain and range is vector space.Particular in norm space.There is also inverse mapping. For inverse operator the same condition is one-to-one and onto. One-to-one means image of different elements is different. And onto means the range covers all the set of domain. If these two conditions hold then we can define inverse oprator.

Notations:

$T: D(T) \to Y$ is said to be injective or one-to-one if for any

$x_1, x_2 \in D(T)$ such that $x_1 \neq x_2 \Rightarrow Tx_1 \neq Tx_2$

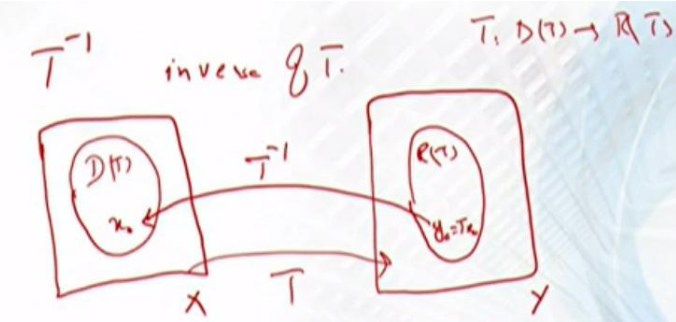
If we take counter inverse then $Tx_1 = Tx_2 \Rightarrow x_1 = x_2$,

Now if $T: D(T) \to R(T)$ then there exists a mapping

$$T': R(T) \to D(T)$$

$y_o \to x_o$ where y_o is any element of R(T)and x_o is element of D(T).i.e. $Tx_o = y_o$

this map T' is called the inverse of T.



$$T'Tx = x \quad \forall\ x \in D(T)$$

and $$TT'y = y \quad \forall\ y \in R(T)$$

Inverse exist if and only if null space has only zero. There is only zero in null space

Module No. 58

Linear Operators

- ***Theorem (Inverse Operator)***

Theorem

Let X,Y be vectors spaces, both real or both complex. Let $T: D(T) \to Y$ be a linear operator with domain $D(T) \subset X$ and range $R(T) \subset Y$ .then:

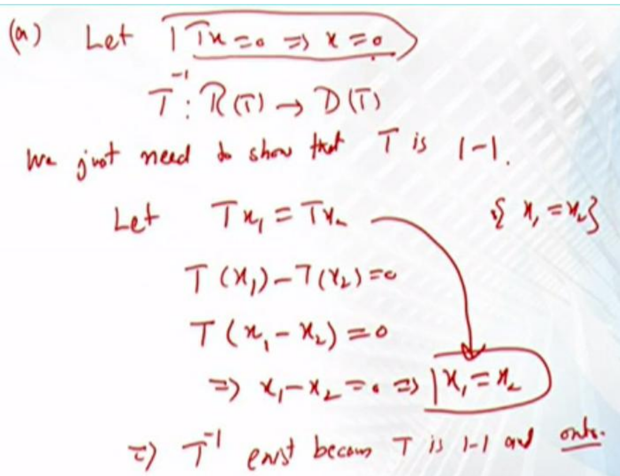
a): **The inverse $T': R(T) \to D(T)$ exists if and only if $Tx=0 \Rightarrow x=0$. (i.e null space has zero elements).**

b): **If T' exists, it is a linear operator.**

c): **if** $\dim D(T) = n < \infty$ *and* T^{-1} **exists, then** $\dim R(T) = \dim D(T)$.

as there is if and only if condition so we have to prove in both ways.

a):



Conversely let T^{-1} exist which mean one –one and onto condition hold.

We have to prove $Tx = 0 \;\; if \;\; and \;\; only \;\; if \;\;\; x = 0$.

One-one means $Tx_1 = Tx_2 \quad \Rightarrow \quad x_1 = x_2$, this is given

Now if we have take $x_2 = 0 \qquad \Rightarrow \quad x_1 = 0 \quad Tx_1 = T_0 = 0 \;\; , \;\; x_1 = 0$

b): If T' exists, it is a linear operator.

We need to show that $T^{-1}$is a linear operator.We assume that T^{-1}exists and we need to show that it is linear operator.

The domain of T^{-1} is basically range of *T* and also *R(T)* is a vector space.

$$x_1, x_2 \in D(T) \Rightarrow y_1 = Tx_1 \quad and \quad y_2 = Tx_2$$

$$y_1 = Tx_1 \qquad \Rightarrow \qquad x_1 = T^{-1} y_1$$

and $\quad y_2 = Tx_2 \qquad \Rightarrow \qquad x_2 = T^{-1} y_2$

T is linear so for any scalar $\alpha \;\; and \;\; \beta$ we have

$$\alpha y_1 + \beta y_2 = \alpha Tx_1 + \beta Tx_2 = T\left(\alpha x_1 + \beta x_2\right) \;\; \because \;\; T \;\; is \;\; linear$$

Applying T^{-1} on above we get

$$T'\left(\alpha y_1 + \beta y_2\right) = \alpha x_1 + \beta x_2$$

Putting values of $x_1 \;\; and \;\; x_2$

$$T'\left(\alpha y_1 + \beta y_2\right) = \alpha \mathrm{T}'\mathrm{y}_1 + \beta \mathrm{T}'\mathrm{y}_2$$

$T^{-1}$is a linear operator

C): if $\dim D(T) = n < \infty \;\; and \;\; T^{-1}$ exists, then $\dim R(T) = \dim D(T)$.

We have proved that $\dim R(T) \le n < \infty$we know

$$\dim R(T) \le \dim D(T) \; \ldots\ldots\ldots\ldots\ldots \mathrm{i}$$

Conversely,

$$T^{-1} : R(T) \to D(T)$$

$$\dim D(T) \le \dim R(T) \ldots\ldots\mathrm{ii}$$

Combining i and ii $\quad \dim R(T) = \dim D(T)$

If inverse exist then both dimensions are equal. That we have to prove.

Module No. 59

Linear Operators

- ***Lemma(Inverse of Product)***

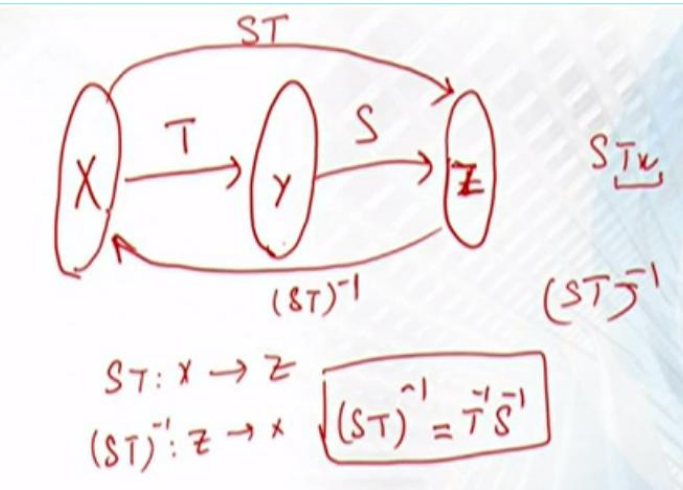
Bijective mean one to one and onto. Here it means inverse of T and S exists.

Lemma

Let $T: X \to Y$ *and* $S: Y \to Z$ be bijective linear operators, where X,Y are vectors spaces.

Then the inverse $(ST)^{-1}: Z \to X$ of the product (the composite) ST exists, and $(ST)^{-1} = T^{-1}S^{-1}$.

Diagram



Mathematically,

If S is bijective and T is bijective then ST is also bijective.

$ST: X \to Z$ bijective

$\Rightarrow$ $(ST)^{-1}$ exist.

It means if $(ST)(ST)^{-1} = I_Z$

If $S: Y \to Z$ then $S^{-1}S = I_Y$

$$S^{-1}ST(ST)^{-1} = S^{-1}I_Z \quad \Rightarrow \quad T(ST)^{-1} = S^{-1}$$

$\Rightarrow \quad T^{-1}T(ST)^{-1} = T^{-1}S^{-1} \qquad \Rightarrow \quad (ST)^{-1} = T^{-1}S^{-1}$

Module No. 60

Linear Operators

- ***Bounded Linear Operator***

Norms spaces are generalization of distances.

Bounded Linear Operator (Definition):

Let X and Y be normed spaces and $T : D(T) \to Y$ a linear operator, where $D(T) \subset X$. The operator T is said to be bounded if there is a real number c such that for all $x \in D(T)$.

$$\|Tx\| \leq c\|x\|$$

If this condition satisfied then we call T to be a bounded linear operator. Bounded function mean range is bounded but here bounded set is mapping over a bounded set so we call this a bounded linear operator.c is fix.

$$\frac{\|Tx\|}{\|x\|} \leq c \quad , \quad x \in D(T) - \{0\}$$

The smallest possible value of c is supremum of left hand side. Then the value of c is called

$$c = \sup_{\substack{x \in D(T),\\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \qquad \text{as} \left(T \text{ norm} = \sup_{\substack{x \in D(T),\\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \right)$$

We call the value as T norm $\qquad c = \|T\|$

$$\text{If } D(T) = \{0\}, \quad \|T\| = 0$$

$$c = \|T\| = \sup_{\substack{x \in D(T),\\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

$$\|Tx\| \leq \|T\|\|x\|$$

This is the formula that we use for bounded linear operator.

Module No. 61

Bounded Linear Operators

➢ *Lemma (Norm)*

First we define the norm and then prove that the norm defined on T satisfies (N1) to (N4).

Lemma:

Let T be a bounded linear operator as defined before.

An alternate formula for the norm of T is $\|T\| = \sup\limits_{\substack{x\in D(T)\\ \|x\|=1}} \|Tx\|$

The norm defined on T satisfies (N1) to (N4).

Proof:

$$\|Tx\| \le c\|x\|$$

$$c = \|T\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} \simeq \sup_{\substack{x\in D(T)\\ \|x\|=1}} \|Tx\|$$

We have to prove $\sup\limits_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} \simeq \sup\limits_{\substack{x\in D(T)\\ \|x\|=1}} \|Tx\|$

Let $\|x\| = a;$ set $y=\frac{x}{a}$, $x \neq 0$,

$$\|y\| = \frac{\|x\|}{a} = 1$$

$$\|T\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{a}$$

as T is linear so, we take constant $a$inside the norm

$$\|T\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \left\|T\left(\frac{1}{a}x\right)\right\| = \sup_{\substack{y\in D(T),\\ \|y\|=1}} \|Ty\| \qquad \text{as } \frac{1}{a} = y$$

Here variable is y which can be any other.

Part a) of lemma is proved.

Part b):

$$\|T\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in D(T),\\ \|x\|=1}} \|Tx\|$$

N1: $\|T\| \geq 0$ is obvious.

N2: $\|T\| > 0 \quad \Rightarrow \quad$ T=0,

$$\|T\| = 0 \quad \Rightarrow \quad Tx = 0, \quad \forall\ x\in D(T) \Rightarrow T = 0$$

N3: $\|\alpha T\| = \sup\limits_{\substack{x\in D(T),\\ \|x\|=1}} \|\alpha Tx\| = \sup\limits_{\|x\|=1}|\alpha|\|Tx\| = |\alpha|\sup\limits_{\|x\|=1}\|Tx\| = |\alpha|\|T\|$ as $\sup\limits_{\|x\|=1}\|Tx\| = \|T\|$

N4: $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$

$$\|T_1+T_2\| = \sup_{\substack{x\in D(T)\\ \|x\|=1}} \|(T_1+T_2)x\|$$
$$\leq \sup_{\|x\|=1}\|T_1x+T_2x\| \leq \sup_{\|x\|=1}\left(\|T_1x\|+\|T_2x\|\right)$$
$$= \sup_{\|x\|=1}\|T_1x\| + \sup_{\|x\|=1}\|T_2x\| = \|T_1\|+\|T_2\|$$

First we define a $T\times T$ norm and then prove the four properties of norm.

Module No. 62

Examples Bounded Linear Operators

- ***Identity Operator***
- ***Zero Operator***
- ***Differentiation Operator***
- ***Integral Operator***

Identity operator:

$$I : X \to X \qquad \Rightarrow \qquad I_x = x \quad \{x \neq \{0\} \text{ normed space}\}$$

$$\|I\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|x\|}{\|x\|} \qquad as \qquad Tx = x$$

$$\|I\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} 1 = 1$$

Zero operator:

The norm space $O : X \to Y$, $\quad O_x = 0 \qquad\qquad x \in X$

$$\|O\| = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = 0 \quad , \|0\| = 0$$

Differentiation operator:

This is defined on normed space of all polynomial on J=[0, 1]

$$\|x\| = \max\{|x(t)|, \quad t \in J\}$$

Value of t varies from 0 to 1 and where the value is maximum, that maximum value is norm of *x*.

applying operator the derivative. Differentiation operator is.

$$Tx(t) = x'(t)$$

Derivation is itself a linear operator.

Now we check that it is bounded or not. $\|Tx(t)\| \leq c\|x(t)\|$. If it is bounded then what is the value of c.

Let $x_n(t) = t^n \quad n \in \mathbb{N}$, what is the norm of $x_n(t)$

$$\|x_n(t)\| = \max\{|x(t)|, \quad t \in [0,1]\} = 1$$

Using operator $Tx_n(t) = nt^{n-1}$

define the norm $\|Tx_n(t)\| = \max\left|nt^{n-1}\right| = 1$

$$\|Tx_n(t)\| = \max(|\, nt^{n-1} \,| : t \in [0,1]) = n.1 = n$$

$$\frac{\|Tx_n\|}{x_n} = \frac{n}{1} = c, \quad n \in \mathbb{N}$$

As n had no bound so, there does not exist any c such that $\frac{\|Tx\|}{\|x_n\|} \leq c$ hold.

Now c is fixed number which does not depend upon N but in this case it depends on N, if we take c as n then next value n+1 will not satisfy this equation. It means that there does not exist any c that this condition $\frac{\|Tx\|}{\|x_n\|} \leq c$ holdhence derivative operative is not bounded.

Integral Operator

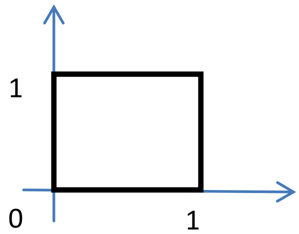
Defined as $T : C[0,1] \to C[0,1]$,

$y=Tx \qquad y(t)=\int_0^1 k(t,\tau)x(\tau)d\tau$

k is integral of T it is fix for different integral operator,

T is linear as integration is linear, also derivation is a linear operator same as integral is linear operator.

K is continuous on $J\times J$. We have two variables t and τ , $k(t,\ \tau)$



Whatever the value of k is, it should be in the square

$k(t,\ \tau)$ is bounded. And if it is bounded then

$k(t,\ \tau)\le k_o,\ \ t,\tau\in J\times J\ ,\ \ k_o\in\mathbb{R}$ where $J\times J$ is this square box.

$|x(t)|\le \max_{t\in J}|x(t)|=\|x\|$

Now example, $\|y\|=\|Tx\|=\max_{t\in J}\left|\int_0^1 k(t,\tau)x(\tau)d\tau\right|$

$\le \max_{t\in J}\int_0^1 |k(t,\tau)||x(\tau)|d\tau$

$\le k_o\|x\|$

$\|Tx\|\le k_o\|x\|$ it has k and k_o is fix so integral operator is a linear operator.

Module No. 63

Examples Bounded Linear Operators

- ***Matrix***

Identity operator:

$T:R^n\to R^r$

$$\begin{bmatrix} a_{11} & . & a_{1n} \\ . & . & . \\ a_{r1} & . & a_{rn}\end{bmatrix}\begin{bmatrix}\xi_1 \\ . \\ \xi_n\end{bmatrix}=\begin{bmatrix}x_1 \\ . \\ x_n\end{bmatrix}$$

$r\times n \qquad n\times 1 \quad r\times 1$

$A \qquad\quad \text{x} \ = \ \text{y}$

The entries are $x=(\xi_j)\ ,\quad y=(\eta_j)$

And the matrix is $A=(\alpha_{ij}),\quad 1\le \text{i}\le r,\quad 1\le \text{j}\le n$

$$\eta_j = \sum_{k=1}^{n} \alpha_{jk} \xi k$$

T is linear because the properties of matrices is it bounded?

$$\|x\| = \left(\sum_{m=1}^{n} \xi_m^2 \right)^{\frac{1}{2}} \qquad , \quad x \in \mathbb{R}^n$$

and
$$\|y\| = \left(\sum_{j=1}^{r} \eta_j^2 \right)^{\frac{1}{2}} \qquad , \quad \mathrm{y} \in \mathbb{R}^n$$

for bounded we have to check norm of T "T(x)".

$$\|Tx\| = \left(\sum_{j=1}^{r} \eta_j^2 \right)^{\frac{1}{2}}$$

$$\|Tx\|^2 = \sum_{j=1}^{r} \eta_j^2$$

$$\|Tx\|^2 = \sum_{j=1}^{r} \left(\sum_{k=1}^{n} \alpha_{jk} \xi_k \right)^2$$

Where $\eta_j = \sum_{k=1}^{n} \alpha_{jk} \xi_k$

Cauchy Schwaz inequality on above $\|Tx\|^2$

$$\leq \sum_{j=1}^{r} \left[\left(\sum_{k=1}^{n} \alpha_{jk}{}^2 \right)^{\frac{1}{2}} \left(\sum_{m=1}^{n} \xi_m^2 \right)^{\frac{1}{2}} \right]^2 = \|x\|^2 \left(\sum_{j=1}^{r} \sum_{k=1}^{n} \alpha_{jk}^2 \right)$$

$$\|Tx\|^2 \leq c^2 \|x\|^2$$

Here is a c which depends upon T.

We can write as

$$\|Tx\| \leq c\|x\|$$

T is already linear and with this value of c we can say matrices is a linear bounded operator.in last four examples three are linear operator but differential was not linear operator.

MTH 641

FUNCTIONAL ANALYSIS

MODULE # 60 To 113

(FINAL TERM SYLLABUS)

Don't look for someone who can solve your problems,

Instead go and stand in front of the mirror,

Look straight into your eyes,

And you will see the best person who can solve your problems!

Always trust yourself.

(BY ABU SULTAN)

Module No. 60

Linear Operators

- ***Bounded Linear Operator***

Norms spaces are generalization of distances. By using Norm spaces we are going to discuss Bounded Linear Operator.

Bounded Linear Operator (Definition):

Let X and Y be normed spaces and $T: D(T) \to Y$ a linear operator, where $D(T) \subset X$. The operator T is said to be bounded if there is a real number c such that for all $x \in D(T)$.

$$\|Tx\| \le c\|x\|$$

If this condition satisfied then we call T to be a bounded linear operator. Bounded function mean range is bounded but here bounded set is mapping over a bounded set so we call this a bounded linear operator. c is fix.

$$\frac{\|Tx\|}{\|x\|} \le c \quad , \quad x \in D(T) - \{0\}$$

The smallest possible value of c is supremum of left hand side. Then the value of c is called

$$c = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \qquad \text{as}$$

We call the value as T norm

$$c = \|T\|$$

$$\text{If } D(T) = \{0\}, \quad \|T\| = 0$$

$$c = \|T\| = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

$$\|Tx\| \le \|T\|\|x\|$$

This is the formula that we use for bounded linear operator.

Module No. 61

Bounded Linear Operators

- ***Lemma (Norm)***

First we define the norm (equivalent definition) and then prove that the norm defined on T satisfies all four properties of Norm i.e. (N1) to (N4).

Lemma (Statement):

Let T be a bounded linear operator as defined before then an alternate formula for the norm of T is

$$\|T\| = \sup_{\substack{x \in D(T) \\ \|x\| = 1}} \|Tx\|$$

The norm defined on T satisfies (N1) to (N4).

Proof: Part (a)

$$\|Tx\| \le c\|x\|$$

$$c = \|T\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} \simeq \sup_{\substack{x\in D(T)\\ \|x\|=1}} \|Tx\|$$

We have to prove

$$\sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} \simeq \sup_{\substack{x\in D(T)\\ \|x\|=1}} \|Tx\|$$

Let $\|x\| = a;$ set $y=\frac{x}{a}$, $x \neq 0,$

$$\|y\| = \frac{\|x\|}{a} = 1$$

$$\|T\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{a}$$

as T is linear so, we take constant a inside the norm

$$\|T\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \left\| T\left(\frac{1}{a}x\right)\right\| = \sup_{\substack{y\in D(T),\\ \|y\|=1}} \|Ty\| \qquad \text{as } \frac{1}{a} = y$$

Here variable is y which can be any other. Part (a) of lemma is proved.

Part (b):

$$\|T\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in D(T),\\ \|x\|=1}} \|Tx\|$$

N1: $\|T\| \geq 0$ is obvious.

N2: $\|T\| > 0 \quad \Rightarrow \quad T=0,$

$\|T\| = 0 \quad \Rightarrow \quad Tx = 0, \quad \forall \; x \in D(T) \;\Rightarrow\; T = 0$

N3: $\|\alpha T\| = \sup_{\substack{x\in D(T),\\ \|x\|=1}} \|\alpha Tx\| = \sup_{\|x\|=1} |\alpha| \|Tx\| = |\alpha| \sup_{\|x\|=1} \|Tx\| = |\alpha| \|T\|$ as $\sup_{\|x\|=1} \|Tx\| = \|T\|$

N4: $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$

$$\|T_1 + T_2\| = \sup_{\substack{x\in D(T)\\ \|x\|=1}} \|(T_1 + T_2)x\|$$

$$\leq \sup_{\|x\|=1} \|T_1x + T_2x\| \leq \sup_{\|x\|=1} \left(\|T_1x\| + \|T_2x\|\right)$$

$$= \sup_{\|x\|=1} \|T_1x\| + \sup_{\|x\|=1} \|T_2x\| = \|T_1\| + \|T_2\|$$

First we define a $T \times T$ norm and then prove the four properties of norm.

Module No. 62

Examples Bounded Linear Operators

- ***Identity Operator***
- ***Zero Operator***
- ***Differentiation Operator***
- ***Integral Operator***

Identity operator:

$$I : X \to X \qquad \Rightarrow \qquad I_x = x \quad \{x \neq \{0\} \text{ normed space}\}$$

$$\|I\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|x\|}{\|x\|} \qquad as \qquad Tx = x$$

$$\|I\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} 1 = 1$$

Zero operator:

The norm space $O : X \to Y$, $\quad O_x = 0 \qquad\qquad x \in X$

$$\|O\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = 0 \quad , \|0\| = 0$$

Differentiation operator:

This is defined on normed space of all polynomial on J=[0, 1]

$$\|x\| = \max\{|x(t)|, \ t \in J\}$$

Value of t varies from 0 to 1 and where the value is maximum, that maximum value is norm of *x*.

applying operator the derivative. Differentiation operator is.

$$Tx(t) = x'(t)$$

Derivation is itself a linear operator.

Now we check that it is bounded or not. $\|Tx(t)\| \leq c\|x(t)\|$. If it is bounded then what is the value of c.

Let $x_n(t) = t^n \quad n \in \mathbb{N}$, what is the norm of $x_n(t)$

$$\|x_n(t)\| = \max\{|x(t)|, \ t \in [0,1]\} = 1$$

Using operator $Tx_n(t) = nt^{n-1}$

define the norm $\qquad \|Tx_n(t)\| = \max\left|nt^{n-1}\right| = 1$

$$\|Tx_n(t)\| = \max(|nt^{n-1}| : t \in [0,1]) = n.1 = n$$

$$\frac{\|Tx_n\|}{x_n} = \frac{n}{1} = c, \quad n \in \mathbb{N}$$

As n had no bound so, there does not exist any c such that $\frac{\|Tx\|}{\|x_n\|} \leq c$ hold.

Now c is fixed number which does not depend upon N but in this case it depends on N, if we take c as n then next value n+1 will not satisfy this equation. It means that there does not exist any c that this condition $\frac{\|Tx\|}{\|x_n\|} \leq c$ holdhence derivative operative is not bounded.

Integral Operator

Defined as $T : C[0,1] \to C[0,1]$,

$$y = Tx \qquad\qquad y(t) = \int_0^1 k(t,\tau)x(\tau)d\tau$$

k is integral of T it is fix for different integral operator, T is linear as integration is linear, also derivation is a linear operator same as integral is linear operator.

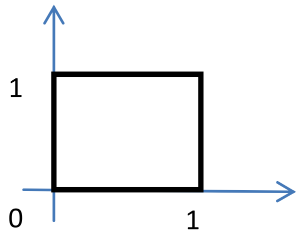
K is continuous on $J \times J$. We have two variables $t \; and \; \tau$, $k(t, \; \tau)$

Whatever the value of k is, it should be in the square

$k(t, \; \tau)$ is bounded. And if it is bounded then

$k(t, \; \tau) \le k_o, \;\; t,\tau \in J \times J \; , \;\; k_o \in \mathbb{R}$ where $J \times J$ is this square box.

$|x(t)| \le \max_{t \in J} |x(t)| = \|x\|$



Now example, $\|y\| = \|Tx\| = \max_{t \in J} \left| \int_0^1 k(t,\tau)x(\tau)d\tau \right|$

$$\le \max_{t \in J} \int_0^1 |k(t,\tau)||x(\tau)|\, d\tau$$

$$\le k_o \|x\|$$

$\|Tx\| \le k_o \|x\|$ it has $k \; and \; k_o$ is fix so integral operator is a linear operator.

MODULE No. 63

EXAMPLES BOUNDED LINEAR OPERATORS

- ***Matrix***

Identity operator:

$$T : R^n \to R^r$$

$$\begin{bmatrix} a_{11} & . & a_{1n} \\ . & . & . \\ a_{r1} & . & a_{rn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ . \\ \xi_n \end{bmatrix} = \begin{bmatrix} x_1 \\ . \\ x_n \end{bmatrix}$$

$$r \times n \qquad n \times 1 \quad r \times 1$$

$$A \qquad \text{x} \; = \; \text{y}$$

The entries are $x = (\xi_j) \; , \quad y = (\eta_j)$

And the matrix is $A = (\alpha_{ij}), \quad 1 \le \text{i} \le r, \quad 1 \le \text{j} \le n$

$$\eta_j = \sum_{k=1}^{n} \alpha_{jk} \xi k$$

T is linear because the properties of matrices is it bounded?

$$\|x\| = \left(\sum_{m=1}^{n} \xi_m^2 \right)^{\frac{1}{2}} \qquad , \quad x \in \mathbb{R}^n$$

and

$$\|y\| = \left(\sum_{j=1}^{r} \eta_j^2 \right)^{\frac{1}{2}} \qquad , \quad \text{y} \in \mathbb{R}^n$$

for bounded we have to check norm of T “T(x)”.

$$\|Tx\| = \left(\sum_{j=1}^{r} \eta_j^2\right)^{\frac{1}{2}}$$

$$\|Tx\|^2 = \sum_{j=1}^{r} \eta_j^2$$

$$\|Tx\|^2 = \sum_{j=1}^{r} \left(\sum_{k=1}^{n} \alpha_{jk} \xi_k\right)^2$$

Where $\eta_j = \sum_{k=1}^{n} \alpha_{jk} \xi_k$

Cauchy Schwaz inequality on above $\|Tx\|^2$

$$\leq \sum_{j=1}^{r} \left[\left(\sum_{k=1}^{n} \alpha_{jk}{}^2\right)^{\frac{1}{2}} \left(\sum_{m=1}^{n} \xi_m^2\right)^{\frac{1}{2}}\right]^2 = \|x\|^2 \left(\sum_{j=1}^{r} \sum_{k=1}^{n} \alpha_{jk}^2\right)$$

$$\|Tx\|^2 \leq c^2 \|x\|^2$$

Here is a c which depends upon T.
We can write as

$$\|Tx\| \leq c\|x\|$$

T is already linear and with this value of c we can say matrices is a linear bounded operator.in last four examples three are linear operator but differential was not linear operator.

MODULE NO. 71

LINEAR FUNCTION (EXAMPLES):

- Space $C[a \ \ b]$
- Space l^2

Space $C[a \ \ b]$**:**

We have define a linear function on space $C[a \ \ b]$ that we have fixed an element t_o from the set J as $t_o \in J$. Now define a functional operator f(x) which is operating on x which is element from $C[a \ \ b]$. $x \in C[a \ \ b]$

This x is not a variable, it is a function. So f_1 which is defined on $C[a \ \ b]$ linear as it is a linear operator. f_1 is bounded. To find the norm

$$|f_1| = |x(b)| \leq \|x\|$$

$$\|x\| = 1 \quad \Rightarrow \quad \|f_1\| \leq 1..........(i)$$

If we take $x_0 = 1$ and substitute in this equation we get

$$|f_1(x_o)| \leq \|f_1\|.\|x\|$$

$$1 \leq \|f_1\|.1 \quad \Rightarrow \quad \|f_1\| \geq 1...........(ii$$

From i) and ii)

$$\|f_1\| = 1$$

So the function defined on C is linear, bounded and Norm is 1.

Space l^2

We choose a fix say $a=(a_j)\in l^2$

$$f(x)=\sum_{j=1}^{\infty}\xi_j a_j \qquad x\in l^2,\ x=(\xi_j)$$

This sequence is linear, converging and bounded.
For boundedness

$$|f(x)|=\left|\sum_{j=1}^{\infty}\xi_j a_j\right|\le\sum_{j=1}^{\infty}|\xi_j a_j|\le\sqrt{\sum_{j=1}^{\infty}|\xi_j|^2}\sqrt{\sum_{j=1}^{\infty}|a_j|^2}=\|x\|.\|a\|$$

It is the same definition of bounded.
M of a complete metric space *X* is itself complete if and only if the set *M* is closed in *X*.

Module No. 72

Linear Function:

- Algebraic Dual Space
- Second Algebraic Dual Space
- Canonical Mapping

Algebraic Dual Space

Set of all linear function defined on a vector space X is itself a vector space and called Algebraic Dual Space and denoted by X^*

Operation on this vector space are
1st Operation Sum

$$f_1+f_2 \qquad f_1, f_2 \text{ linear functional}$$

$$\left(f_1+f_2\right)x=f_1(x)+f_2(x) \quad x\in X$$

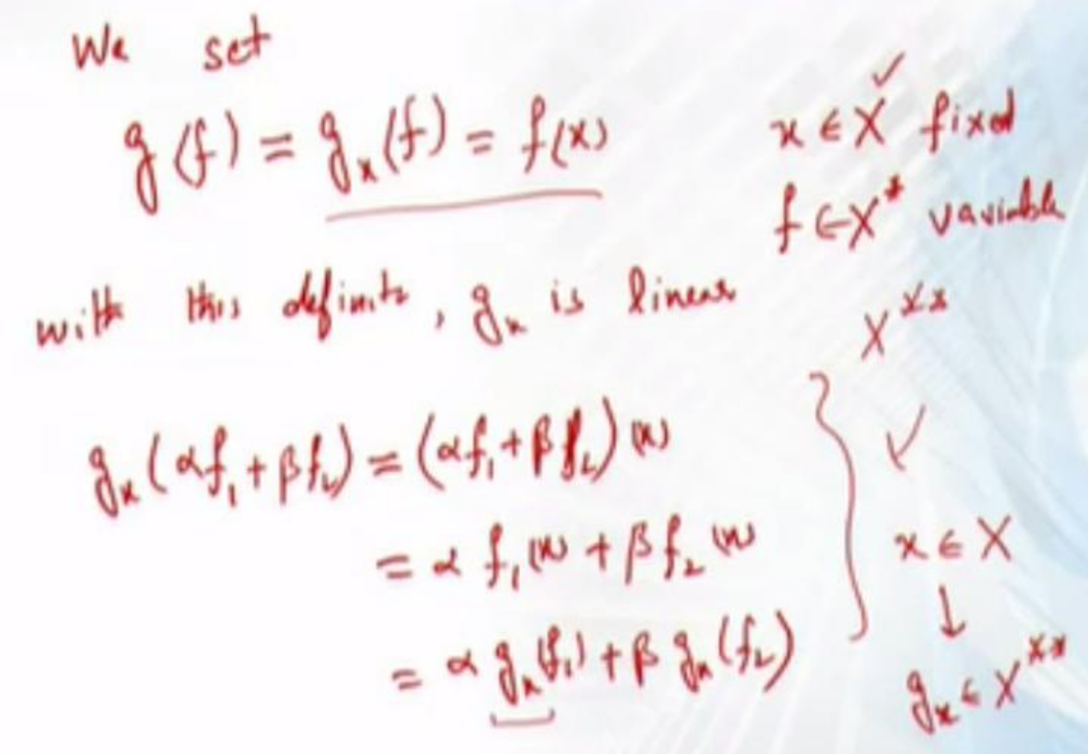
2nd Operation Scalar Multiplication

$$\left(af\right)x=af(x)$$

Second Algebraic Dual Space X^{**}

Space	element	Vector at a point
X	$x\in X$	
X^*	g	f (x)
X^{**}	G	g(x)

For each $x,g\in X^{**}$



Conical Mapping:

$C: X \to X^{**}$ this mapping is called canonical mapping of X into X^{**} defined as

$x \mapsto g_x$.

$C(\alpha x+\beta y)(f) = g_{\alpha x+\beta y}(f)$

$= f(\alpha x+\beta y) = \alpha f(x) + \beta f(y) = \alpha g_x(f) + \beta g_{y\partial}(f)$

$= \alpha(Cx)(f) + \beta(Cy)(f)$

So, this is a linear function as well. Canonical mapping is a relation between X and X^{**}.

Module No. 73

Linear Function:

- Algebraically Reflexive
- Second Algebraic Dual Space
- Canonical Mapping

Isomorphism:

It is one-one and onto map.

Algebraically Reflexive:

$T:(X,d) \to (\tilde{X},\tilde{d})$ bijective

$\tilde{d}(T_x, T_y) = d(x,y)$

$C: X \to X^{**}\ x \mapsto g_x$.

If C is surjective (on b) bijection. $\Re(C) = X^{**}$

We call X to be algebraically reflexive.

Set of all linear function defined on a vector space X is itself a vector space and called

Module No. 74

Linear Operators and functional on finite Dimensional Spaces:

Finite dimensions mean basis which have finite many elements.

Let X and Y bef.gfinite dimension vector spaces over the same field.

Let $T: X \to Y$ be a linear operator. let $E = \{e_1, ..., e_n\}$ be the basis for X and $B = \{b_1, ..., b_n\}$ be the basis for Y.

$x \in X$, x=$\xi_1 e_1 + \xi_2 e_2 + ... + \xi_n e_n$

$$y = Tx = T\left(\sum_{k=1}^{n} \xi_k e_k\right) = \sum_{k=1}^{n} T(\xi_k e_k) = \sum_{k=1}^{n} \xi_k T(e_k)$$

T is uniquely determinal if the image $y_k = Te_k$ of n basis vectors $e_1, ..., e_n$ are prescribed.

y=Tx ; $y \in Y$ $\{b_1, ..., b_n\}$

y=$\eta_1 b_1 + \eta_2 b_2 + ... + \eta_r b_r$

$Te_k \in Y$, Te$_1$=$\tau_{11} b_1 + \tau_{12} b_2 + ... + \tau_{1r} b_r$

$$Te_k = \sum_{j=1}^{r} \tau_{kj} b_j$$

$$y = \sum_{j=1}^{r} \eta_j b_j = \sum_{k=1}^{n} \xi_k Te_k = \sum_{k=1}^{n} \xi_k \sum_{j=1}^{r} \tau_{kj} b_j$$

Comibinig these two summation

$$y = \sum_{j=1}^{r}\left(\sum_{k=1}^{n} \tau_{kj} \xi_k\right) b_j$$

$$\eta_j = \sum_{k=1}^{n} \tau_{kj} \xi_k$$

The image y=Tx=$\sum \eta_j b_j$ of $x = \sum \xi_k Te_k$ can be obtained from

$$\eta_j = \sum_{k=1}^{n} \tau_{kj} \xi_k$$

Module No. 75

Operators on finite Dimensional Spaces:

Remarks:

As in the case of linear operators on a finite dimensional normed space, every linear functional defined on a finite dimensional normed space is bounded and hence continuous. Since for linear funcionals range is either $\mathbb{R}$ or $\mathbb{C}$, which are complete. So X^* as the space of all bounded linear functionals defined on X, is also complete and hence is Banach space. This is true even if X is not a Banach space.

"Algebraic Dual Space of X": set of all linear funcionals defined on X.

"Dual or Conjugate Space of X": X^* set of all continuous or bounded linear functionals defined on X.

We take algebraic dual when there is no condition of continuous or bounded linear functions.

Theorem:

Let X be an n-dimensional vector space and X^* be its dual space. Then

$$\dim X^* = \dim X = n.$$

X^* is collection of linear functions or linear operator while X may be any space.

Proof:

Let $\dim X = n.$

Let basis of X be $B = \{e_1, ..., e_2\}$

We define a function.

$$f_j(e_1) = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j :\ i, j = 1, ..., n \end{cases}$$

e.g. j=1 , $f(e_1) = 1, f(e_2) = 0, f(e_3) = 0, ..., f(e_n) = 0$

j=2 , $f(e_1) = 0, f(e_2) = 1, f(e_3) = 0, ..., f(e_n) = 0$

but each n-tuples f_j in this case can be extended as linear functions on X.

Module No. 76

Operators on Finite Dimensional Spaces:

Lemma(Zero Vector):

Let X be a finite deimensional vector space. If $x_0 \in X$ has the property that $f(x_0) = 0$ for all $f \in X^*$ then $x_0 = 0$.

B^* is the basis of X^*

$\{f_1, f_2, ..., f_n\}$

$$\Rightarrow f_j(e_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$= \delta_{ij}$$

Proof:

For all $x_0 = 0$,

$$x_0 = \sum_{i=1}^{n} x_i e_i \quad ; \quad \mathrm{f} \in \mathrm{X}^* \quad ,$$

$$\mathrm{f}(\mathrm{x}_0) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} f\left(\sum_{i=1}^{n} x_i e_i\right) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} x_i f(e_i) = 0 \quad , \quad \text{j=1,......,n}$$

$$\Rightarrow \quad \mathrm{x}_\mathrm{j} = 0 \quad , \quad \forall \text{ j=1,........,n}$$

$$x_0 = \sum_{i=1}^{n} x_i e_i = 0 \quad \Rightarrow \quad x_0 = \delta$$

Module No. 77

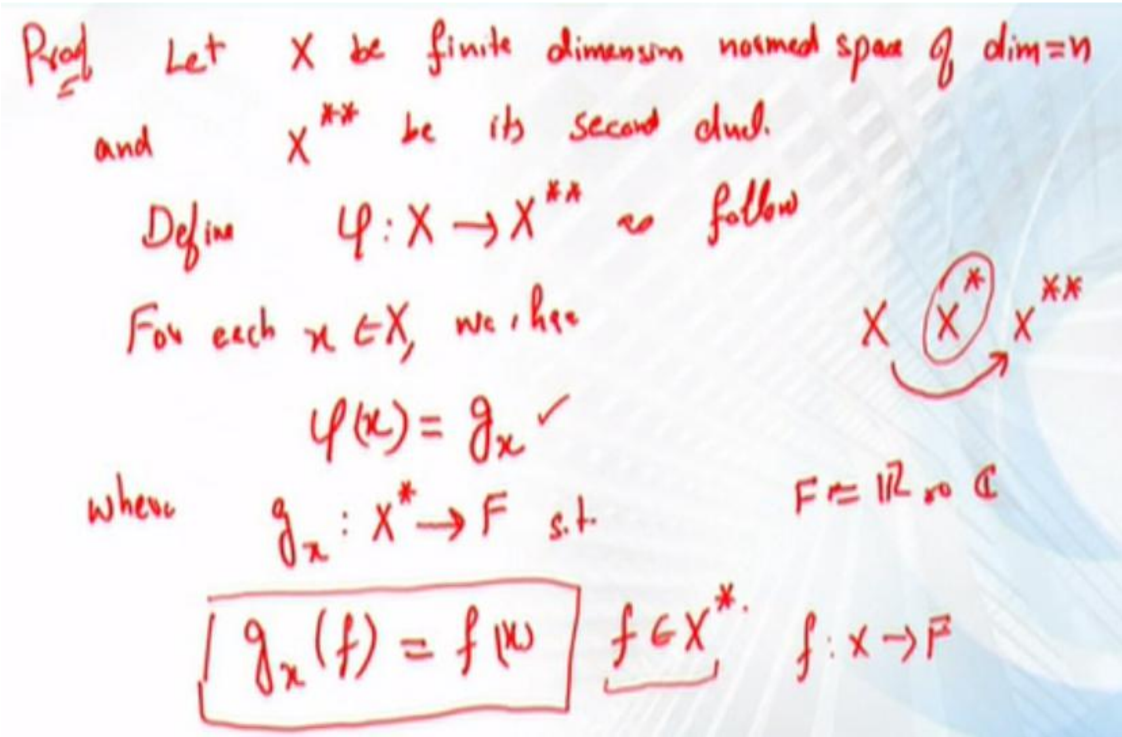
Operators on Finite Dimensional Spaces:

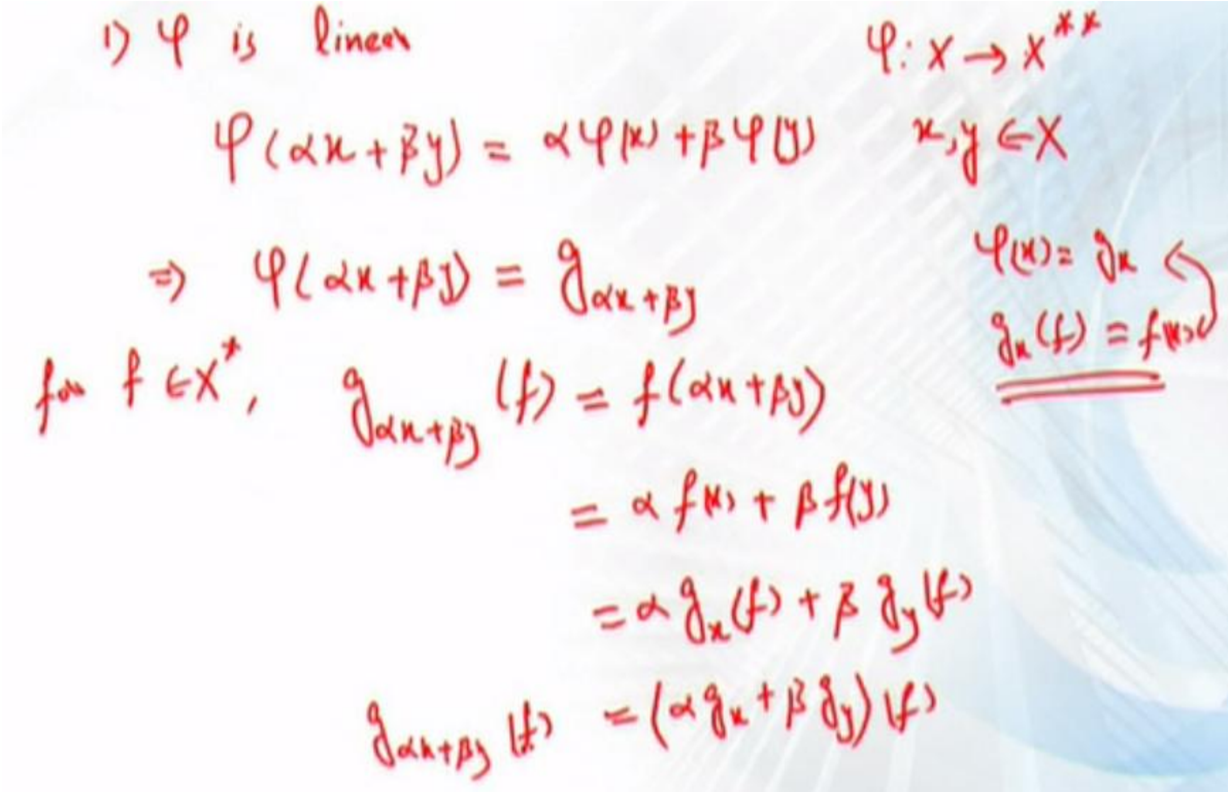
Theorem(Reflexivity):

A normed space X is said to algebraically reflexive if there is an isometric isomorphism between X and X^{**}.
Ordinarily a normed spacer may not be reflexive.
If X is an incomplete normed space even then X^{*} and X^{**} are Banach spaces. So in this case X cannot be a reflexive space.
However there are Banach spaces which are not reflexive.

Theorem:

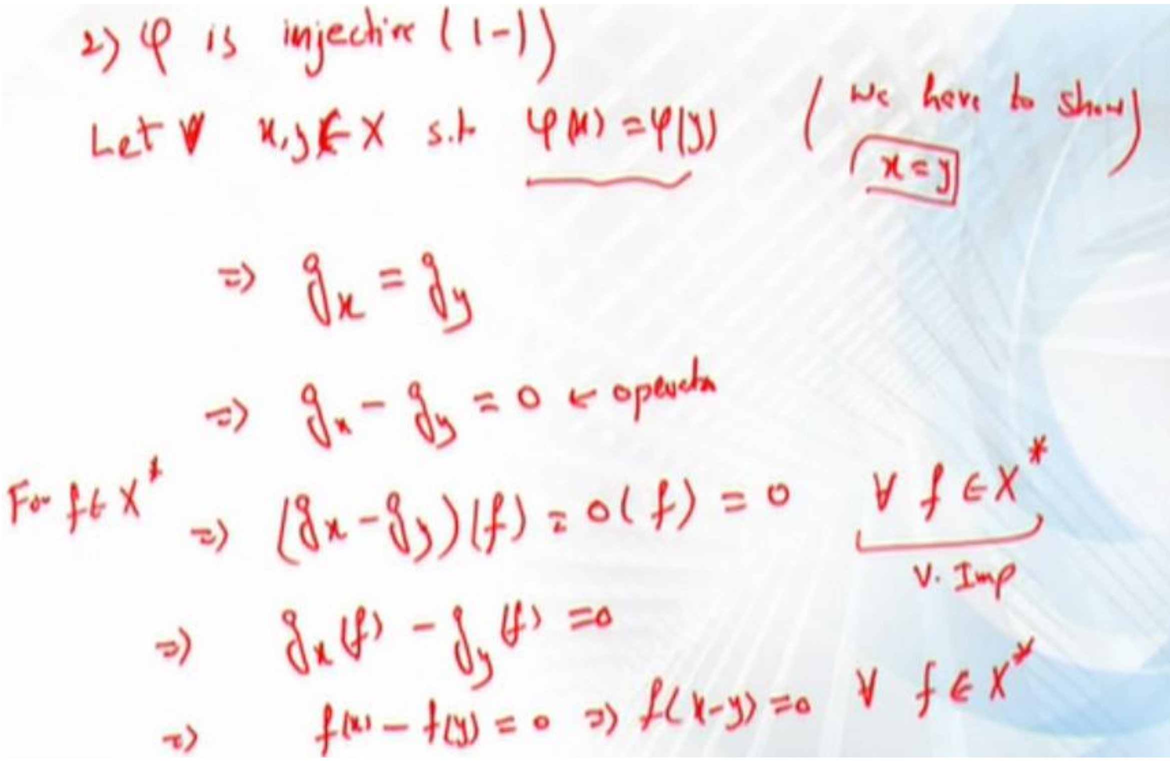
A finite dimensional vector space is reflexive.
Equivalently, A finite dimensional normed space is isomorphic space is isomorphic to its second dual.

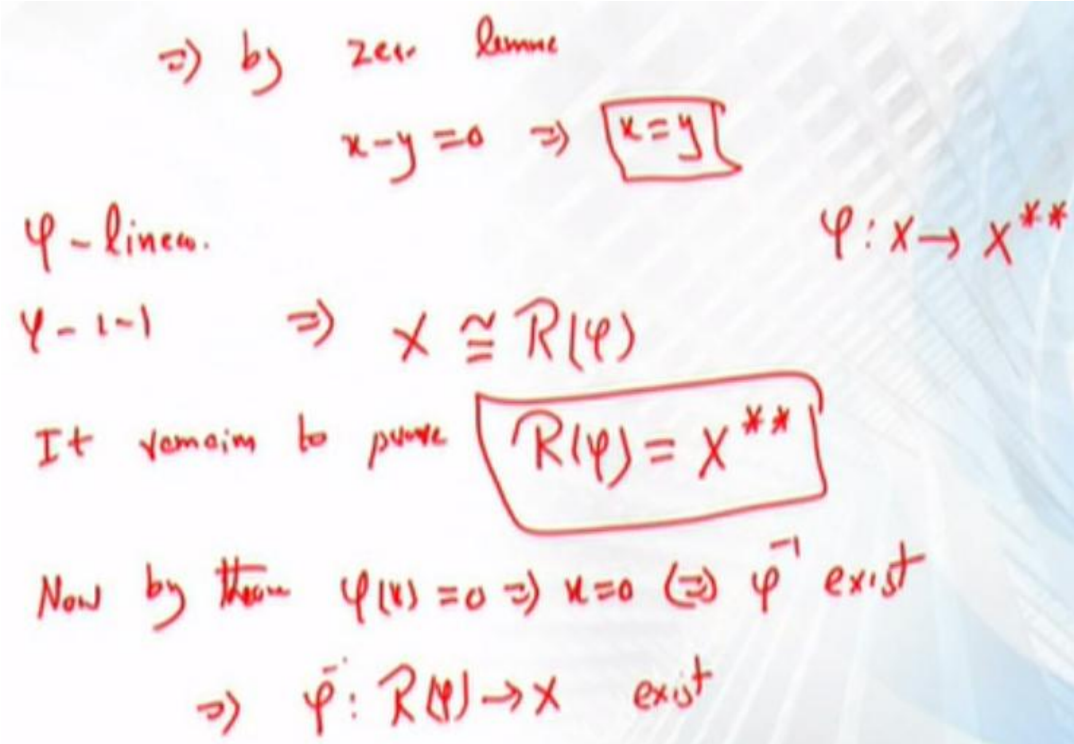


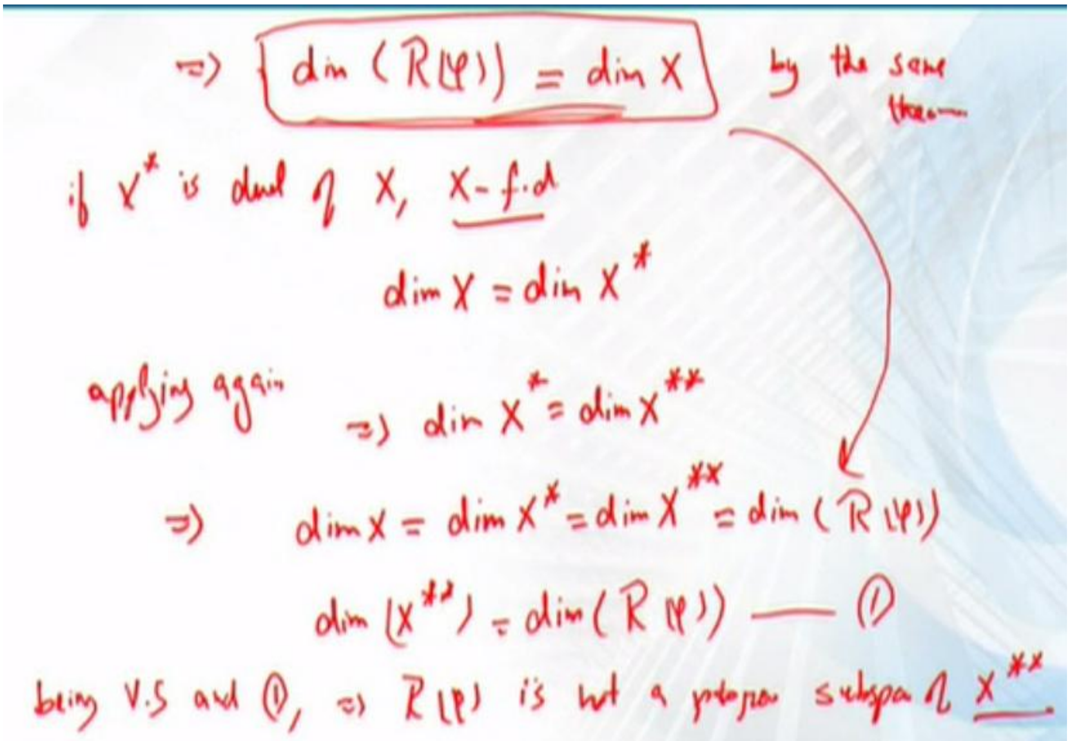


$g_{\alpha x+\beta y} = \alpha g_x + \beta g_y$

$\varphi(\alpha x+\beta y) = \alpha\varphi(x) + \beta\varphi(y)$







$R(\varphi) = X^{**}$ $\quad$ φ *is onto*

$X \cong X^{**}$ $\quad$ X *reflexive*

MODULE NO. 78

LINEAR TRANSFORMATION:

Q No.1:

Find the null space of $T:\mathbb{R}^3 \to \mathbb{R}^2$ represented by

$$\begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ -2x_1 + x_2 \end{bmatrix}$$

$2\times 3 \quad 3\times 1 \qquad 2\times 1$

What is meant by null space, it means we have to find those values of $x \in \mathbb{R}^3$ say $x = (x_1, x_2, x_3)$ such that we operate T the answer is zeros as

All those x are element of null space.

$$\begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ -2x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Also we can also say that

$$x_1 + 3x_2 + 2x_3 = 0$$
$$-2x_1 + x_2 = 0$$

We can solve it by using any linear algebra method that will give us solution like echelon form or reduced echelon form and the base of that solution is called basis of null space. Basis mean when apply the element of $\mathbb{R}^3$ the answer should be zero and get a system of linear equation. Find the solution of this system of linear equation. And after finding the solution find the basis that basis are basis of null space.

Example.

Q.NO2

Find the null space of $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $(\xi_1, \xi_2, \xi_3) \leftrightarrow (\xi_1, \xi_2, -\xi_1 - \xi_2)$

1) Basis of $\mathbb{R}(T)$
2) Basis of N(T)
3) Matrix representation.

MODULE NO.79

EXERCISES

DUAL BASIS

Example 1:

a): Find the dual basis of X when basis of X are $B = \{(1,-1,3),(0,1,-1),(0,3,-2)\}$,

Find $B^* = ?, X^* = ?$ do it yourself

b): let $\{f_1, f_2, f_3\}$ be basis of dual space for X and if X is given by

$e_1 = (1,1,1), \quad e_2 = (1,1,-1), \quad e_3 = (1,-1,1)$

Find $f_1(x), f_2(x), f_3(x)$ *when* $x = (0,1,0)$

MODULE NO.80

NORMED SPACES OF OPERATORS

- Examples of Dual Spaces
- $\mathbb{R}^n$

Isometric Isomorphism

A linear operator $\phi : X \to Y$. X, Y normed spaces, is said to be Isometric Isomorphism if

ϕ isbijective.

ϕ preserve norms.

That is for any

$x \in X$, $\|\phi(x)\| = \|x\|$ is

Module No.81

Examples Spaces of Operators

- Examples of Dual Spaces
- l^1

Space l^1

The dual space of l^n is l^∞ means that it is bijective, it is linear and it preserve norm.
After defining the map we shall prove these properties one by one.

Proof:

Module No.82

Bounded Linear Operators

Quiz: Complete norm spaces are called Banach spaces.

Theroem

Let*B(X, Y)* be the set of all bounded linear operators form a normed space *X* to a normed space *Y*.

If ***Y*** is a Banach space, then ***B(X, Y)*** is also a Banach.

Proof:

Let $\{T_n\}$ be an arbitrary Cauchy seq. in *B(X,Y)*.

We will show that $\{T_n\}$ converges to an operator *T* in *B(X, Y)*.Since $\{T_n\}$ is Cauchy for every

$\varepsilon > 0$ $\exists$ N such that $\|T_n - T_m\| < \varepsilon$ (m,n>N)

For all $x \in X$ and (m,n>N) we have

$$\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\|$$
$$\le \|T_n - T_m\|\|x\| < \varepsilon\|x\|$$

Thus for a fixed *x* and given $\overline{\varepsilon}$

Thus for a fixed x and given $\bar{\varepsilon}$ we may choose $\varepsilon = \varepsilon_x$ so that

$$\varepsilon_x \|x\| < \bar{\varepsilon}$$

$$\Rightarrow \|T_n(x) - T_m(x)\| < \varepsilon\|x\| = \varepsilon_x\|x\| < \bar{\varepsilon}$$

$\Rightarrow \{T_n(x)\}$ is a Cauchy seq. in Y.

Since Y is complete

Let $T_n(x) \xrightarrow{\text{converge}} y$ for some $y \in Y$

We can define a map

$$T: X \to Y$$

$$T(x) = y$$

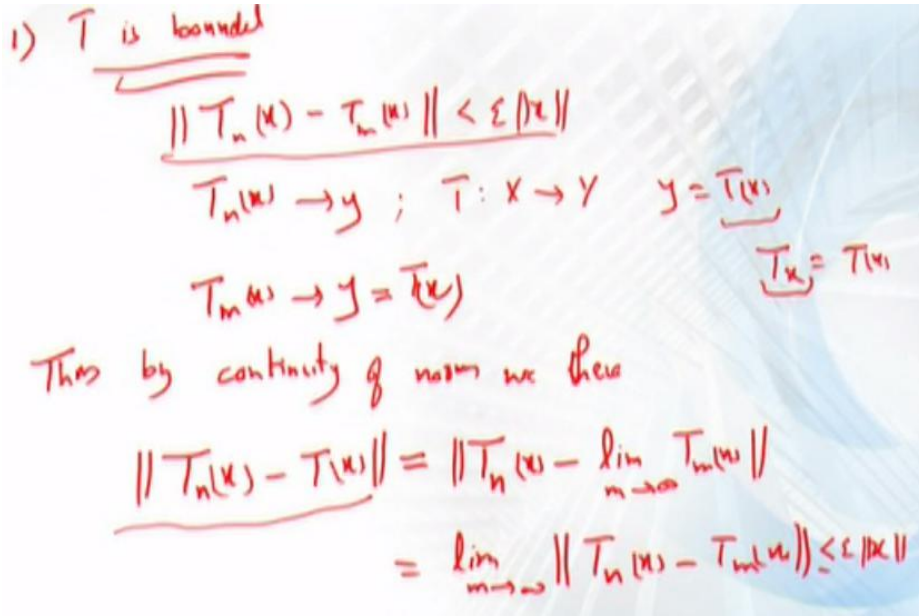
We will show that T is the required bounded linear operator.

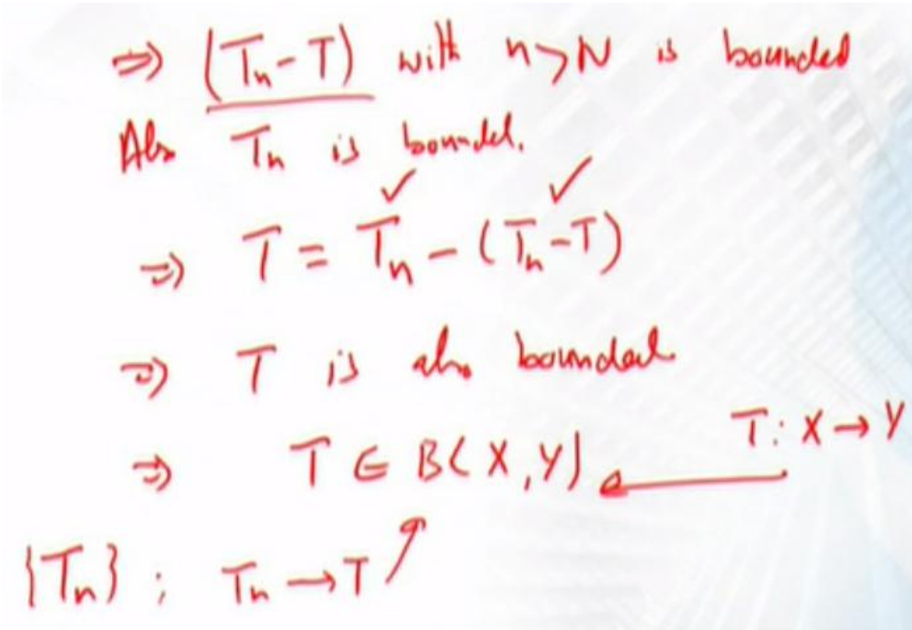
to show ① T is bounded ② $T_n \to T$

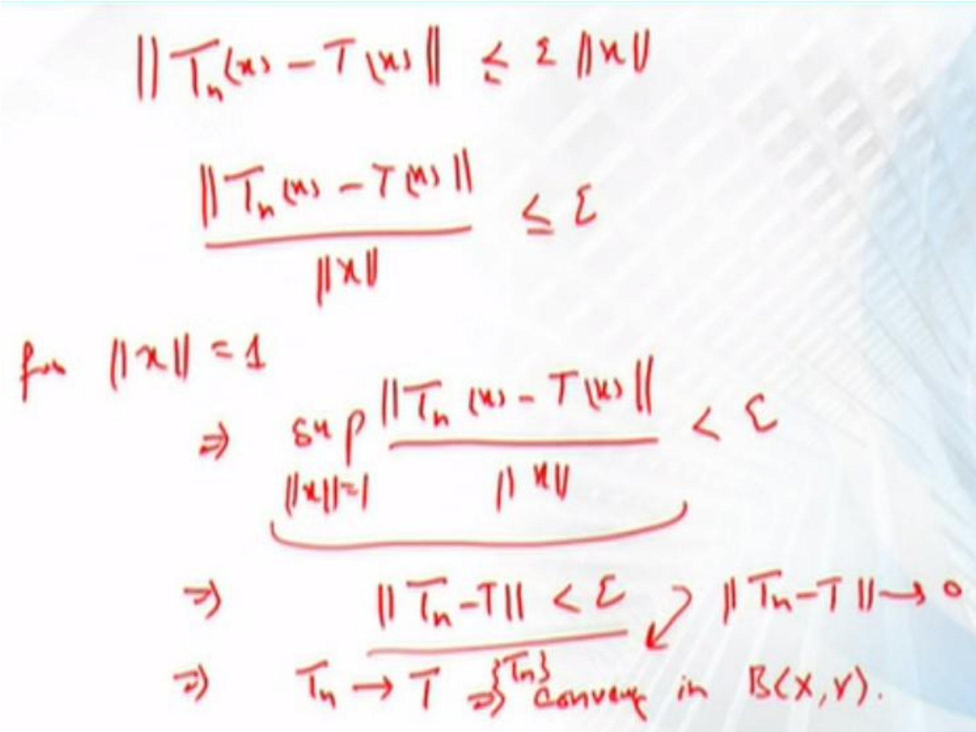
T is linear $T(\alpha x + \beta z)$: $x, z \in X$

$T(x) = y$, $T(z) = w$

$$\begin{aligned} T(\alpha x + \beta z) &= \lim_{n\to\infty} T_n(\alpha x + \beta y) \\ &= \lim_{n\to\infty} T_n(\alpha x) + \lim_{n\to\infty} T_n(\beta y) \\ &= \alpha \lim_{n\to\infty} T_n(x) + \beta \lim_{n\to\infty} T_n(y) \\ &= \alpha y + \beta w \\ &= \alpha T(x) + \beta T(z) \Rightarrow T \text{ is linear.} \end{aligned}$$


1) T is bounded
||Tn(x) - Tm(x)|| < ε||x||
Tn(x) → y ; T: X → Y y = T(x)
Tx = T(x)
Tm(x) → y = T(x)
Thus by continuity of norm we have
||Tn(x) - T(x)|| = ||Tn(x) - lim m→∞ Tm(x)||
= lim m→∞ ||Tn(x) - Tm(x)|| ≤ ε||x||


⇒ (Tn - T) with n > N is bounded
Also Tn is bounded.
⇒ T = Tn - (Tn - T)
⇒ T is also bounded
⇒ T ∈ B(X, Y)
T: X → Y
{Tn} ; Tn → T



Hence B(X, Y) is complete and Banach space.

MODULE NO.83

FINITE HILBERT SPACES

Functional analysis course consist of three major parts parts

1. Metric space is set and we define a space on it that has a certain properties. If it is completer then it is complete space means it should converge within the space
2. Normed Spaces: Norm is a vector space and we define a norm on vector space. Norm is a generalization of distance function.
3. Finite Hilbert Spaces (Inner Product Space)

Hilbert Space

Quiz: Complete inner product space is called a Hilbert Space.

In inner product the generalization is dot product.

Inner product Space

Let V be a vector space over a field F where F is $\mathbb{R}$ or $\mathbb{C}$.

An inner product in V is a function $<\bullet,\bullet>: V \times V \to F$ satisfying the following conditions:

Quiz:

Let $x, y, z \in V$; $\alpha \in$ F where α may be real or complex.

i. $<x,x> \geq 0; \quad <x,x> = 0 \quad \Leftrightarrow x = 0$

ii. $<\alpha x, y> = \alpha <x,y>$; but not true for second value as $<x, \alpha y> \neq \alpha <x,y>$

iii. $<x+y, z> = <x,z> + <y,z>$

iv. $<x,y> = \overline{<y,x>}$

$<\bullet,\bullet>: V \times V \to F$ inner product.

Inner Product Space

The pair $(V, <\bullet,\bullet>)$ is called an inner product space.

a): $<ax+by, z>$ $\quad$ *where* $\quad x, y, z \in V$, $a, b \in F$

Using (iii) property $<ax+by,z>=<ax,z>+<by,z>$

Using (ii) property $a<x,z>+b<y,z>$

$$<0,z>=<0.x,z>=0<x,z>=0$$

b): ***Quiz:***

for all $x,y\in V$, $a\in F$

$$<x,ay>=\overline{<ay,x>}=\overline{a<y,x>}$$
$$=\bar{a}\overline{<y,x>}=\bar{a}<x,y>$$

Module No.84

Cauchy Schwarz Inequality

Theorem:

For any two elements *x,y* is an inner product space *V*,

$|<x,y>|\le\|x\|.\|y\|$, the define norm is $\|x\|=\sqrt{<x,x>}$, x,y $\in$ V

Proof:

If x=y=0 then 0=0

Let at least one of x and y is not equal to zero

Let $|<x+\lambda y,x+\lambda y>|\ge 0$ by definition

$$<x,x+\lambda y>+<\lambda y,x+\lambda y>$$
$$<x,x+\lambda y>+y<y,x+\lambda y>$$

Module No.85

Norm on Inner Product space

Theorem:

In an inner product space V, the function $\|\ .\ \|:V\to\mathbb{R}^+$ given by

$\|x\|=\sqrt{<x,y>}$, $x\in V$ defines a norm in V.

Proof:

N1: $\|x\|\ge 0$

For a $x\in V$, $\|x\|=\sqrt{<x,x>}\ge 0$ as $<x,x>\ge 0$

N2:

$$\|x\|=0 \quad\Leftrightarrow\quad \sqrt{<x,x>}=0 \quad\Leftrightarrow\quad <x,x>=0 \quad\Leftrightarrow\quad x=0$$

N3: $\|\alpha x\|=|\alpha|\|x\|$

now $\|\alpha x\|=\sqrt{<\alpha x,\alpha x>}\ \Rightarrow\ \|\alpha x\|^2=<\alpha x,\alpha x>$

$\Rightarrow$ $\|\alpha x\|^2=\alpha\bar{\alpha}<x,x>=|\alpha|^2\|x\|^2$

N4: $\|x+y\|\le\|x\|+\|y\|\quad\forall$ x, y $\in$ V

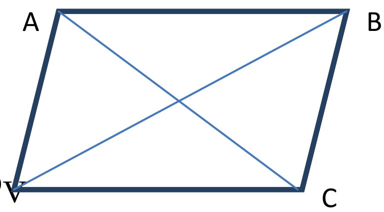
$$\|x+y\|^2 = <x+y, x+y>$$
$$= <x, x+y> + <y, x+y>$$
$$= \overline{<x+y, x>} + \overline{<x+y, y>}$$
$$= \overline{<x,x> + <y,x>} + \overline{<x,y> + <y,y>}$$
$$= \overline{<x,x>} + \overline{<y,x>} + \overline{<x,y>} + \overline{<y,y>}$$

Now $= <x,x> + <x,y> + \overline{<x,y>} + <y,y>$

$$= \|x\|^2 + 2\operatorname{Re} <x,y> + \|y\|^2 \qquad \because \operatorname{Re}(z) \le |z|$$
$$\le \|x\|^2 + 2|<x,y>| + \|y\|^2$$
$$\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \qquad \because \; |<x,y>| \le \|x\|\|y\|$$
$$= (\|x\| + \|y\|)^2$$
$$\|x+y\|^2 \le \|x\| + \|y\|$$

Module No.86

Parallelogram Law



$$\overline{AC}^2 + \overline{BD}^2 = 2\left(\overline{AB}^2 + \overline{BC}^2\right)$$ **Quiz**

Theorem:

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \qquad \text{for all } x,y \in V$$

Proof:

$$\|x+y\|^2 = <x+y, x+y>$$
$$= <x,x> + <x,y> + \overline{<x,y>} + <y,y>$$
$$= \|x\|^2 + 2\operatorname{Re} <x,y> + \|y\|^2 \qquad \text{....(i}$$

Replace y=-y

$$\|x-y\|^2 = <x+y, x+y>$$
$$= <x,x> - <x,y> - \overline{<x,y>} + <y,y>$$
$$= \|x\|^2 - 2\operatorname{Re} <x,y> + \|y\|^2 \qquad \text{.....(ii}$$

Adding (i and (ii

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

That we have to prove.

Special Case:

Another result from above equations is

Subtracting (ii from (i

$$\|x+y\|^2 - \|x-y\|^2 = 4\operatorname{Re} <x,y>$$

If V is a real inner product space

Re(z)=z or Re<x,y>=<x,y>

$$<x,y> = \frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2\right\}$$

The above form is when V is a real inner product space not complex space.

Module No.87

- Polarization Identity
- Appolonius Identity

Polarization Identity

For any x, y in complex inner product space

$$<x,y>=\frac{1}{4}\left\{\|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2\right\}$$

We have to prove this complex inner product space.

Proof:

Let $x, y \in V$

$$\|x+y\|^2=<x+y,x+y>$$
$$=\|x\|^2+2\operatorname{Re}<x,y>+\|y\|^2$$
$$=\|x\|^2+<x,y>+\overline{<x,y>}+\|y\|^2$$
$$=\|x\|^2+<x,y>+<y,x>+\|y\|^2 \quad ....(i$$

If we replace y=-y

$$\|x+y\|^2=\|x\|^2+<x,-y>+<-y,x>+\|-y\|^2$$
$$=\|x\|^2-<x,y>-<y,x>+\|y\|^2 \quad ....(ii$$

Replace $y=iy$ in eq(i

$$\|x+iy\|^2=\|x\|^2+<x,iy>+<iy,x>+\|iy\|^2$$
$$=\|x\|^2+\bar{i}<x,y>+i<y,x>+\|y\|^2 \qquad \because \|iy\|^2=<iy,iy>=i\bar{i}<y,y>=-i^2<y,y>$$
$$=\|x\|^2-i<x,y>+i<y,x>+\|y\|^2 \quad ......(iii$$

Replace $y=-iy$ in eq(i

$$\|x-iy\|^2=\|x\|^2+<x,-iy>+<-iy,x>+\|-iy\|^2$$
$$=\|x\|^2+i<x,y>-i<y,x>+\|y\|^2 \quad ......(iv$$

Subtracting (ii from (i

$$\|x+y\|^2-\|x-y\|^2=4\operatorname{Re}<x,y> \quad ....(v$$

Subtracting (iv from (iii

$$\|x+iy\|^2-\|x-iy\|^2=2\{i<y,x>-i<x,y>\}$$
$$=-2i\{<x,y>-<y,x>\}=-2i\left\{<x,y>-\overline{<x,y>}\right\}$$
$$=-2i(2i)\operatorname{Im}<x,y>=4\operatorname{Im}<x,y> \quad .....(vi$$

Now we solve $4\operatorname{Re}<x,y>+4\operatorname{Im}<x,y>$

$$\|x+y\|^2-\|x-y\|^2+i\|x+y\|^2-i\|x-y\|^2=4\{<x,y>\}$$

Appolonius Identity

$$\|z-x\|^2+\|z-y\|^2=\frac{1}{2}\|x-y\|^2+2\left\|z-\frac{1}{2}(x+y)\right\|^2 \quad , \quad x,y,z\in V$$

Using parallelogram law

$$\|x'+y'\|^2+\|x'-y'\|^2=2\|x'\|^2+2\|y'\|^2 \quad \text{put} \qquad x'=z-x,\ y'=z-y$$

Self-assignment

Module No.88

- **Space** $C\left[0,\frac{\pi}{2}\right]$
- **Space** l^p

Counter example 1: Space $C\left[0,\frac{\pi}{2}\right]$

Inner product define a norm and under this norm

Every inner product space is a norm space.

Every norm space is not an inner product space. This is not true always.

If a space is inner product then it satisfied the parallelogram law otherwise it is not an inner product space.

We take a norm and built an inner product space and then prove that this inner product space does not satisfy the parallelogram law.

The given set is $C\left[0,\frac{\pi}{2}\right]$ real valued continuous function defined on C[a, b].

The norm of function $f\in C\left[0,\frac{\pi}{2}\right]$, is

$$\|f\|=\sup_{x\in\left[0,\frac{\pi}{2}\right]}|f(x)| \qquad ,$$

Let $f,g\in C\left[0,\frac{\pi}{2}\right]$; $f(t)=\sin t, \quad g(t)=\cos t$

We know that sin and cos are continuous functions. Let $C\left[0,\frac{\pi}{2}\right]$ is an inner product space

where the inner product $<\bullet,\bullet>$ define by

$$\|f\|=\sqrt{<f,f>} \quad \Rightarrow \quad <f,f>=\|f\|^2$$

$$\|f\|=\sup_{x\in\left[0,\frac{\pi}{2}\right]}|f(x)|$$

$$\|f+g\|^2+\|f-g\|^2=2\|f\|^2+2\|g\|^2$$

As $f(t)=\sin t, \quad g(t)=\cos t$

$$\|f\|=\sup_{x\in\left[0,\frac{\pi}{2}\right]}|\sin(x)|=1=\|g\|$$

$$\|f+g\| = \sup_{x\in\left[0,\frac{\pi}{2}\right]} |f(x)+g(x)|$$

$$= \sup_{x\in\left[0,\frac{\pi}{2}\right]} |\sin x+\cos x| = \sqrt{2}$$

$$\|f-g\| = 1$$

Now

$$\|f+g\|^2+\|f-g\|^2 = 2\|f\|^2+2\|g\|^2$$

$$\left(\sqrt{2}\right)^2+(1)^2 = 2\times1^2+2\times1^2$$

$$2+1=2+2$$

$$3=4$$

But $3\neq4$ so our supposition is wrong. This inner product space does not satisfied parallelogram law. Hence every norm space is not inner product space.

Counter example2: Space l^p

l^p Collection of all bounded sequences,
$P>1, P\neq2$ if p=2 then it will give inner product space

$$\{x_i\},\quad \|x\| = \sqrt[p]{\sum_{i=1}^{\infty}|x_i|^p}$$

We will see that $<x,x>=\|x\|^2$ is an inner product space or not. We will check this if it satisfied the parallelogram or not.

Let

$$x=(1,1,0,0,.........)\ ;\ y=(1,-1,0,0,...........)$$

$$\|x\| = \sqrt[p]{1^p+1^p+0+0+....} = \sqrt[p]{2} = 2^{\frac{1}{p}}$$

$$\|y\| = \sqrt[p]{1^p+(-1)^p+0+0+....} = \sqrt[p]{2} = 2^{\frac{1}{p}}$$

$$x+y=(2,0,0,0,.....)\quad\Rightarrow\quad \|x+y\| = \sqrt[p]{2^p} = 2^{p\times\frac{1}{p}} = 2$$

$$x-y=(0,2,0,0,.....)\quad\Rightarrow\quad \|x-y\| = \sqrt[p]{2^p} = 2^{p\times\frac{1}{p}} = 2$$

$$\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2$$

$$2^2+2^2=2\times2^{\frac{1}{p}}+2\times2^{\frac{1}{p}}$$

$$8=4\times2^{\frac{2}{p}}\quad \text{as } p>1,\ p\neq2$$

The values on both sides are also not equal so this does not satisfied the parallelogram law. Contradict to our supposition. So norm space is not an inner product space.

MODULE NO.90

➢ THEOREM (CONTINUITY OF INNER PRODUCT)

Theorem:

Let V be any inner product space. For any sequences $\{x_n\}$ and $\{y_n\}$ in V

$x_n \to x$, $y_n \to y$ implies $< x_n, y_n > \to < x, y >$

Proof:

$$|< x_n, y_n > - < x, y >|$$
$$= |< x_n, y_n > - < x_n, y > + < x_n, y > - < x, y >|$$
$$= |< x_n, y_n - y > + < x_n - x, y >|$$
$$\leq |< x_n, y_n - y >| + |< x_n - x, y >|$$

Now from Cauchy Swarzinequality

$$|x, y| \leq \|x\|\|y\|$$
$$\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\|$$

Given that $x_n \to x$, $y_n \to y$ so,

$$\|y_n - y\| = \|y - y\| = 0 \quad , \quad \|x_n - x\| = \|x - x\| = 0 \quad \text{as n} \to \infty$$

As $n \to \infty$

$$|< x_n, y_n > - < x, y >| \leq 0$$
$$< x_n, y_n > \to < x, y > \quad \text{as } n \to \infty$$

Theorem:

If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in V, then the inner product $< x_n, y_n >$ is a Cauchy sequence in F.

Proof:

$\{x_n\}, \{y_n\}$ are Cauchy sequence

To show $< x_n, y_n >$ is also Cauchy Sequence.

$$\Rightarrow \quad \|x_n - x_m\| \to 0 \quad ; \quad \|y_n - y_m\| \to 0, \ \text{m,n} \to \infty$$
$$|< x_n, y_n > - < x_m, y_m >| = |< x_n, y_n > - < x_n, y_m > + < x_n, y_m > - < x_m, y_m >|$$
$$= |< x_n, y_n - y_m > + < x_n - x_m, y_m >|$$
$$\leq |< x_n, y_n - y_m >| + |< x_n - x_m, y_m >|$$
$$\leq \|x_n\|\|y_n - y_m\| + \|x_n - x_m\|\|y_m\|$$
$$\Rightarrow \quad |< x_n, y_n > - < x_m, y_m >| \to 0, \text{ as } \ \text{n,m} \to \infty$$
$$\Rightarrow \quad < x_n, y_n > \text{ is a Cauchy Sequence}$$

MODULE NO.91

Examples of Inner product spaces

➢ SPACE $\mathbb{R}^n$

➢ SPACE $\mathbb{C}^n$

- **SPACE** $\mathbb{C}[a,b]$
- **SPACE** l^n
- **SPACE** P_n (Collection of all polynomials of degree n)

Proof:

1. $\mathbb{R}^n$, the elements are of the form

$$x=(x_1,x_2,.........x_n)\ \ ;\ y=(y_1,y_2,.........y_n)$$

The inner product form is $<x,y>=\sum_{i=1}^{n} x_i y_i$ (Note: check all axiom self-assignment)

The Norm is $\|x\|=\sqrt{<x,x>}=\sqrt{\sum_{i=1}^{n} x_i x_i}=\sqrt{\sum_{i=1}^{n} x_i^2}$

2. $\mathbb{C}^n$

The elements are $z=(z_1,z_2,........,z_n)\ \ ;\ z'=(z_1',z_2',.........,z_n')$ if conjugate does not define then it does not satisfied the second or third axiom of inner product space.

The inner product form is $<z,z'>=\sum_{i=1}^{n} z_i \bar{z}'_i$ (Note: check all axiom self-assignment)

3. $\mathbb{C}[a,b]$ be the space of all continuous function defined on [a, b].

$$<f,g>=\int_a^b f(t).\overline{g(t)}dt$$ define an inner product on C[a, b]

(Note: complex function can also be including. In previous example the C[a, b] was not inner product space with define function definition).

$$<\bullet,\bullet>:V\times V\rightarrow F$$

We will check all four properties of inner product as

i): $<f,f>=0 \quad \Leftrightarrow \quad f=0$

ii): $<f+g,h>=<f,h>+<g,h>$

iii): $<\alpha f,g>=\alpha<f,g>$

iv): $<g,f>=\overline{<f,g>}$

it define inner product and is define inner product space.

4. l^n is a space of sequences.

$$l^2 : x\{x_i\}$$

The condition or norm is

$$\sum_{i=1}^{\infty} |x_i|^2<\infty$$

Let defined the inner product of $y=\{y_i\}$ is

$$<x,y>=\sum_{i=1}^{\infty} x_i \overline{y_i}$$

Checd all four axioms as exercise for inner product.

5. P_n

Let P_n be the collection of all polynomial of degree n(or less than n).

We can write this as $a_n x^n + a_{n-1}x^{n-1} + ...+ a_1 x + a$ e.g $3x^2 - 2x + 1$ of degree two.

Let $u(x), v(x) \in P_n$

The inner product is

$$< u(x), v(x) >= \int_a^b u(x)v(x)dx \quad , \quad x \in [a,b]$$

with this define P_n is an inner product space.

We have not defined conjugate of v(x) as the interval defined is a real valued so its conjugate is also real valued.

Module No.92

Orthogonal Systems

➢ Pythagorean Theorem

The dot product of two vectors when they are perpendicular is zero. Similarly in inner product if two vectors are perpendicular then their inner product is zero.

Theorem:

In an inner product space V and x, y in V if $x \perp y$ then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

Proof:

$$\|x+y\|^2 =< x+y, x+y >$$

$$= < x,x > + < x,y > + < y,x > + < y,y >$$

As x and y are perpendicular so $< x,y >= 0, < y,x >= 0$

$$\|x+y\|^2 =< x,x > + < y,y >= \|x\|^2 + \|y\|^2$$

Generalized form:

$\{x_1, x_2, ...x_n\}$ be nonzero vectors in V inner product space such that

$< x_i, x_j >= 0 \quad , \quad i \neq j$

This system $\{x_1, x_2, ...x_n\}$ is called orthogonal system as all vectors inside it are perpendicular to each other.

The generalized statement is $\|x_1 + x_2 + ...+ x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + ...+ \|x_n\|^2$

The idea of proof is

$$\left\|\sum_{i=1}^n x_i\right\|^2 = \langle \left\|\sum_{i=1}^n x_i, \sum_{i=1}^n x_i\right\| \rangle$$

$$=< x_1 + ...+ x_n, x_1 + ...+ x_n >$$

$$=< x_1, x_1 + ...+ x_n > + ...+ < x_n, x_1 + ...+ x_n >$$

$$= \sum_{i=1}^n \sum_{j=1}^n < x_i, x_j >$$

$< x_i, x_j >= \|x_i\|^2 \quad$, if $i \neq j \quad < x_i, x_j >= 0$ and for $i = j$ then $< x_i, x_j >= \|x_i\|^2$

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n}\|x_i\|^2$$

Module No.93

Orthogonal Systems

➢ Theorem (Linearly Independence)

Any sequence $\{x_n\}$ of non-zero mutually orthogonal vectors in an inner product space V is linearly independent.

Proof: do it yourself

Let $x = (x_1, x_2, ...x_n)$ be the orthogonal sequence.

Remark:

If $< x, x_1 >= 0$, $\forall$ i=1,2,.......,n $\Rightarrow$ $\left\langle \sum_{i=0}^{n} \alpha_i x_i, x \right\rangle = 0$

$$\left\langle \sum_{i=0}^{n} a_i x_i, x \right\rangle = \langle a_1 x_1 + a_2 x_2 + ... + a_n x_n, x\rangle = a_1 \langle x_1, x\rangle + ... + a_n \langle x_n, x\rangle = 0$$

MTH 641

Functional Analysis

MODULE NO. 29 TO 63

(MID TERM SYLLABUS)

THESE ARE JUST SHORT HINT FOR THE PREPARATION OF MTH 641

Don't look for someone who can solve your problems,
Instead go and stand in front of the mirror,
Look straight into your eyes,
And you will see the best person who can solve your problems!
Always trust yourself.

A gift from Unknown to Juniors VU Mathematics Students

Module No. 29

Theorem (Complete Subspace):

Theorem:

A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X.

As this condition is if and only if so vice versa.From previous theorem we have

Theorem:

Let M be a nonempty subset of a metric space $d(X,d)$ *and* $\bar{M}$ its closure as defined before then,

a): $x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that $x_n \to x$.

b): M is closed if and only if the situation $x_n \in M,\ \ x_n \to x$ implies that $x \in M$.

Proof:

Let M is subspace of X over d is then (X,d) complete.

$$M \subset (X,d),$$

M is complete if and only if M is closed, and M is closed if and only if

$$M = \bar{M}.$$

Now we can say that

$$M \subset (X,d) \Leftrightarrow M = \bar{M}.$$

Suppose M is complete and we need to show that $M = \bar{M}$.

Now by definition $M \subseteq \bar{M}$. Now we need to prove that $\bar{M} \subseteq M$ (to be proved).

"Let M be a nonempty subspace of a metric space $d(X,d)$ *and* $\bar{M}$ its closure as defined before then,

From the part "a" of previous theorem

a):

$x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that $x_n \to x$.

Now $x \in \bar{M}$

As M is a subspace of a complete metric space $d(X, d)$ *and* x_n is also in *X so,*

$\Rightarrow$ there is a sequence (x_n) in X such that $x_n \to x$.

Since every convergent sequence in a metric space is Cauchy, then (x_n) is Cauchy.

Our supposition is that M is complete. So, (x_n) converges in M

$\Rightarrow$ $x_n \to x \in M$

$\Rightarrow$ $\bar{M} \subseteq M$

we start from $x \in \bar{M}$ and obtained $x \in M$

$\Rightarrow$ $M = \bar{M}$

Hence M is closed.

Conversely:

M is closed

$\Rightarrow$ $M = \bar{M}$

and we need to show that M is complete.

For this we need to show that every Cauchy sequence in *M* converges in

$M, \; x \in M$.

Let (x_n) be a Cauchy sequence in M such that $x_n \to x$,

By the previous theorem $x \in \bar{M}$

but $\bar{M} = M$ $\Rightarrow$ $x \in M$

Since (x_n) is an arbitrary sequence,

$\Rightarrow$ true for all Cauchy sequences in *M*,

Hence proved

Module No. 30

Theorem (Continuous Mapping):

Theorem:

A mapping $T: X \to Y$ of a metric space (X, d) into a metric space $(Y, \tilde{d})$ is continuous at a point $x_o \in X$ if and only if $x_n \to x_o$ implies $Tx_n \to Tx_o$.

Proof:

Suppose T is continuous, we will prove that if $x_n \to x_o$ implies $Tx_n \to Tx_o$.

T is continuous means $T: X \to Y$

a given $\varepsilon > 0$ there exist $\delta > 0$ such that

$$d(x, x_o) < \delta \qquad \tilde{d}(Tx,\ Tx_o) < \varepsilon$$

So, let $x_n \to x_o$ there exist a $\mathbb{N}$ such that for all n> $\mathbb{N}$ we have

$$d(x_n,\ x_o) < \delta$$

This is δ of convergence.

$$\tilde{d}(Tx,\ Tx_o) < \varepsilon \quad , \quad n > \mathbb{N}$$

By definition $\qquad Tx_n \to Tx_o$

Converse:

Let $x_n \to x_o$ implies $Tx_n \to Tx_o$ for all x_o .

We have to show that T is continuous by contradiction.

We suppose that it is not true then there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is some $x \neq x_o$ such that

$$d(x, x_o) < \delta \qquad \Rightarrow \qquad \tilde{d}(Tx, Tx_o) \geq \varepsilon$$

In particular $\quad \delta = \dfrac{1}{n} \qquad d(x, x_o) < \dfrac{1}{n}$

$$\Rightarrow \qquad x_n \to x_o$$

$$\Rightarrow \qquad Tx\ not \to Tx_o$$

$$\Rightarrow \qquad \tilde{d}(Tx, Tx_o) \geq \varepsilon$$

Module No. 31

Exmaples (Completeness):

- $\mathbb{R}$

We will show that $\mathbb{R}$ and $\mathbb{C}$ are completes. In this module we show only that $\mathbb{R}$ is a complete metric space which means every sequence in $\mathbb{R}$ is convergent in $\mathbb{R}$ and every Cauchy sequence is convergent.

Lemma a:

Every Cauchy sequence in a metric space is bounded.

This is for every metric space.

Lemma b:

If a Cauchy sequence has a subsequence that converges to $\bar{x}$, then the sequence converges to $\bar{x}$.

Proposition:

Every sequence of real numbers has a monotone subsequence.

Proof:

Suppose the sequence $\{x_n\}$ has no monotone increasing subsequence, we will show that it has a monotone decreasing sequence. The sequence $\{x_n\}$ must have a first term, say x_{n_1} such that all subsequent terms are smaller

$n > n_1$ means that n comes after n_1, $\Rightarrow x_n < x_{n_1}$.

Otherwise, $\{x_n\}$ would have a monotone increasing subsequence.

Similarly, the remaining sequence $\{x_{n_2}, x_{n_3}........\}$ it must have some first term.

Let first term of remaining sequence is x_{n_2} , Now this x_{n_2} is less than x_{n_1}, $x_{n_2} < x_{n_1}$.

Now we take the remaining sequence $\{x_{n_3}........\}$, whose first term is x_{n_3}, now this $x_{n_3} < x_{n_2}$.

Hence this process will continue $x_{n_1} > x_{n_2} > x_{n_3}.............$,

and is a monotonic decreasing subsequence.

We have proved that every sequence of Real numbers has a monotone subsequence.

Now using lemma a, b and proposition we have a theorem.

Theorem:

$\mathbb{R}$ is a completer metric space, i.e., every Cauchy sequence of real numbers converges.

Proof:

Let $\{x_n\}$ be a Cauchy sequence.

Remark a implies that $\{x_n\}$ is bounded. Now if the given Cauchy sequence is bounded then its subsequence is also bounded.

Every subsequence of $\{x_n\}$ is bounded.

Also $\{x_n\}$ has a monotone subsequence.Now $\{x_n\}$ is monotone as well as bounded.

Monotone Convergence Theorem:

If a sequence $\{x_n\}$ is monotone and bounded this implies that it is convergent.

This implies that subsequence is convergent. Now using remarks 2 if we have a Cauchy sequence has a subsequence is convergent than the original sequence will also convergent. $\{x_n\}$ is convergent. As this general sequence $\{x_n\}$ from $\mathbb{R}$ so, every Cauchy sequence from $\mathbb{R}$ is convergent which means that $\mathbb{R}$ is complete.

Module No. 32

Exmaples (Completeness):

- $\mathbb{R}^n$

Here we prove that $\mathbb{R}^n$ is complete

Example:

The Euclidean space $\mathbb{R}^n$ is complete.

Proof:

Let $\mathbb{R}^n$, the elements of $\mathbb{R}^n$ are n-tuples say

$$x = (a_1, a_2, ...a_n) \quad ; \quad a_i, b_i \in \mathbb{R}$$
$$y = (b_1, b_2, ...b_n)$$

The distance function in $\mathbb{R}^n$ is

$$d(x, y) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + ...+ (a_n - b_n)^2}$$

Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}^n$

$$x_m = (a_1^{(m)}, a_2^{(m)}, ...a_n^{(m)})$$

(i.e .

$$x_1 = (a_1^{(1)}, a_2^{(1)}, ...a_n^{(1)})$$
$$x_1 = (a_1^{(2)}, a_2^{(2)}, ...a_n^{(2)})$$
$$. \qquad)$$
$$.$$
$$.$$
$$x_r = (a_1^{(r)}, a_2^{(r)}, ...a_n^{(r)})$$

The distance function is

$$d(x_m, x_r) = \sqrt{(a_1^{(m)} - a_1^{(r)})^2 + (a_2^{(m)} - a_2^{(r)})^2 + ...+(a_n^{(m)} - a_n^{(r)})^2} < \varepsilon \quad , \quad \forall m, r > N$$

Taking power two, we have

$$(a_1^{(m)} - a_1^{(r)})^2 + (a_2^{(m)} - a_2^{(r)})^2 + ...+(a_n^{(m)} - a_n^{(r)})^2 < \varepsilon^2$$

$$(a_j^{(m)} - a_j^{(r)})^2 < \varepsilon^2,$$
$$\left|a_j^{(m)} - a_j^{(r)}\right| < \varepsilon, \quad \forall m, r > N, \quad j = 1, 2, ...n$$

For a fixed j $(a_j^{(1)} + a_j^{(2)} + ...)$ is a Cauchy sequence, this implies it is converging in $\mathbb{R}$ because $\mathbb{R}$ is a complete metric space.

$$\Rightarrow \qquad a_j^{(m)} \to a_j^{(r)}, \quad m \to \infty, \quad a_j \in \mathbb{R}, \text{ j=1,2,......,N}$$

$$a_1^{(m)} \to a_1$$
$$a_2^{(m)} \to a_2$$
$$.$$
$$.$$
$$a_n^{(m)} \to a_n$$

All these values $a_1, a_2, ...a_n$ called x, As $x = (a_1, a_2, ...a_n) \in \mathbb{R}^n$

$$\Rightarrow \qquad d(x_m, x) \le \varepsilon, \qquad r \to \infty, \quad x_m \to x$$

$\Rightarrow$ x is a limit of $\langle x_m \rangle$,

$\Rightarrow$ $\langle x_m \rangle$ was general element

$\Rightarrow$ $\mathbb{R}^n$ is completer

Module No. 33

Exmaples (Completeness):

- $\mathbb{C}[a,b]$

Here we prove that $\mathbb{C}[a,b]$ is complete metric space

Example:

The function space $\mathbb{C}[a,b]$ is complete; here $[a,b]$ is any given closed interval on $\mathbb{R}$.

Let (x_m) be any Cauchy sequence in $\mathbb{C}[a,b]$.

The metric space in $\mathbb{C}[a,b]$ is

$$d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)| \ , \quad \text{where [a,b]=J}$$

There is an N such that for all m,n>N

$$d(x_m, x_n) = \max_{t\in J} |x_m(t) - x_n(t)| < \varepsilon$$

Hence for any fixed $t = t_o \in J$

$$|x_m(t_o) - x_n(t_o)| < \varepsilon$$

$\Rightarrow$ $x_1(t_o), x_2(t_o), ...$ is a Cauchy sequence of real numbers and $\mathbb{R}$ is complete.

$\Rightarrow$ sequence converges $x_m(t_o) \to x(t_o)$ as $m \to \infty$

In this way to each $t \in J$, a unique real number x(t). This definespointwise function on J.

Now we well show that $x(t) \in \mathbb{C}[a,b]$ *and* $x_m \to x$

$$\max_{t\in J} |x_m(t) - x(t)| \le \varepsilon$$

We are comparing with $\max_{t\in J} |x_m(t) - x_n(t)| < \varepsilon,$ *as* $n \to \infty$

$\Rightarrow$ for every $t \in J$ $|x_m(t) - x(t)| \le \varepsilon$

$\Rightarrow$ $x_m(t)$ converges to x(t) uniform;

If a sequence (x_m) of continuous function on [a,b] converges on [a,b] and the convergence is uniform on [a,b], then the limit function x is continuous on [a,b]

$\Rightarrow$ $x(t)$ is continuous on [a,b]

$\Rightarrow$ $x(t) \in \mathbb{C}[a,b]$.

Module No. 34

Exmaples (Completeness):

- l^{∞}

Here we prove that l^{∞} is complete metric space

Example:

The function space l^{∞} is complete; here $[a,b]$ is any given closed interval on $\mathbb{R}$.

Proof:

Let (x_m) be any Cauchy sequence in l^{∞} such that

In l^{∞} the elements are of the form

$$x=(a_1,a_2,.......) \quad \Rightarrow \quad |a_j|<c_x$$

$$y=(b_1,b_2,.......), \quad \Rightarrow \quad |b_j|<c_y$$

The distance or metric function is

$$d(x,y)=\sup_{j\in\mathbb{N}}|a_j-b_j|$$

Here $\quad x_m=(a_1^{(m)},a_2^{(m)},........),\quad$ as

$$x_1=(a_1^{(1)},a_2^{(1)},........),$$
$$x_2=(a_1^{(2)},a_2^{(2)},........) \quad \text{so on}$$

For any q>0 , there exist $\mathbb{N}$ such that for all m,n> $\mathbb{N}$.

$$d(x_m,y_n)=\sup_{j\in\mathbb{N}}|a_j^{(m)}-b_j^{(n)}|$$

So, if $\sup<\varepsilon$ for a fixed j

$$|a_j^{(m)}-a_j^{(n)}|<\varepsilon \quad , \quad m,n\geq\mathbb{N}$$

$\Rightarrow$ for every fixed j, the sequence $(a_j^{(1)},a_j^{(2)},........)$ is a Cauchy sequence of real numbers $\mathbb{R}$.

Since $\mathbb{R}$ is complete, $a_j^{(m)}$ is convergent in $\mathbb{R}$.

$$a_j^{(m)}\to a_j\in\mathbb{R} \quad as \quad m\to\infty \quad for\ j=1,2,..........$$

For these infinite limits $a_1, a_2, \ldots\ldots\ldots\ldots\ldots$ such that $a_1^{(m)} \to a_1, \quad a_2^{(m)} \to a_2, \ldots\ldots\ldots\ldots\ldots$

We define $\quad x = (a_1, a_2, \ldots\ldots\ldots\ldots) \in \mathbb{R}$

We need to prove $\quad x = (a_1, a_2, \ldots\ldots\ldots\ldots) \in l^\infty$

$$\left|a_j^{(m)} - a_j^{(n)}\right| < \varepsilon$$

$\Rightarrow \quad \left|a_j^{(m)} - a_j\right| < \varepsilon \quad$ as $n \to \infty$. then $x_m \to x$

From above inequality,

$$d(x, y) = \sup\left|a_j^{(m)} - a_j\right| < \varepsilon$$

Which means $\quad x_m \to x$

Since $\quad x_m = (a_j^{(m)}) \in l^\infty$

$$\left|a_j^{(m)}\right| < k_m \qquad \textit{for all } j$$

$$\begin{aligned} \left|a_j\right| &= \left|a_j - a_j^{(m)} + a_j^{(m)}\right| \\ &\leq \left|a_j - a_j^{(m)}\right| + \left|a_j^{(m)}\right| \\ &< \varepsilon + k_m \end{aligned}$$

$\Rightarrow \quad a_j$ is bounded , $\quad x = \left|a_j\right| \in l^\infty$

Module No. 35

Exmaples (Completion of Metric Spaces):

- Space $\mathbb{Q}$
- Space of Polynomials
- Isometric mappings/spaces

here we prove that l^∞ is complete metric space

Isometric Mappings:

Let $\quad X = (X, d)$ and $\tilde{X} = (\tilde{X}, d)$ be metric spaces.

A mapping $T : X \to \tilde{X}$ is said to be isometric or isometry if T preserve distance.

Preseve distance mean after applying the mapping the distance is preserve, i.e. for all $x, y \in X$

$$\tilde{d}(T_x, T_y) = d(x, y)$$

Isometric Spaces:

The space X is said to be isometric with space $\tilde{X}$ if there exist a bijectiveisometry of X onto $\tilde{X}$.

X and $\tilde{X}$ are then called isometric spaces.

Theorem(Completion)

For a metric space $X=(X,d)$ there exists a complete metric space $\hat{X}=(\hat{X},d)$ which has a subspace W that is isometric with X and is dense in $\hat{X}$.

This space $\hat{X}$ is unique except for isometries, that is if $\tilde{X}$ is any complete metric space having a dense subspace $\tilde{W}$ isometric with X, then $\tilde{X}$ and $\hat{X}$ are isometric.

Module No. 36

Vector Space

Definition:

A vector space (or linear space) over a field K is a nonempty set X of elements x,y,……………….(called vectors) together with two algebraic operations.

These operations are called vector addition and multiplication of vectors by scalars, that is, by elements of K.

Vector Addition associates with every ordered pair (x,y) of vectors a vector x+y, called the sum of x and y, in such a way that the following properties hold

Vector addition is commutative and associative.

There exists a vector 0, called the zero vector, and for every vector x there exists a vector –x, such that for all vectors.

Vector Space

$$x+0=x$$

$$x+(-x)=0$$

Multiplication by scalar associates with every vector x and scalar α a vector αx (also written $x\alpha$), called the product of α and x, in such a way that for all vectors x, y and scalar α, β we have

$$\alpha(\beta x)=(\alpha\beta)x \qquad \text{or} \qquad 1x=x$$

and the distributive laws hold.

Module No. 37

Examples(Vector Space)

- ***Space*** $\mathbb{R}^n$
- ***Space*** $\mathbb{C}^n$
- ***Space*** $\mathbb{C}[a,b]$
- ***Space*** l^2

1. ***Space*** $\mathbb{R}^n$

$$x = (\xi_1, ..., \xi_n) \;, \qquad \xi_i \in \mathbb{R}$$

$$y = (\eta_1, ..., \eta_n) \;, \qquad \eta_i \in \mathbb{R}$$

Addition:

$$\mathbf{x+y} = (\xi_1 + \eta_1, ..., \xi_n + \eta_n)$$

scalar Multiplication:

let α be a scalar then

$$\alpha x = (\alpha \xi_1, ..., \alpha \xi_n)$$

Now addition and scalar multiplication in $\mathbb{R}^n$ is a vector space.

2. ***Space*** $\mathbb{C}^n$

Addition:

Let $$x = (\xi_1, ..., \xi_n) \;, \qquad \xi_i \in \mathbb{C}$$

$$y = (\eta_1, ..., \eta_n) \;, \qquad \eta_i \in \mathbb{C}$$

Scalar Multiplication:

addition and scalar multiplication is same as in $\mathbb{R}^n$, so $\mathbb{C}^n$ is a vector space.

3. ***Space*** $\mathbb{C}[a,b]$

Let $x \in \mathbb{C}[a,b]$ *and* $y \in \mathbb{C}[a,b]$

where x and y are fucntions and operating on t

Addition:

$$(x + y)(t) = x(t) + y(t)$$

Scalar Multiplication:

$$(\alpha x)(t) = \alpha x(t)$$

So under addition and scalar multiplication $\mathbb{C}[a,b]$ is vector space over a field $\mathbb{R}$ or $\mathbb{C}$.

4. Space l^2:

In this space we have sequences,if $x \in l^2$ then x is a sequence, say

$$x = (\xi_1, \ldots\ldots\ldots\ldots, \xi_n) \ , \qquad x \in l^2$$

and $$y = (\eta_1, \ldots\ldots\ldots\ldots, \eta_n) \ , \qquad y \in l^2$$

Addition:

$$x + y = (\xi_1 + \eta_1, \ldots\ldots\ldots\ldots, \xi_n + \eta_n)$$

Scalar Multiplication:

$$\alpha x = (\alpha \xi_1, \ldots\ldots\ldots\ldots, \alpha \xi_n)$$

So under addition and scalar multiplication the space l^2 is vector space over a field $\mathbb{R}$ or $\mathbb{C}$

MODULE NO. 38

VECTOR SPACE

- ***Subspace***
- ***Basis of a Vector Space***

Subspace:

A subspace of a vector space X is a nonempty subset Y of X such that addition and scalar multiplication are closed in Y.

Hence T is itself a vector space, the two algebraic operations being those induced from X.

Two Types of subspaces

- Improper Subspace: If the span of a subspace is equal to that vector space ;
- Proper Subspace: If the span of a subspace is not equal to that vector space

Linear Combination

A linear combination of vectors $x_1, \ldots\ldots\ldots, x_n$ of a vector space X is an axpression of the form

$a_1 x_1 + \ldots\ldots\ldots\ldots a_m x_m$ where the coefficients $a_1, \ldots\ldots\ldots\ldots, a_m$ are any scalars.

Span of a Set:

For any nonempty subset $M \subset X$ the set of all linear combinations of vectors of M is called the span of M.

Written as "span M".

Obviously, this is a subspace Y of X, and we say that Y is spanned or generated by M.

Linear Independence:

If two vectors have same direction and different in magnitude then on vector is multiple of other which means that one is dependent to other.

If two vectors have not same direction then one vector is independent to other.

Mathematically:

linearly independent.

$$c_1x_1 + c_2x_2 + ... + c_mx_m = 0$$

if and only if all constant are zero

$$c_1 = c_2 = ... = c_m = 0$$

We call $x_1, x_2, ..., x_m$ linearly independent.

linearly dependent.

If vectors are dependent then their coefficients are not equal to 0 as

let

$$x_1 = 2x_2$$

$$\Rightarrow \quad x_1 - 2x_2 = 0$$

Here coefficient $1 \neq 2 \neq 0$, so x_1 is dependent of x_2 .

Basis of a Vector Space:

As span of M is also a subspace, if the subspace (collection of vectors) is improper subspace(means span of M is equal to that vector space) and linearly independent(coefficients are equal to zero) then that particular subspace is a Basis of a Vector Space.

So, for basis the subspace have to improper subspace and linear independent.

MODULE NO. 39

VECTOR SPACE

Dimension (definition):

The number of elements in subspace of a basis is called dimension of that vector space.

- ***Dimension***
  - i. ***Finite dimensional vector space***
  - ii. ***Infinite dimensional vector space***

Examples:

In $\mathbb{R}^n$ space

Elements of basis of $\mathbb{R}^n$ are $e_1, e_2, \ldots\ldots\ldots\ldots, e_n,$

$$
\begin{aligned}
e_1 &= (1, 0 \ldots\ldots\ldots\ldots, 0) \\
e_2 &= (0, 1 \ldots\ldots\ldots\ldots, 0) \\
&. \\
&. \\
e_n &= (0, 0 \ldots\ldots\ldots\ldots, 1)
\end{aligned}
$$

Sometimes it is called Canononical basis of $\mathbb{R}^n$ basis $\mathbb{R}^n$.

Similarly in $\mathbb{C}^n$ space n-dimension

$C[a,b]$ is infinite dimension vector space because there is no finite set which can span the set of function.

In l^2 space, there are sequences, this is also infinite dimensional vector space.

Result :

Every nonempty vector space $X \neq \{0\}$ has a basis.

Theorem:

Let X be an n dimensional vector space. Then any proper subspace Y of X has dimension less than n.

Proof:

If n=0 this implies X={0}

There is no proper subspace. Hence we can't continue.

If dimension of Y is zero.

Dim Y=0

and $X \neq Y$ Y={0}

{Y is proper subspace of X}

dim Y < dim X

suppose dim Y=n

$\Rightarrow$ Y would have a basis of n elements.

$\Rightarrow$ that basis would also be a basis for X, as element in basis are same, they span and linearly independent.

dim X=n when basis are same then X=Y

but it is contradict to our supposition as we suppose that Y is a proper subset of Xi.e $Y \subset X$ which means X and Y are not equal.

$\Rightarrow$ any linearly independent set of vectors in Y must have less elements then n.

$\Rightarrow$ dim Y< n

That we have to prove.

Module No. 40

Normed Space, Banach Space

- ***Norm***
- ***Normed Space***
- ***Banach Space***

Norm (definition):

A norm on a (real or complex) vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by $\|x\|$.

(This like the notation of mod but it has two vertical lines on left and right side.)

It has following properties:

i): $\|x\| \geq 0$ (N1)

ii): $\|x\| = 0 \Leftrightarrow x = 0$ (N2)

Norm is equal to zero if and only if x=0. Length is always positive or zero but not –ve.

iii): $\|\alpha x\| = |\alpha| \|x\|$ (N3)

if we multiply the length of norm with α (any number) then it will increase the length of Norm α times.

iv): $\|x + y\| \le \|x\| + \|y\|$ (N4) triangular inequality

if x and y are two vectors then their sum of Norms is equal to individual sum of their norm.

Norm metric:

A norm on X defines a metric d on X which is given by

$$d(x, y) = \|x - y\| \qquad \textit{where } x, y \in X$$

and is called the metric induced by the norm as this metric depend on norm so we call it metric induced by norm.

from the property $\|x + y\| \le \|x\| + \|y\|$

we can write $\big|\|y\| - \|x\|\big| \le \|y - x\|$

The norm is real valued function so it is continuous function. Continuous function mean if we define norm on x then it will give us the value of norm x as

$$x \to \|x\|$$

and this mapping is continuous and is mapped $(X, \|.\|) \to \mathbb{R}$.

Norm is always a continuous function.

Norm Space:

A normed space X is a vector space with a norm defined on it.

A normed space is denoted by $(X, \|.\|)$ or simply by X.

Banach Space:

A Banach space is a complete normed space, (Complete in the metric defined by the norm).

Module No. 41

Examples (Normed Space)

- ***Euclidean Space*** $\mathbb{R}^n$
- ***Unitary Space*** $\mathbb{C}^n$
- ***Space*** l^p
- ***Space*** l^∞
- ***Space*** $\mathbb{C}[a,b]$

Euclidean Space $\mathbb{R}^n$

This is a metric space and elements in $\mathbb{R}^n$ is in n-tuples form,

$$x = (\xi_1, \xi_2, \ldots\ldots\ldots, \xi_n) \qquad \text{where } \xi_i \in \mathbb{R}\ , \quad x \in X$$

$$\|x\| = \sqrt{|\xi_1|^2 + \ldots\ldots + |\xi_n|^2}$$

$$= \left(\sum_{i=1}^{n} |\xi_i|^2 \right)^{\frac{1}{2}}$$

$$y = (\eta_1, \eta_2, \ldots\ldots\ldots, \eta_n) \qquad \text{where } \eta_i \in \mathbb{R}$$

The distance function $\quad d(x,y) = \|x - y\|$

$$d(x,y) = \sqrt{|\xi_1 - \eta_1|^2 + \ldots\ldots + |\xi_n - \eta_n|^2}$$

Unitary Space $\mathbb{C}^n$

This is a metric space and elements in $\mathbb{C}^n$ is in n-tuples form,

$$x = (\xi_1, \xi_2, \ldots\ldots\ldots, \xi_n) \qquad \text{where } \xi_i \in \mathbb{C}\ , \quad x \in X$$

$$\|x\| = \sqrt{|\xi_1|^2 + \ldots\ldots + |\xi_n|^2}$$

$$= \left(\sum_{i=1}^{n} |\xi_i|^2 \right)^{\frac{1}{2}}$$

$$y = (\eta_1, \eta_2, \ldots\ldots\ldots, \eta_n) \qquad \text{where } \eta_i \in \mathbb{C}$$

The distance function

$$d(x,y) = \|x - y\|$$

$$= \sqrt{|\xi_1 - \eta_1|^2 + \ldots\ldots + |\xi_n - \eta_n|^2}$$

Space l^p

$$x = (\xi_1, \xi_2, \ldots\ldots\ldots) \ ,$$

$$y = (\eta_1, \eta_2, \ldots\ldots\ldots)$$

$$\|x\| = \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{\frac{1}{p}}$$

The distance function $d(x, y) = \|x - y\|$

$$= \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p \right)^{\frac{1}{p}}$$

Space l^∞

$$x \in l^\infty$$

The metric is given by $\|x\| = \sup_j |\xi_j|$

Space $\mathbb{C}[a,b]$:

This is a space of all real valued continuous functions defined on closed interval [a,b]

The norm of the function is $\|x\| = \max_{t \in J} |x(t)|$, with this metric space it is a norm space.

Module No. 42

Unit Sphere

- ***Unit Sphere***

Unit Sphere

The sphere with center 0 and radius 1, S(0;1), this we define in $\mathbb{R}^2$, but in any metric space

Those points from x whose norm is 1. $\{x \in X \mid \|x\| = 1\}$,

In a normed space X is called the unit sphere. In norm space the collection of all those points which are equal to 1 is called a Unit Sphere.

Let $\|x\|$ be a norm, and space is $\mathbb{R}^2$, the element in $\mathbb{R}^2$ are $x = (\xi_1, \xi_2)$

Example:

(i.e x=(2,-3), $\|x\| = |2| + |-3| = 2 + 3 = 5$)

$$\|x\| = |\xi_1| + |\xi_2|$$

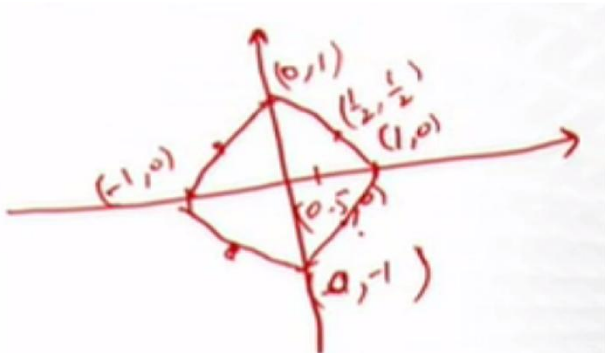
Norm of (1,0) is 1, and similarly norm of point (0,1) is also 1.

Similarly for Norm of (-1,0) is 1, and also norm of point (0,-1) is also 1.

This norm is according to function $\|x\| = |\xi_1| + |\xi_2|$,

for x=(1,0)

$$\|(1,0)\| = 1 + 0 = 1$$



Another Example.

The norm is defined as $\|x\| = |\xi_1^2 + \xi_2^2|^{1/2}$ similar to equation of circle.

In unit sphere we have the condition that norm of x is 1, $\|x\| = 1$

$$1 = \left(\xi_1^2 + \xi_2^2\right)^{1/2}$$

$$1 = \xi_1^2 + \xi_2^2$$

Another Example.

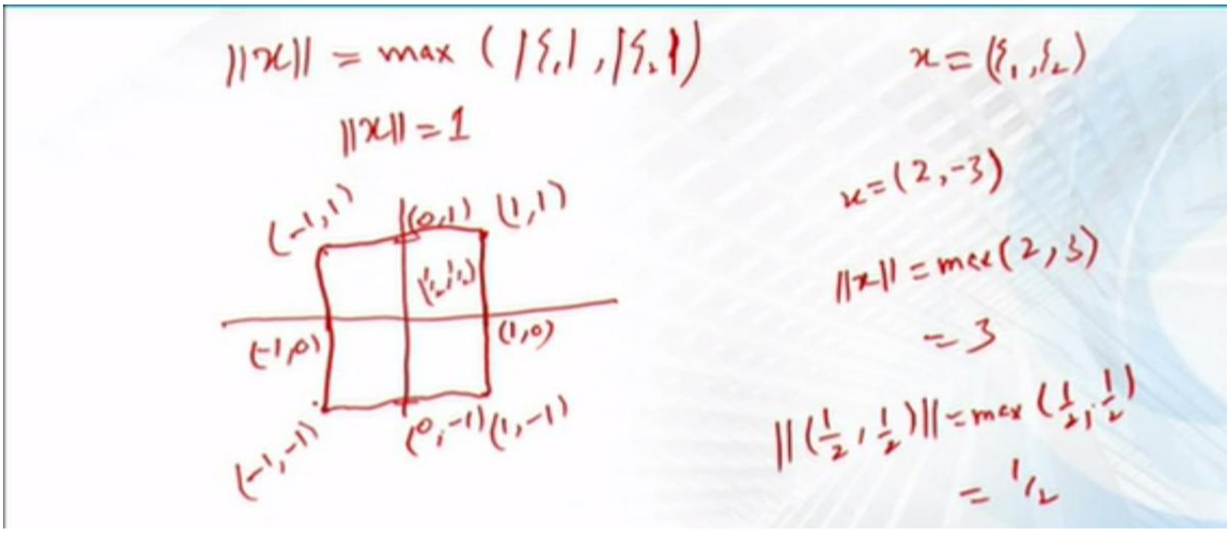
The norm is defined as $\|x\| = \max\left(|\xi_1|, |\xi_2|\right)$ similar to equation of circle.

Suppose $x \in \mathbb{R}^2$, such that $x = (\xi_1, \xi_2)$,

Let say $x = (2, -3)$

According to given condition,

$$\|x\| = \max(|2|, |-3|) = \max(2, 3) = 3$$



Here the sphere is a square.

We have discussed only $\mathbb{R}^2$ norm space and also its sketches, but it can be $\mathbb{R}^n$, $\mathbb{C}^n$ or any other space like space of functions C[a,b].

When we defined different norm then the shape of the unit sphere is depends on the norm define.

Module No. 43

Normed Spaces

- ***Subspace***

Subspace (definition)

A subspace Y of a normed space X is a subspace of X considered as a vector space, with the norm obtained by restricting the norm on X to the subset Y.

This norm on Y is said to be induced by the norm on X.

If Y is closed in X, then Y is called a closed subspace of X.

Subspace l^p :

A subspace Y of a Banach space X is a subspace of X considered as a normed space.

Hence we do not require Y to be complete.

Theorem :

A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.

Convergence in Normed Spaces.

The metric function is $d(x,y) = \|x - y\|$

For convergence we define as

i): A sequence (x_n) in a normed space X is convergent if X contains an x such that

$$\lim_{n\to\infty}\|x_n - x\| = 0$$

$$x_n \to x \ , \qquad x \ \text{lim}\mathit{it\ of}\ (x_n)$$

Now this definition define for Cauchy sequence

ii): A sequence (x_n) in a normed space X is a Cauchy sequence if for every $\varepsilon > 0$ there is an N such that

$$\|x_m - x_n\| < \varepsilon \quad \mathit{for\ all}\ m, n > N$$

Module No. 44

Normed Spaces

- ***Convergence of Infinite Series***
- ***Basis in Normed Spaces***
- ***Completion in Normed Spaces (Theorem***

Convergence of Infinite Series

A sequence (x_k) is associate with a sequence of partial sum s_n .

$$s_n = x_1 + x_2 + ...x_n \qquad \text{where } n=1,..........,$$

If s_n convergent, $s_n \to s$, then

$$\sum_{i=1}^{\infty} x_i = x_1 + x_2 + ... \text{is also convergent.}$$

if $\|s_n - s\| \to 0$ then $s_n \to s$.

If we have following series

$$\|x_1\| + \|x_2\| + ... \text{converges,}$$

$$\Rightarrow \qquad \sum_{i=1}^{\infty} x_i \quad \text{absolutely convergent.}$$

So , we have transform the convergence and absolutely convergence in term of norm.

Basis:

In a normed space X is a Cauchy sequence if for every $\varepsilon > 0$ there is an N such that

Elements of basis of $\mathbb{R}^n$ are $e_1, e_2, ..., e_n$, such that

$$e_1 = (1, 0................, 0)$$
$$e_2 = (0, 1................, 0)$$
$$.$$
$$.$$
$$e_1 = (0, 0................, 1)$$

Sometimes it is called Canononical basis of $\mathbb{R}^n$.

Elements are spanning and are linearly independent.

Any element $x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$ in the form of norm is

$$\|x - \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n\| \to 0$$

and if this condition is hold then we say that it is a basis in the norm space.

Theorem Completion:

Let $X = (x, \|.\|)$ be a normed space then there is a Banach space $\hat{X}$ and an isometry A from X onto a subspace W of $\hat{X}$ which is dense in $\hat{X}$.

The space $\hat{X}$ is unique, except for isometries.

Module No. 45

Fininte Dimensional Normed Spaces

- ***Lemma (Linear Combination)***

Lemma

Let $\{x_1, ..., x_n)$ be a linearly independent set of vectors in a normed space X (of any dimension).

Then there is a number c>0 such that for every choice of scalars $\alpha_1, ..., \alpha_n$ we have

$$\|\alpha_1 x_1 + ... + \alpha_n x_n\| \geq c\left(|\alpha_1| + ... + |\alpha_n|\right)$$

Proof:

$$S=|\alpha_1|+............+|\alpha_n|=\left(|\alpha_1|+............+|\alpha_n|\right)$$

$$\|\alpha_1x_1+.........+\alpha_nx_n\|\geq c\left(|\alpha_1|+............+|\alpha_n|\right),\quad \text{where } c>0$$

Now we have two cases:

i): If S=0

It means $|\alpha_i|=0 \quad \Rightarrow \quad \alpha_i=0 \;\; for\;\; all\;\; i=1,.......,n$

ii): If S>0

$$\|\alpha_1x_1+.........+\alpha_nx_n\|\geq cS \qquad \text{as S>0 so we can divide it}$$

$$\frac{\|\alpha_1x_1+.........+\alpha_nx_n\|}{S}\geq c$$

$$\left\|\frac{\alpha_1x_1}{S}+.........+\frac{\alpha_nx_n}{S}\right\|\geq c$$

$$\|\beta_1x_1+.........+\beta_nx_n\|\geq c$$

If we define $\beta_i=\frac{\gamma_i}{S}$ then from S we have

$$\frac{|\alpha_1|+............+|\alpha_n|}{S}=1$$

$$\frac{|\alpha_1|}{S}+............+\frac{|\alpha_n|}{S}=1$$

$$\sum_{i=1}^{n}|\beta_i|=1$$

To prove $\|\beta_1x_1+.........+\beta_nx_n\|\geq c$ We have to prove $\sum_{i=1}^{n}|\beta|_i=1$

We do this by contradiction.

Suppose it is false that $\|\beta_1x_1+.........+\beta_nx_n\|\geq c$

So we can find a sequence $\langle y_m\rangle$ of vectors $y_m=\beta_1^{(m)}x_1+......+\beta_n^{(m)}x_n$ such that

$$\|y_m\|\to 0 \qquad as \quad m\to\infty$$

as we suppose that $\|\beta_1x_1+.........+\beta_nx_n\|\leq c$

so we will find values smaller than c.

$$\sum_{j=1}^{n}\left|\beta_j^{(m)}\right| = 1 \qquad \Rightarrow \qquad \left|\beta_j^{(m)}\right| \le 1$$

Thus for each fixed $\left\langle \beta_j^{(m)} \right\rangle = \left(\beta_j^{(1)} + \beta_n^{(2)}........\right)$ is bounded.

By Bolzano-Weisrtren theorem has a convergent subspace.

For all j=1,2,…………,n

$\Rightarrow \quad \left\langle \beta_1^{(m)} \right\rangle$ has converged subsequence say $\gamma_1^{(m)}$ converges to β_1

$$y_m = \beta_1^{(m)} x_1 + ...+ \beta_n^{(m)} x_n$$
$$y_{m,1} = \gamma_1^{(m)} x_1 + ...+ \beta_n^{(m)} x_n$$

$$\beta_2^{(m)} \to \gamma_2^{(m)} \to \beta_2$$

This is also true for

$$y_{m,2} = \gamma_2^{(m)} x + \gamma_2^{(m)} x +,.........,+\beta_n^{(m)} x_n$$

..

..

$$y_{m,n} = \sum_{j=1}^{n} \gamma_j^{(m)} x_j \quad \text{for all} \quad \sum_{j=1}^{n}\left|\gamma_j^{(m)}\right| = 1.$$

$$\gamma_j^{(m)} \to \beta_j \qquad \text{as} \qquad \text{m} \to \infty$$

$$y_{m,n} \to y = \sum_{j=1}^{n} \beta_j x_j \quad \text{with} \sum \beta_j = 1 \quad \Rightarrow \quad \text{all } \beta_j \ne 0$$

Using the linearly independence condition $\{x_1,.........,x_n\}$ are linearly independent.

This implies $\beta_1 x_1 + ...+ \beta_n x_n \ne 0 \qquad \Rightarrow \qquad y \ne 0$

Now $y_{m,n} \to y \qquad \|y_{m,n}\| \to \|y\|$ where $\|.\|$ is continuous

Hence $\|y_m\| \to 0$ and $y_{m,n}$ is a subsequence of y_m but we have supposed that $y \ne 0$

$$\|\text{y}_{\text{m,n}}\| \to 0 = \|\text{y}\| \to \text{y} = 0 \qquad \text{N2 proved}$$

Hence proved

Module No. 46

Normed Spaces

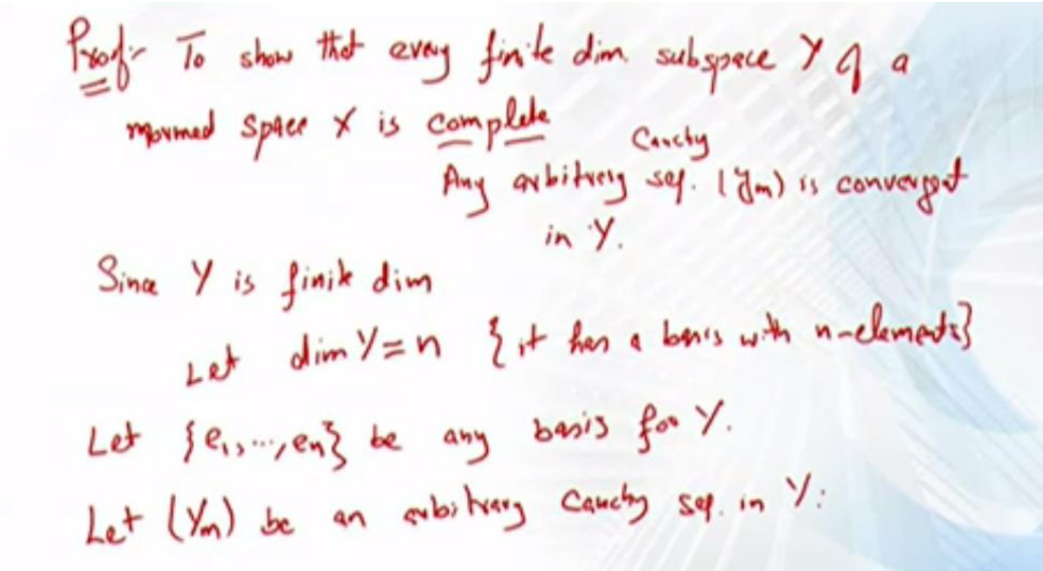
- ***Theorem (Completeness)***

Theorem

Every finite dimensional subspace Y of a normed space X is complete. In particular, every finite dimensional normed space is complete.

Proof:

Prove it yourself:



Proof:- To show that every finite dim subspace Y of a normed space X is complete

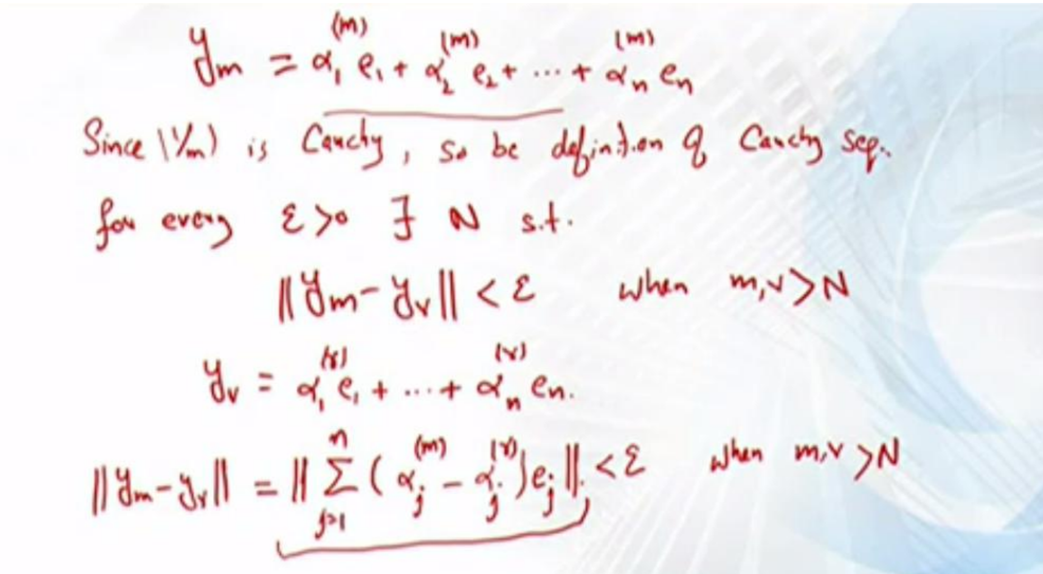
Any arbitrary Cauchy seq. (y_m) is convergent in Y.

Since Y is finite dim

Let $\dim Y = n$ {it has a basis with n-elements}

Let $\{e_1, \dots, e_n\}$ be any basis for Y.

Let (y_m) be an arbitrary Cauchy seq. in Y:



$$y_m = \alpha_1^{(m)} e_1 + \alpha_2^{(m)} e_2 + \dots + \alpha_n^{(m)} e_n$$

Since (y_m) is Cauchy, so by definition of Cauchy seq.

for every $\varepsilon > 0$ $\exists$ N s.t.

$$\|y_m - y_v\| < \varepsilon \quad \text{when } m, v > N$$

$$y_v = \alpha_1^{(v)} e_1 + \dots + \alpha_n^{(v)} e_n$$

$$\|y_m - y_v\| = \left\| \sum_{j=1}^{n} \left(\alpha_j^{(m)} - \alpha_j^{(v)}\right) e_j \right\| < \varepsilon \quad \text{when } m, v > N$$

$$\varepsilon > \left\| \sum_{j=1}^{n} (\alpha_j^{(m)} - \alpha_j^{(r)}) e_j \right\| \geq c \sum_{j=1}^{n} |\alpha_j^{(m)} - \alpha_j^{(r)}| \quad \{\because \text{by Lemma } 45$$

$$\Rightarrow \quad c \sum_{j=1}^{n} |\alpha_j^{(m)} - \alpha_j^{(r)}| < \varepsilon \qquad c > 0$$

$$\Rightarrow \quad \sum_{j=1}^{n} |\alpha_j^{(m)} - \alpha_j^{(r)}| < \frac{\varepsilon}{c} \qquad m, r > N$$

For fixed j $\quad |\alpha_j^{(m)} - \alpha_j^{(r)}| < \frac{\varepsilon}{c} \Rightarrow$ A Cauchy seq. of $\mathbb{R}$ or $\mathbb{C}$.

Hence it is convergent. Let α_j denote the limit of each seq.. So for these 'n' sequences, let $\alpha_1, \ldots, \alpha_n$ be the limit.

Set $\quad y = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$

Also

$$\|y_m - y\| = \left\| \sum_{j=1}^{n} (\alpha_j^{(m)} - \alpha_j) e_j \right\|$$

$$\leq \sum_{j=1}^{n} |\alpha_j^{(m)} - \alpha_j| \|e_j\|$$

$\Rightarrow \alpha_j^{(m)} \longrightarrow \alpha_j$ as $m \to \infty$

$\|y_m - y\| \to 0$

$y_m \to y \in Y$

(y_m) is convergent to $y \in Y$

since (y_m) is an arbitrary, $\Rightarrow$

Y is complete.

Module No. 47

Normed Spaces

- ***Theorem (Closedness)***

As we have already proved that every finite dimensional subspace is complete and we also know that a subspace is complete if and only if it is closed.

Theorem

Every finite dimensional subspace Y of a normed space X is closed in X. This result is true for finite dimensional subspace but for infinite space it is not true.

Infinite dimensional subspaces are like C[0,1], l^2 are infinite dimensional normed space which are not closed space. We use dense, limit points to prove this.

Module No. 48

Normed Spaces

- ***Theorem (Equivalent Norms)***

Definition

A norm $\|.\|$ on a vector space X is said to be equivalent to a norm $\|.\|_o$ on X if there are positive numbers a and b such that for all $x \in X$ we have

$$a\|x\|_o \leq \|x\| \leq \|x\|_o b$$

This property should hold for every element x of vector space X.($a\|x\|_o$ read a times x not norm).

If we prove about condition then we say that these two norms are equivalent.

Equivalent norms on X define the same topology for X.

Theorem (Equivalent norms)

One finite dimensional vector space X, any norm $\|.\|$ is equivalent to any other norm $\|.\|_o$

Proof:

Proof: $\|\cdot\| \cong \|\cdot\|_0$

$\forall\ x \in X,\ \exists\ a, b$ s.t.

$$a\|x\|_0 \le \|x\| \le b\|x\|_0$$

Let $\dim X = n$, $\{e_1, \dots, e_n\}$ be any basis of X.

Then every $x \in X$ has a unique representation

$$x = \alpha_1 e_1 + \dots + \alpha_n e_n \quad \text{——} (1)$$

Now by Lemma (45) $\exists\ c > 0$ s.t.

$$\|x\| \ge c(|\alpha_1| + \dots + |\alpha_n|) = c\sum_{j=1}^{n} |\alpha_j| \quad \text{——} (2)$$

by applying $\|\cdot\|_0$ we get by (1)

$$\|x\|_0 = \|\alpha_1 e_1 + \dots + \alpha_n e_n\|_0 \le \sum_{j=1}^{n} |\alpha_j| \|e_j\|$$

Let

$$K = \max_{j=1,\dots,n} \|e_j\|$$

$$\le K\sum_{j=1}^{n} |\alpha_j| \Rightarrow \frac{\|x\|_0}{K} \le \sum_{j=1}^{n} |\alpha_j|$$

$$\|x\| \ge c\sum_{j=1}^{n} |\alpha_j| \ge c\frac{\|x\|_0}{K}$$

$$\Rightarrow \|x\| \ge \frac{c}{K}\|x\|_0 \Rightarrow \|x\| \ge a\|x\|_0, \quad a = \frac{c}{K}$$

by interchanging both norms $\Rightarrow \|x\|_0 \ge \frac{1}{b}\|x\|$

$$\Rightarrow b\|x\|_0 \ge \|x\|$$

$$a\|x\|_0 \le \|x\| \le b\|x\|_0 \quad \text{required}$$

Module No. 49

Compactness and finite Dimension

- ***Lemma (Compactness)***

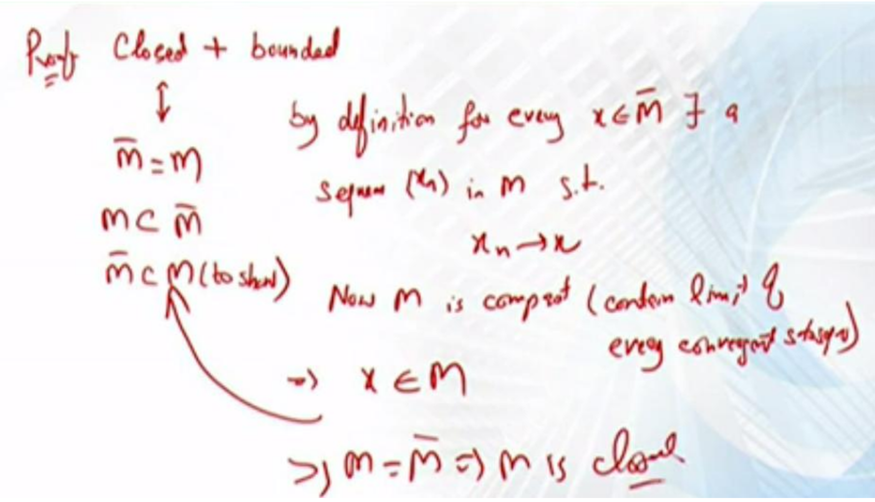
Definition

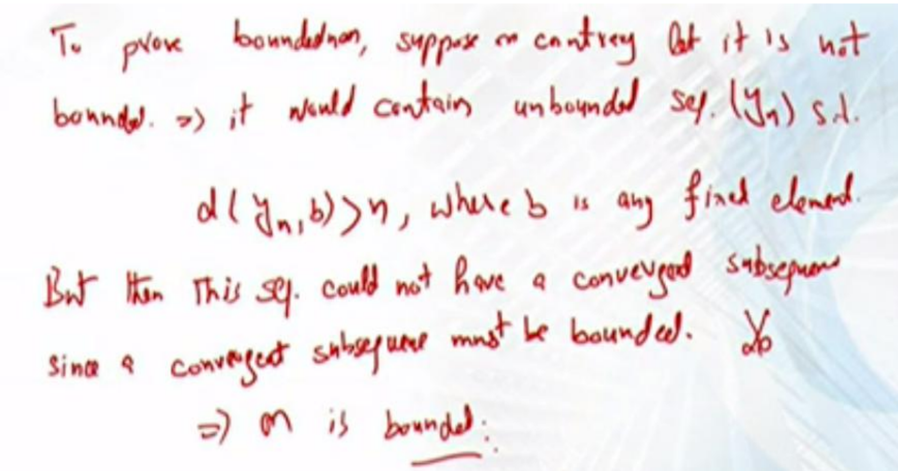
A metric space X is said to be compact if every sequence in X has a convergent subsequence. A subset M of X is said to be compact if M is compact considered as a subspace of X, that is if every sequence in M has a convergent subsequence whose limit is an element of M.

Lemma (Compactness)

A compact subset M of a metric space is closed and bounded.

For close of M we show that $\bar{M} = M$. Now we have to prove closed and bounded

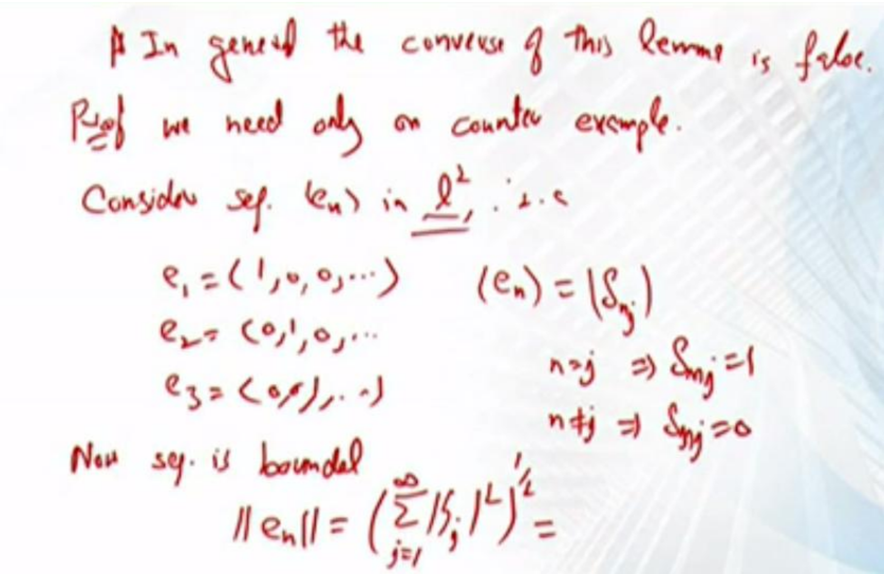


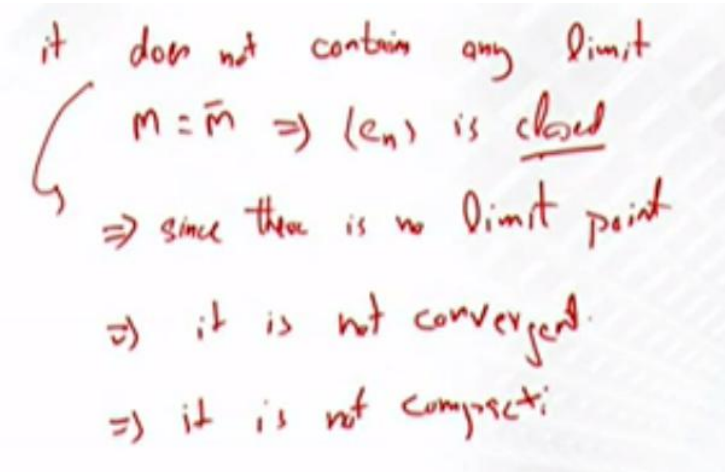


Conversely

In general the converse of this lemma is false.

Proof





The above example is closed and bounded but not compact so the converse is false that a closed and bounded metric space is not compact.

Module No. 50

Theorem (Compactness)

- ***Lemma (Compactness)***

In case of finite dimensional subset M is a compact set if and only if it is closed and bounded. Here we prove both directions.

Theorem (Compactness)

In a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded.

Proof:

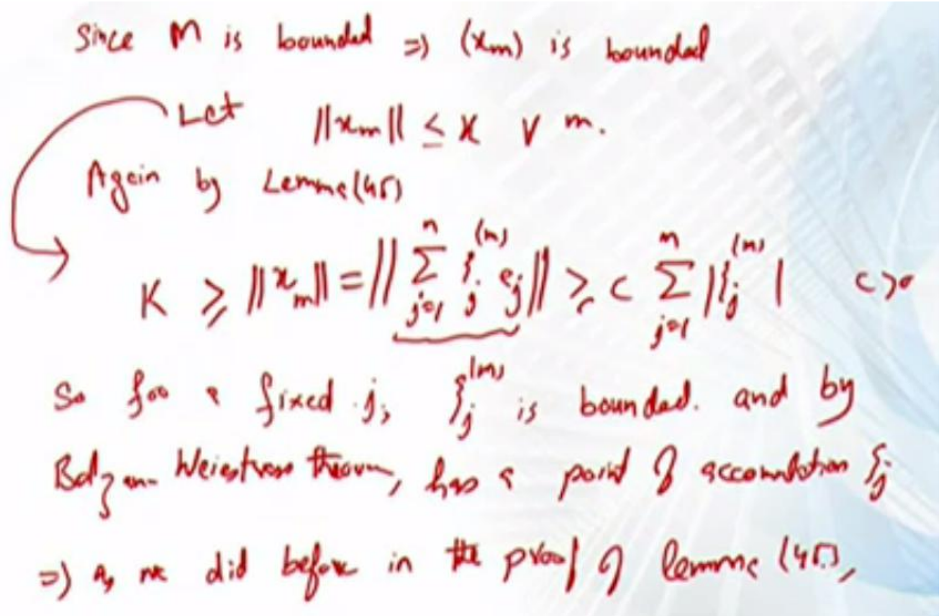
We have to prove that compact implies closed and bounded. This we have proved already. Now we prove the converse only. We have to prove only compact (for finite dimensional only).

Let M be closed and bounded, we need to show that M is compact (i.e. every sequence in M has a subseq which converges in M).

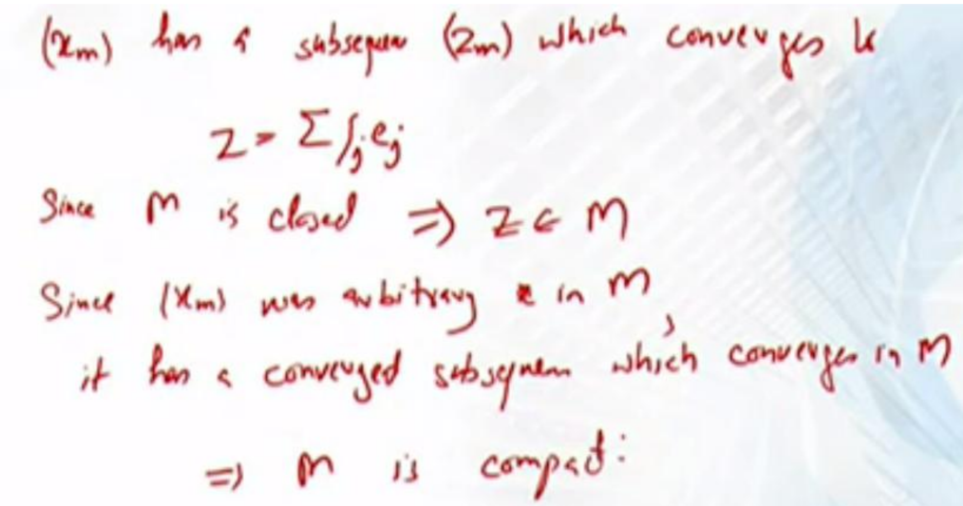
Let it is finite dimension so, say n, as $\dim X = n$ and $\{e_1 + \ldots\ldots\ldots. + e_n\}$ be a basis for X

Let $\langle x_m \rangle$ be any sequence in M.

$$\Rightarrow \quad x_m = \xi_1^{(m)} e_1 + \ldots\ldots. + \xi_n^{(m)} e_n$$



Lemma 45 lecture,



Module No. 51

Compactness and Finite Dimension

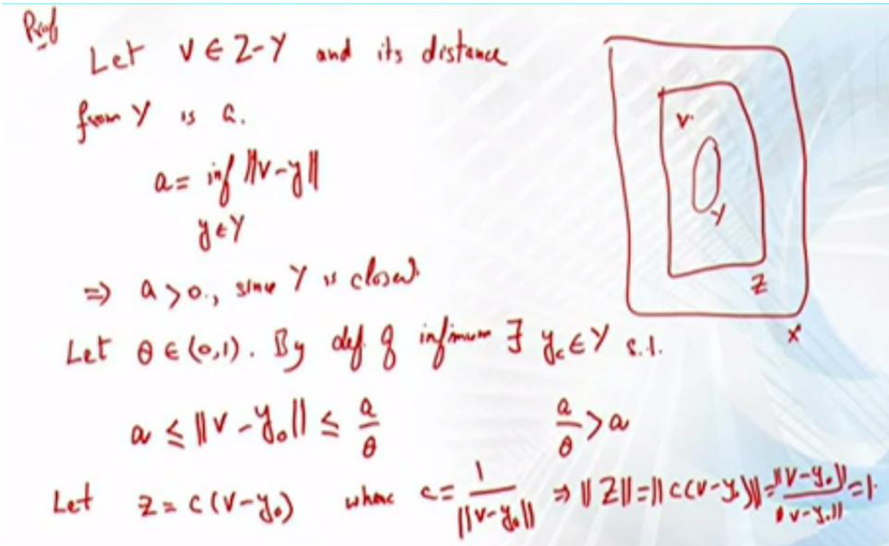
- ***F. Riesz's Lemma***

F. Riesz's Lemma

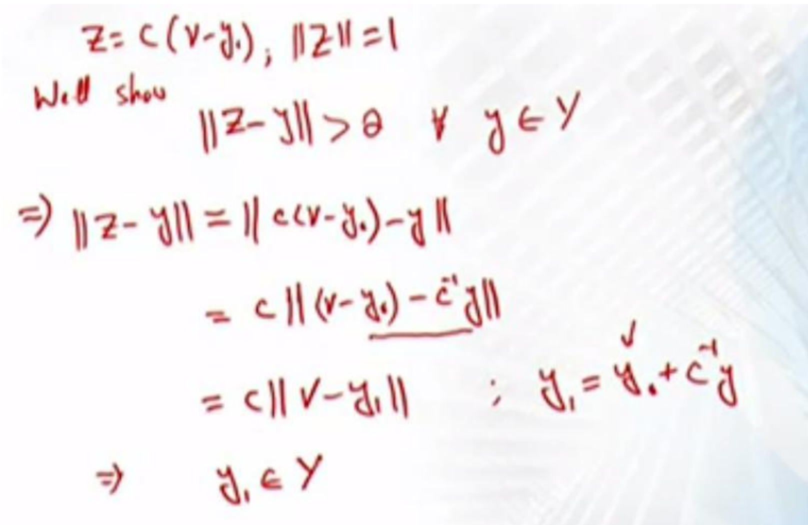
Let Y and Z be subspaces of a normed space X (of any dimension), and suppose that Y is closed and is a proper subset of Z, then for every real number θ in the interval (0,1) there is a $z \in Z$ such that

$$\|z\| = 1$$
$$\|z - y\| \geq \theta \quad for\ all\ y \in Y$$

First part $\|z\| = 1$ we prove as



Second part: $\|z - y\| \geq \theta \quad for\ all\ y \in Y$



$z = c(v - y_0)$; $\|z\| = 1$

We'll show $\|z - y\| > \theta \quad \forall\, y \in Y$

$\Rightarrow \|z - y\| = \|c(v - y_0) - y\|$

$= c\|(v - y_0) - c^{-1}y\|$

$= c\|v - y_1\| \quad ; \; y_1 = y_0 + c^{-1}y$

$\Rightarrow \quad y_1 \in Y$

Module No. 52

Finite Dimension

- ***Theorem (Finite Dimension)***

Theorem

If a normed space X has the property that the closed unit ball $M = \{x \mid \|x\| \le 1\}$ is compact, then X is finite dimensional.

Proof: Suppose on contrary that M is compact but $\dim X = \infty$.

Let $x_1 \in X$ s.t. $\|x_1\| = 1$

it generates one dimensional subspace X_1 of X

$\Rightarrow$ finite dim $\Rightarrow$ compact $\Rightarrow$ closed. Since it is proper subspace of X_1 by Riesz's Lemma $\exists$ a $x_2 \in X$ with $\|x_2\| = 1$ s.t.

$$\|x_2 - x_1\| \ge \theta = \tfrac{1}{2} \text{ (say)} \quad \theta \in (0,1)$$

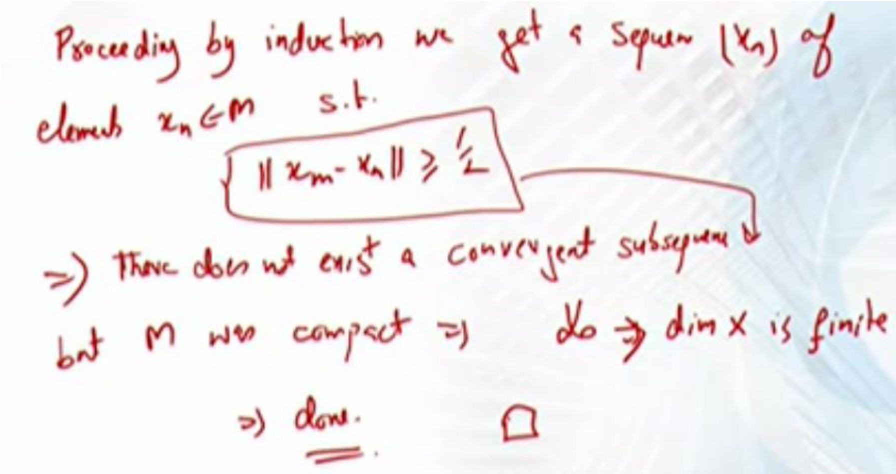
Again x_1, x_2 generate a two dimensional proper closed subspace X_2 of X. Again by Riesz's Lemma $\exists$ $x_3 \in X$ s.t. $\|x_3\| = 1$ and $\forall\, x \in X_2$ we have

$$\|x_3 - x\| \ge \tfrac{1}{2}$$

in particular since $x_1, x_2 \in X_2$

$\Rightarrow \|x_3 - x_1\| \ge 1/2$

$\|x_3 - x_2\| \ge 1/2.$



Module No. 53

Compactness and Finite Dimension

- ***Theorem (Continuous Mapping)***
- ***Corollary (Maximum and minimum)***

Theorem

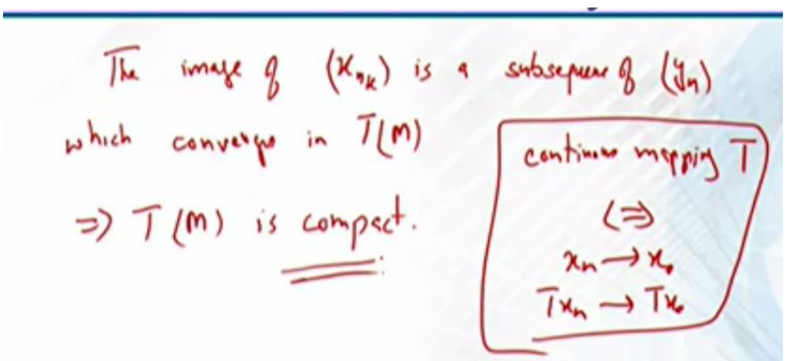
Let X and Y be metric spaces and $T: X \to Y$ be a continuous mapping.

Then the image of a compact subset M of X under T is compact.

Proof:

By definition of compactness we need to show that every sequence $\langle y_n \rangle$ in the image $T(M) \subset Y$ continuous a subsequence which converges in T(M).

Now since $y_n \in T(M)$, we have x_n such that $y_n = Tx_n$, for some $x_n \in M$. since M is compact, (x_n) contains subsequence $\langle x_{n_k} \rangle$ which converges in M.



Corollary (maximum and minimum)

A continuous mapping T of a compact subset M of a metric space *X* into R assumes a maximum and a minimum at some points of M.

$$T : M \to R$$

$\Rightarrow$ $T(M) \subset \mathbb{R}$

$T(M),\quad \left.\begin{matrix} M\text{ - compact} \\ T - \text{continuous} \end{matrix}\right]$ by previous result

$\Rightarrow$ T(M) is compact.

which means it is closed and bounded because compactness implies close and bounded.

$\Rightarrow$ $\inf T(M) \in T(M), \quad and \quad \sup\ T(M) \in T(M)$

Inverse image of these two points consist of points of M at which Tx is minimum or maximum respectively. And that we have to prove.

Module No. 54

Functional Analysis

- ***Linear Operators***

In functional analysis if we define a metric on a set then it is a metric space and if we define a norm on a vector then it is called a norm space. In mapping if we take a and b as norms then we define a linear operator on the mapping and it should satisfied the certain properties.

Operator

In the case of vector spaces and, in particular, normed spaced, a mapping is called an operator.

Linear Operator

A linear operator T is an operator such that

i): the domain $\mathcal{D}$(T) of T is a vector space and the range *R*(T) lies in a vector space over the same field.

ii): for all $x, y \in D(T)$ and scalar α

$$T(x+y)=Tx+Ty \qquad \text{also} \qquad T(\alpha x) = \alpha Tx$$

By combining above two equations

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty \quad where\ \alpha\ and\ \beta\ are\ both\ scalar$$

$T(x) \simeq Tx$ is same.

Some more notations.

$\mathcal{D}$(T) domain of T

$\mathcal{R}$(T) range of T

$\mathcal{N}$(T) denotes the null space of T.

Null space are those element from the domain of T such that on which we operate gives the answer zero. $x \in D(T)$ such that Tx=0

Also null space of T is similar to kernel of T.

Let $D(T) \subset X$ *and* $R(T) \subset Y,\ X, Y$ *vector space.*

(vector spaces can be real and complex spaces).

Then T is an operator from $\mathcal{D}$(T) onto $\mathcal{R}$(T), the notation is

$T : D(T) \to R(T)$, D(T) covers all range so it is onto.

Or $\mathcal{D}$(T) into y $T : D(T) \to Y$ $R(T) \subset Y$

if $\mathcal{D}$(T) is the whole space X, then we write $T : X \to Y$

moreoverif we take $\alpha = 0 \Rightarrow$ T0=0.

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty \quad \text{where } \alpha \text{ and } \beta \text{ are both scalar}$$

T is a homomorphism when it is a linear operator.

$T : X \to Y$, wherewe have two kind of vector space, one vector space is X and other vector space is Y. we apply operations on X and also operation on Y.These operation may or may same on both vector spaces.

Module No. 55

Linear Operators

- ***Examples.***

Operator is a mapping whose domain and range is a vector space. It is subset of vector space. Below are different linear operators.

Identity Operator

Identity mean it operate on the same vector space. $I_x : X \to X$

$\Rightarrow$ $I_x(x) = x \quad \forall\ x \in X$

$\Rightarrow$ $I_x(\alpha x + \beta y)$ we have to prove

Zero Operator:

$$O : X \to Y \text{ such that } Ox = 0 \quad \forall \; x \in X$$

here the 0 on right side is belong to vector space Y.

Differentiation:

Let *X* be a vector space of all polynomials on [a,b]. A set of polynomial in denoted by *x(t)*

$$Tx(t) = x'(t) \qquad \forall \; x(t) \in X$$

When we apply T on polynomial x(t) then $x'(t)$ is also a polynomial. So this operatorT maps X onto itself. There is no polynomial whose derivative we can't find.

Integration:

Linear operator T for C[a,b] into itself can be defined by

$$Tx(t) = \int_a^t x(\tau) d\tau$$

taa τ is just a variable and C[a, b] is collection of all continuous function on a and b.

Multiplication by t:

Let C[a, b] be a collection of continuous functions defined on a and b.

$$Tx(t) = tx(t)$$

This operator plays an important role in quantum theory of physics.

Elementary vector algebra:

Here we have different types of maps we have

$T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ cross product of two vectors is also a vector.

For cross vector we need two vectors. Then each element is also a vector.

$$T_1 = \underline{a} \times \underline{x}$$

Similarly for dot product:

Dot product of two vector is a scalar, so the map on real numbers $\mathbb{R}$ as

$$T_2 : \mathbb{R}^3 \to \mathbb{R}$$

$$T_2(x) = \underline{a}.\underline{x} = a_1x_1 + a_2x_2 + a_3x_3 \in \mathbb{R} \quad \text{where } x \in \mathbb{R}^3$$

For different map we fix a.

Matrices:

We denote matrix by capital letter say A. whose elements are in rows and column.

$$A=(\alpha_{jk})$$

Let with r rows and n column we define a linear operator which is

$$T:\mathbb{R}^n \to \mathbb{R}^r$$

Where $\mathbb{R}^n = \{(x_1,......,x_n) \mid x_i \in \mathbb{R}\}$, in column form so that we use matrices multiplication

$$\begin{bmatrix} x_1 \\ . \\ . \\ x_n \end{bmatrix}$$

such as say

$$\underset{r\times 1}{\begin{bmatrix} y_1 \\ . \\ . \\ y_n \end{bmatrix}} = \underset{r\times n}{\begin{bmatrix} \alpha_{11} & . & . & . & \alpha_{1n} \\ . & & & & \\ . & & & & \\ \alpha_{r1} & . & . & . & \alpha_{rn} \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ . \\ . \\ x_n \end{bmatrix}}$$

For matrix multiplication number of first matrix column is equal to number of rows of second column.rxn is a fix matrix

To check the linear condition we use

$$T(\alpha x+\beta y)=\alpha Tx+\beta Ty$$

Matrix multiplication satisfied this condition, hence this operator is a linear operator.

Module No. 56

Linear Operators

- ***Theorem (Range and Null space)***

Null space is the collection of those elements from the domain on which we apply the operator and the answer is zero.

Theorem

Let *T* be a linear operator. Then:

- The range *R(T)* is a vector space. (domain is also a vector space as discussed)
- If $\dim D(T)=n<\infty$, *then* $\dim R(T)\le n$ (dimension of domain vector space is finite then range is equal or less than the dimension of domain or equal.
- The null space *N(T)* is a vector space.

The first two results are about range and third result is about null space.

Proof: (a) R(T) is a vector space.

$$y_1, y_2 \in R(T)$$
$$\Rightarrow \quad \alpha y_1 + \beta y_2 \quad \in \text{R(T)}, \quad \text{where } \alpha, \beta \text{ are scalar}$$

Since

$$y_1, y_2 \in R(T) \quad \text{and} \quad x_1, x_2 \in D(T)$$
$$T : D(T) \to Y$$
$$y_1 \in Tx_1 \quad , \quad y_2 \in Tx_2$$

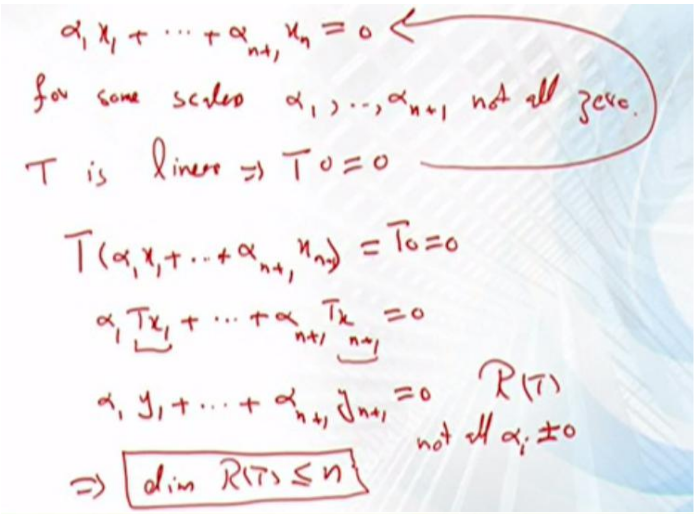
Also domain of T "*D(T")* is a vector space so, $\alpha x_1 + \beta x_2 \quad \in D(T)$ this is by definition of vector space.Since T is linear

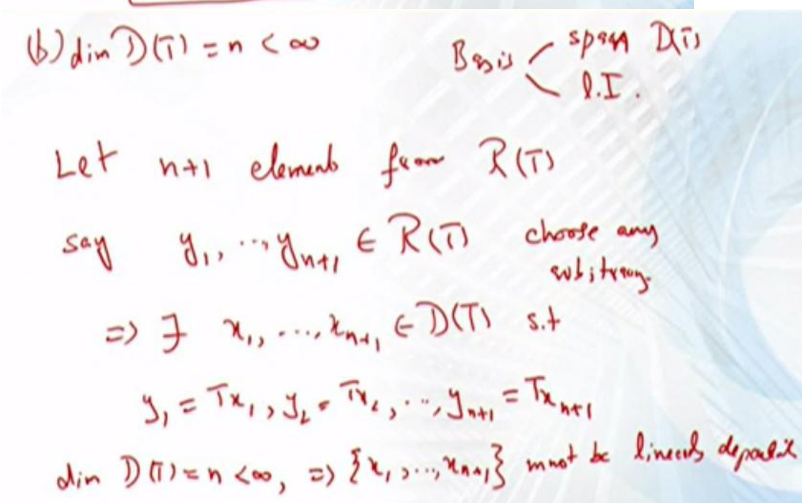
$$T\left(\alpha x_1 + \beta x_2\right) = \alpha \; Tx_1 + \beta Tx_2 = \alpha \; y_1 + \beta y_2 \in R(T)$$

Here $\alpha x_1 + \beta x_2$ is domain and gives $\alpha \; y_1 + \beta y_2$ range of T.Hence *R(T)* is a vector space.

Part (b):

Basis should span *D(T)*and it should linearly independent. More one than condition is if n element linearly independent then the elements other than n will be linearly dependent.





Linear operators preserve linearly dependence.

Part (c):

$$x_1, x_2 \in N(T)$$
$$Tx_1 = Tx_2 = 0$$

To prove it a vector space, we have to prove $\alpha x_1 + \beta x_2 \in N(T)$

$$T(\alpha x_1 + \beta x_2) = \alpha\ Tx_1 + \beta Tx_2 = \alpha \times 0 + \beta \times 0 = 0$$

$\Rightarrow \quad \alpha x_1 + \beta x_2 \in N(T)$

$\Rightarrow \quad$ N(T) is a vector space (proved)

Module No. 57

Linear Operators

- ***Inverse Operators***

Operator is a mapping whose domain and range is vector space.Particular in norm space.There is also inverse mapping. For inverse operator the same condition is one-to-one and onto. One-to-one means image of different elements is different. And onto means the range covers all the set of domain. If these two conditions hold then we can define inverse oprator.

Notations:

$T : D(T) \to Y$ is said to be injective or one-to-one if for any

$x_1, x_2 \in D(T)$ such that $x_1 \neq x_2 \Rightarrow Tx_1 \neq Tx_2$

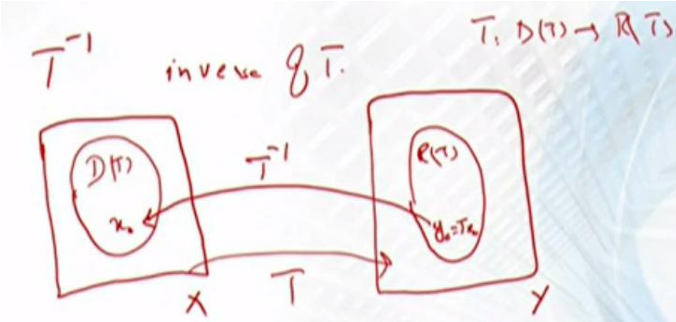
If we take counter inverse then $Tx_1 = Tx_2 \Rightarrow x_1 = x_2$,

Now if $T : D(T) \to R(T)$ then there exists a mapping

$$T' : R(T) \to D(T)$$

$y_o \to x_o$ where y_o is any element of R(T)and x_o is element of D(T).i.e. $Tx_o = y_o$

this map T' is called the inverse of T.



$$T'Tx = x \quad \forall \ x \in D(T)$$

and $$TT'y = y \quad \forall \ y \in R(T)$$

Inverse exist if and only if null space has only zero. There is only zero in null space

Module No. 58

Linear Operators

- ***Theorem (Inverse Operator)***

Theorem

Let X,Y be vectors spaces, both real or both complex. Let $T:D(T)\to Y$ be a linear operator with domain $D(T)\subset X$ and range $\mathrm{R(T)}\subset \mathrm{Y}$ .then:

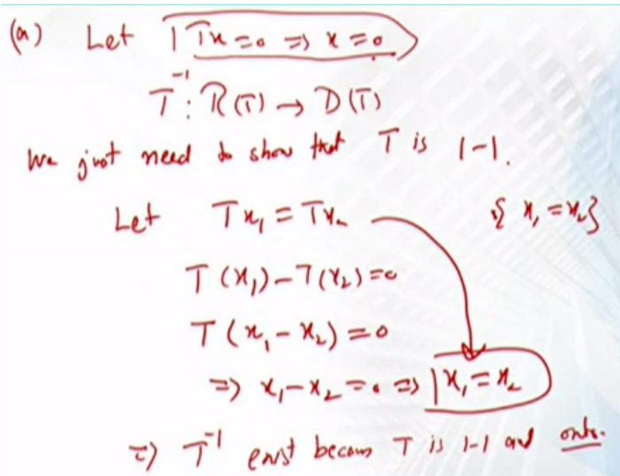
a): **The inverse** $T':R(T)\to D(T)$ **exists if and only if *Tx=0*$\Rightarrow$*x=0*. (i.e null space has zero elements).**

b): **If** T' **exists, it is a linear operator.**

c): **if** $\dim D(T)=n<\infty \ \ and \ \ T^{-1}$ **exists, then** $\dim R(T)=\dim D(T).$

as there is if and only if condition so we have to prove in both ways.

a):



Conversely let T^{-1} exist which mean one –one and onto condition hold.

We have to prove $Tx = 0$ *if and only if* $x = 0$.

One-one means $Tx_1 = Tx_2 \Rightarrow x_1 = x_2$, this is given

Now if we have take $x_2 = 0 \Rightarrow x_1 = 0 \quad Tx_1 = T_0 = 0 \;,\; x_1 = 0$

b): If T' exists, it is a linear operator.

We need to show that T^{-1} is a linear operator.We assume that T^{-1} exists and we need to show that it is linear operator.

The domain of T^{-1} is basically range of *T* and also *R(T)* is a vector space.

$$x_1, x_2 \in D(T) \Rightarrow y_1 = Tx_1 \quad and \quad y_2 = Tx_2$$

$$y_1 = Tx_1 \qquad \Rightarrow \qquad x_1 = T^{-1}y_1$$

and $$y_2 = Tx_2 \qquad \Rightarrow \qquad x_2 = T^{-1}y_2$$

T is linear so for any scalar α *and* β we have

$$\alpha y_1 + \beta y_2 = \alpha Tx_1 + \beta Tx_2 = T(\alpha x_1 + \beta x_2) \quad \because \; T \text{ is linear}$$

Applying T^{-1} on above we get

$$T'(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2$$

Putting values of x_1 *and* x_2

$$T'(\alpha y_1 + \beta y_2) = \alpha T' y_1 + \beta T' y_2$$

T^{-1} is a linear operator

C): if $\dim D(T) = n < \infty$ *and* T^{-1} exists, then $\dim R(T) = \dim D(T)$.

We have proved that $\dim R(T) \le n < \infty$ we know

$$\dim R(T) \le \dim D(T) \; \ldots\ldots\ldots\ldots\ldots \text{i}$$

Conversely,

$$T^{-1} : R(T) \to D(T)$$

$$\dim D(T) \le \dim R(T) \ldots\ldots\text{ii}$$

Combining i and ii $\quad \dim R(T) = \dim D(T)$

If inverse exist then both dimensions are equal. That we have to prove.

Module No. 59

Linear Operators

- ***Lemma(Inverse of Product)***

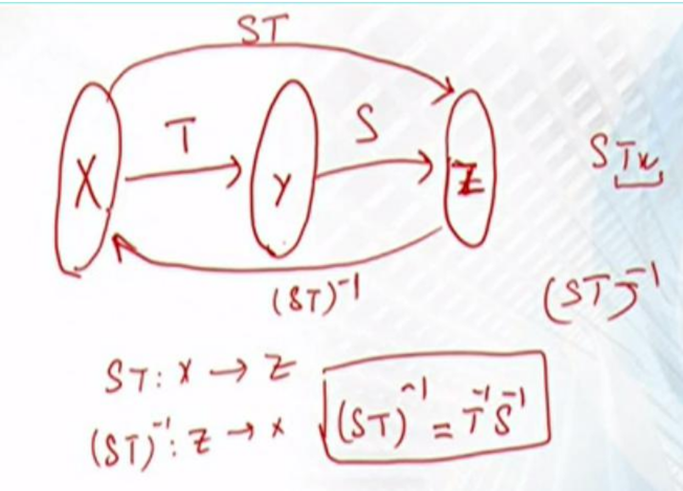
Bijective mean one to one and onto. Here it means inverse of T and S exists.

Lemma

Let $T:X\to Y$ *and* $S:Y\to Z$ be bijective linear operators, where X,Y are vectors spaces.

Then the inverse $(ST)^{-1}:Z\to X$ of the product (the composite) ST exists, and $(ST)^{-1}=T^{-1}S^{-1}$.

Diagram



Mathematically,

If S is bijective and T is bijective then ST is also bijective.

$ST:X\to Z$ bijective

$\Rightarrow$ $(ST)^{-1}$ exist.

It means if $(ST)(ST)^{-1}=I_Z$

If $S:Y\to Z$ then $S^{-1}S=I_Y$

$S^{-1}ST(ST)^{-1}=S^{-1}I_Z$ $\Rightarrow$ $T(ST)^{-1}=S^{-1}$

$\Rightarrow \quad T^{-1}T(ST)^{-1} = T^{-1}S^{-1} \qquad \Rightarrow \quad (ST)^{-1} = T^{-1}S^{-1}$

MODULE NO. 60

LINEAR OPERATORS

- ***Bounded Linear Operator***

Norms spaces are generalization of distances.

Bounded Linear Operator (Definition):

Let X and Y be normed spaces and $T : D(T) \to Y$ a linear operator, where $D(T) \subset X$. The operator T is said to be bounded if there is a real number c such that for all $x \in D(T)$.

$$\|Tx\| \le c\|x\|$$

If this condition satisfied then we call T to be a bounded linear operator. Bounded function mean range is bounded but here bounded set is mapping over a bounded set so we call this a bounded linear operator.c is fix.

$$\frac{\|Tx\|}{\|x\|} \le c \quad , \quad x \in D(T) - \{0\}$$

The smallest possible value of c is supremum of left hand side. Then the value of c is called

$$c = \sup_{\substack{x\in D(T), \\ x\neq 0}} \frac{\|Tx\|}{\|x\|} \qquad \text{as} \left(T \text{ norm} = \sup_{\substack{x\in D(T), \\ x\neq 0}} \frac{\|Tx\|}{\|x\|} \right)$$

We call the value as T norm $\qquad c = \|T\|$

$$\text{If } D(T) = \{0\}, \quad \|T\| = 0$$

$$c = \|T\| = \sup_{\substack{x\in D(T), \\ x\neq 0}} \frac{\|Tx\|}{\|x\|}$$

$$\|Tx\| \le \|T\|\|x\|$$

This is the formula that we use for bounded linear operator.

MODULE NO. 61

Bounded Linear Operators

➢ *Lemma (Norm)*

First we define the norm and then prove that the norm defined on T satisfies (N1) to (N4).

Lemma:

Let T be a bounded linear operator as defined before.

An alternate formula for the norm of T is $\|T\| = \sup\limits_{\substack{x\in D(T) \\ \|x\|=1}} \|Tx\|$

The norm defined on T satisfies (N1) to (N4).

Proof:

$$\|Tx\| \le c\|x\|$$

$$c = \|T\| = \sup_{\substack{x\in D(T), \\ x\neq 0}} \frac{\|Tx\|}{\|x\|} \simeq \sup_{\substack{x\in D(T) \\ \|x\|=1}} \|Tx\|$$

We have to prove $\sup\limits_{\substack{x\in D(T), \\ x\neq 0}} \frac{\|Tx\|}{\|x\|} \simeq \sup\limits_{\substack{x\in D(T) \\ \|x\|=1}} \|Tx\|$

Let $\|x\| = a;$ set $y=\frac{x}{a}$, $x \neq 0$,

$$\|y\| = \frac{\|x\|}{a} = 1$$

$$\|T\| = \sup_{\substack{x\in D(T), \\ x\neq 0}} \frac{\|Tx\|}{a}$$

as T is linear so, we take constant $a$inside the norm

$$\|T\| = \sup_{\substack{x\in D(T), \\ x\neq 0}} \left\| T\left(\frac{1}{a}x\right) \right\| = \sup_{\substack{y\in D(T), \\ \|y\|=1}} \|Ty\| \qquad \text{as } \frac{1}{a} = y$$

Here variable is y which can be any other.

Part a) of lemma is proved.

Part b):

$$\|T\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in D(T),\\ \|x\|=1}} \|Tx\|$$

N1: $\|T\| \geq 0$ is obvious.

N2: $\|T\| > 0 \quad \Rightarrow \quad$ T=0,

$$\|T\| = 0 \quad \Rightarrow \quad Tx = 0, \quad \forall\ x \in D(T) \ \Rightarrow\ T = 0$$

N3: $\|\alpha T\| = \sup_{\substack{x\in D(T),\\ \|x\|=1}} \|\alpha Tx\| = \sup_{\|x\|=1} |\alpha| \|Tx\| = |\alpha| \sup_{\|x\|=1} \|Tx\| = |\alpha| \|T\|$ as $\sup_{\|x\|=1} \|Tx\| = \|T\|$

N4: $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$

$$\begin{aligned}\|T_1 + T_2\| &= \sup_{\substack{x\in D(T)\\ \|x\|=1}} \|(T_1 + T_2)x\| \\ &\leq \sup_{\|x\|=1} \|T_1 x + T_2 x\| \leq \sup_{\|x\|=1} \left(\|T_1 x\| + \|T_2 x\|\right) \\ &= \sup_{\|x\|=1} \|T_1 x\| + \sup_{\|x\|=1} \|T_2 x\| = \|T_1\| + \|T_2\|\end{aligned}$$

First we define a $T \times T$ norm and then prove the four properties of norm.

Module No. 62

Examples Bounded Linear Operators

- ***Identity Operator***
- ***Zero Operator***
- ***Differentiation Operator***
- ***Integral Operator***

Identity operator:

$$I : X \to X \qquad \Rightarrow \qquad I_x = x \quad \{x \neq \{0\} \text{ normed space}\}$$

$$\|I\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|x\|}{\|x\|} \qquad as \qquad Tx = x$$

$$\|I\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} 1 = 1$$

Zero operator:

The norm space $O : X \to Y$, $\quad O_x = 0 \qquad\qquad x \in X$

$$\|O\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = 0 \quad , \|0\| = 0$$

Differentiation operator:

This is defined on normed space of all polynomial on J=[0, 1]

$$\|x\| = \max\{|x(t)|, \quad t \in J\}$$

Value of t varies from 0 to 1 and where the value is maximum, that maximum value is norm of *x*.

applying operator the derivative. Differentiation operator is.

$$Tx(t) = x'(t)$$

Derivation is itself a linear operator.

Now we check that it is bounded or not. $\|Tx(t)\| \le c\|x(t)\|$. If it is bounded then what is the value of c.

Let $x_n(t) = t^n \quad \text{n} \in \mathbb{N}$, what is the norm of $x_n(t)$

$$\|x_n(t)\| = \max\{|x(t)|, \quad t \in [0,1]\} = 1$$

Using operator $Tx_n(t) = nt^{n-1}$

define the norm $\|Tx_n(t)\| = \max\left|nt^{n-1}\right| = 1$

$$\|Tx_n(t)\| = \max(|\, nt^{n-1}\,| : t \in [0,1]) = n.1 = n$$

$$\frac{\|Tx_n\|}{x_n} = \frac{n}{1} = c, \quad \text{n} \in \mathbb{N}$$

As n had no bound so, there does not exist any c such that $\frac{\|Tx\|}{\|x_n\|} \le c$ hold.

Now c is fixed number which does not depend upon N but in this case it depends on N, if we take c as n then next value n+1 will not satisfy this equation. It means that there does not exist any c that this condition $\frac{\|Tx\|}{\|x_n\|} \le c$ holdhence derivative operative is not bounded.

Integral Operator

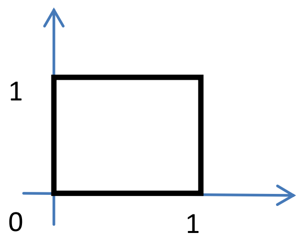
Defined as $T : C[0,1] \to C[0,1]$,

$$y = Tx \qquad y(t) = \int_0^1 k(t,\tau)x(\tau)d\tau$$

k is integral of T it is fix for different integral operator,

T is linear as integration is linear, also derivation is a linear operator same as integral is linear operator.

K is continuous on $J \times J$. We have two variables $t \text{ and } \tau$, $k(t, \tau)$



Whatever the value of k is, it should be in the square

$k(t, \tau)$ is bounded. And if it is bounded then

$k(t, \tau) \le k_o, \; t,\tau \in J \times J \;, \; k_o \in \mathbb{R}$ where $J \times J$ is this square box.

$|x(t)| \le \max_{t \in J} |x(t)| = \|x\|$

Now example, $$\|y\| = \|Tx\| = \max_{t \in J} \left| \int_0^1 k(t,\tau)x(\tau)d\tau \right|$$

$$\le \max_{t \in J} \int_0^1 |k(t,\tau)||x(\tau)|\,d\tau$$

$$\le k_o \|x\|$$

$\|Tx\| \le k_o \|x\|$ it has $k \text{ and } k_o$ is fix so integral operator is a linear operator.

Module No. 63

Examples Bounded Linear Operators

- ***Matrix***

Identity operator:

$$T : R^n \to R^r$$

$$\begin{bmatrix} a_{11} & . & a_{1n} \\ . & . & . \\ a_{r1} & . & a_{rn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ . \\ \xi_n \end{bmatrix} = \begin{bmatrix} x_1 \\ . \\ x_n \end{bmatrix}$$

$r \times n \qquad n \times 1 \quad r \times 1$

$A \qquad\quad \text{x} \;\; = \;\; \text{y}$

The entries are $x = (\xi_j)$, $y = (\eta_j)$

And the matrix is $A = (\alpha_{ij}),\quad 1 \le \mathrm{i} \le r,\quad 1 \le \mathrm{j} \le n$

$$\eta_j = \sum_{k=1}^{n} \alpha_{jk} \xi k$$

T is linear because the properties of matrices is it bounded?

$$\|x\| = \left(\sum_{m=1}^{n} \xi_m^2 \right)^{\frac{1}{2}} \qquad , \quad x \in \mathbb{R}^n$$

and

$$\|y\| = \left(\sum_{j=1}^{r} \eta_j^2 \right)^{\frac{1}{2}} \qquad , \quad \text{y} \in \mathbb{R}^n$$

for bounded we have to check norm of T "T(x)".

$$\|Tx\| = \left(\sum_{j=1}^{r} \eta_j^2 \right)^{\frac{1}{2}}$$

$$\|Tx\|^2 = \sum_{j=1}^{r} \eta_j^2$$

$$\|Tx\|^2 = \sum_{j=1}^{r} \left(\sum_{k=1}^{n} \alpha_{jk} \xi_k \right)^2$$

Where $\eta_j = \sum_{k=1}^{n} \alpha_{jk} \xi_k$

Cauchy Schwaz inequality on above $\|Tx\|^2$

$$\leq \sum_{j=1}^{r} \left[\left(\sum_{k=1}^{n} \alpha_{jk}{}^2 \right)^{\frac{1}{2}} \left(\sum_{m=1}^{n} \xi_m^2 \right)^{\frac{1}{2}} \right]^2 = \|x\|^2 \left(\sum_{j=1}^{r} \sum_{k=1}^{n} \alpha_{jk}^2 \right)$$

$$\|Tx\|^2 \leq c^2 \|x\|^2$$

Here is a c which depends upon T.

We can write as

$$\|Tx\| \leq c \|x\|$$

T is already linear and with this value of c we can say matrices is a linear bounded operator.in last four examples three are linear operator but differential was not linear operator.

MTH 641

FUNCTIONAL ANALYSIS

MODULE # 60 To 113

(FINAL TERM SYLLABUS)

Don't look for someone who can solve your problems,

Instead go and stand in front of the mirror,

Look straight into your eyes,

And you will see the best person who can solve your problems!

Always trust yourself.

(BY ABU SULTAN)

Module No. 60

Linear Operators

- ***Bounded Linear Operator***

Norms spaces are generalization of distances. By using Norm spaces we are going to discuss Bounded Linear Operator.

Bounded Linear Operator (Definition):

Let X and Y be normed spaces and $T : D(T) \to Y$ a linear operator, where $D(T) \subset X$. The operator T is said to be bounded if there is a real number c such that for all $x \in D(T)$.

$$\|Tx\| \le c\|x\|$$

If this condition satisfied then we call T to be a bounded linear operator. Bounded function mean range is bounded but here bounded set is mapping over a bounded set so we call this a bounded linear operator. c is fix.

$$\frac{\|Tx\|}{\|x\|} \le c \quad , \quad x \in D(T) - \{0\}$$

The smallest possible value of c is supremum of left hand side. Then the value of c is called

$$c = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \qquad \text{as}$$

We call the value as T norm

$$c = \|T\|$$

$$\text{If } D(T) = \{0\}, \quad \|T\| = 0$$

$$c = \|T\| = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

$$\|Tx\| \le \|T\|\|x\|$$

This is the formula that we use for bounded linear operator.

Module No. 61

Bounded Linear Operators

- ***Lemma (Norm)***

First we define the norm (equivalent definition) and then prove that the norm defined on T satisfies all four properties of Norm i.e. (N1) to (N4).

Lemma (Statement):

Let T be a bounded linear operator as defined before then an alternate formula for the norm of T is

$$\|T\| = \sup_{\substack{x \in D(T) \\ \|x\| = 1}} \|Tx\|$$

The norm defined on T satisfies (N1) to (N4).

Proof: Part (a)

$$\|Tx\| \le c\|x\|$$

$$c = \|T\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} \simeq \sup_{\substack{x\in D(T)\\ \|x\|=1}} \|Tx\|$$

We have to prove

$$\sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} \simeq \sup_{\substack{x\in D(T)\\ \|x\|=1}} \|Tx\|$$

Let $\|x\| = a$; set $y=\frac{x}{a}$, $x \neq 0$,

$$\|y\| = \frac{\|x\|}{a} = 1$$

$$\|T\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{a}$$

as T is linear so, we take constant a inside the norm

$$\|T\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \left\| T\left(\frac{1}{a}x\right)\right\| = \sup_{\substack{y\in D(T),\\ \|y\|=1}} \|Ty\| \qquad \text{as } \frac{1}{a} = y$$

Here variable is y which can be any other. Part (a) of lemma is proved.

Part (b):

$$\|T\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in D(T),\\ \|x\|=1}} \|Tx\|$$

N1: $\|T\| \geq 0$ is obvious.

N2: $\|T\| > 0 \Rightarrow$ T=0,

$\|T\| = 0 \Rightarrow Tx = 0, \quad \forall\ x \in D(T) \Rightarrow T = 0$

N3: $\|\alpha T\| = \sup_{\substack{x\in D(T),\\ \|x\|=1}} \|\alpha Tx\| = \sup_{\|x\|=1} |\alpha| \|Tx\| = |\alpha| \sup_{\|x\|=1} \|Tx\| = |\alpha| \|T\|$ as $\sup_{\|x\|=1} \|Tx\| = \|T\|$

N4: $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$

$$\|T_1 + T_2\| = \sup_{\substack{x\in D(T)\\ \|x\|=1}} \|(T_1+T_2)x\|$$

$$\leq \sup_{\|x\|=1} \|T_1x + T_2x\| \leq \sup_{\|x\|=1} \left(\|T_1x\| + \|T_2x\|\right)$$

$$= \sup_{\|x\|=1} \|T_1x\| + \sup_{\|x\|=1} \|T_2x\| = \|T_1\| + \|T_2\|$$

First we define a $T \times T$ norm and then prove the four properties of norm.

MODULE NO. 62

EXAMPLES BOUNDED LINEAR OPERATORS

- ***Identity Operator***
- ***Zero Operator***
- ***Differentiation Operator***
- ***Integral Operator***

Identity operator:

$$I : X \to X \qquad \Rightarrow \qquad I_x = x \quad \{x \neq \{0\} \text{ normed space}\}$$

$$\|I\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|x\|}{\|x\|} \qquad as \qquad Tx = x$$

$$\|I\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} 1 = 1$$

Zero operator:

The norm space $O : X \to Y$, $\quad O_x = 0 \qquad\qquad x \in X$

$$\|O\| = \sup_{\substack{x\in D(T),\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = 0 \quad , \|0\| = 0$$

Differentiation operator:

This is defined on normed space of all polynomial on J=[0, 1]

$$\|x\| = \max\{|x(t)|, \quad t \in J\}$$

Value of t varies from 0 to 1 and where the value is maximum, that maximum value is norm of *x*.
applying operator the derivative. Differentiation operator is.

$$Tx(t) = x'(t)$$

Derivation is itself a linear operator.
Now we check that it is bounded or not. $\|Tx(t)\| \leq c\|x(t)\|$. If it is bounded then what is the value of c.
Let $x_n(t) = t^n \quad n \in \mathbb{N}$, what is the norm of $x_n(t)$

$$\|x_n(t)\| = \max\{|x(t)|, \quad t \in [0,1]\} = 1$$

Using operator $Tx_n(t) = nt^{n-1}$

define the norm $\qquad \|Tx_n(t)\| = \max\left|nt^{n-1}\right| = 1$

$$\|Tx_n(t)\| = \max(|\,nt^{n-1}\,| : t \in [0,1]) = n.1 = n$$

$$\frac{\|Tx_n\|}{x_n} = \frac{n}{1} = c, \quad n \in \mathbb{N}$$

As n had no bound so, there does not exist any c such that $\frac{\|Tx\|}{\|x_n\|} \leq c$ hold.

Now c is fixed number which does not depend upon N but in this case it depends on N, if we take c as n then next value n+1 will not satisfy this equation. It means that there does not exist any c that this condition $\frac{\|Tx\|}{\|x_n\|} \leq c$ holdhence derivative operative is not bounded.

Integral Operator

Defined as $T : C[0,1] \to C[0,1]$,

$$y = Tx \qquad\qquad y(t) = \int_0^1 k(t,\tau)x(\tau)d\tau$$

k is integral of T it is fix for different integral operator, T is linear as integration is linear, also derivation is a linear operator same as integral is linear operator.

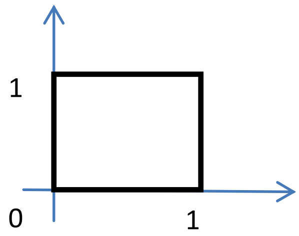
K is continuous on $J \times J$. We have two variables $t\ and\ \tau$, $k(t,\ \tau)$

Whatever the value of k is, it should be in the square

$k(t,\ \tau)$ is bounded. And if it is bounded then

$k(t,\ \tau) \leq k_o,\ \ t,\tau \in J \times J\ ,\ \ k_o \in \mathbb{R}$ where $J \times J$ is this square box.

$|x(t)| \leq \max_{t\in J} |x(t)| = \|x\|$



Now example,

$$\|y\| = \|Tx\| = \max_{t\in J} \left| \int_0^1 k(t,\tau)x(\tau)d\tau \right|$$

$$\leq \max_{t\in J} \int_0^1 |k(t,\tau)||x(\tau)|\,d\tau$$

$$\leq k_o \|x\|$$

$\|Tx\| \leq k_o \|x\|$ it has $k\ and\ k_o$ is fix so integral operator is a linear operator.

MODULE NO. 63

EXAMPLES BOUNDED LINEAR OPERATORS

- ***Matrix***

Identity operator:

$$T : R^n \to R^r$$

$$\begin{bmatrix} a_{11} & . & a_{1n} \\ . & . & . \\ a_{r1} & . & a_{rn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ . \\ \xi_n \end{bmatrix} = \begin{bmatrix} x_1 \\ . \\ x_n \end{bmatrix}$$

$$r \times n \qquad n\times 1 \quad r\times 1$$

$$A \qquad \text{x} \quad = \quad \text{y}$$

The entries are $x = (\xi_j)$, $y = (\eta_j)$

And the matrix is $A = (\alpha_{ij}),\quad 1 \leq \text{i} \leq r,\quad 1 \leq \text{j} \leq n$

$$\eta_j = \sum_{k=1}^{n} \alpha_{jk} \xi k$$

T is linear because the properties of matrices is it bounded?

$$\|x\| = \left(\sum_{m=1}^{n} \xi_m^2 \right)^{\frac{1}{2}} \qquad , \quad x \in \mathbb{R}^n$$

and

$$\|y\| = \left(\sum_{j=1}^{r} \eta_j^2 \right)^{\frac{1}{2}} \qquad , \quad \text{y} \in \mathbb{R}^n$$

for bounded we have to check norm of T "T(x)".

$$\|Tx\| = \left(\sum_{j=1}^{r} \eta_j^2\right)^{\frac{1}{2}}$$

$$\|Tx\|^2 = \sum_{j=1}^{r} \eta_j^2$$

$$\|Tx\|^2 = \sum_{j=1}^{r} \left(\sum_{k=1}^{n} \alpha_{jk}\xi_k\right)^2$$

Where $\eta_j = \sum_{k=1}^{n} \alpha_{jk}\xi_k$

Cauchy Schwaz inequality on above $\|Tx\|^2$

$$\le \sum_{j=1}^{r} \left[\left(\sum_{k=1}^{n} \alpha_{jk}{}^2\right)^{\frac{1}{2}} \left(\sum_{m=1}^{n} \xi_m^2\right)^{\frac{1}{2}}\right]^2 = \|x\|^2 \left(\sum_{j=1}^{r}\sum_{k=1}^{n} \alpha_{jk}^2\right)$$

$$\|Tx\|^2 \le c^2 \|x\|^2$$

Here is a c which depends upon T.
We can write as

$$\|Tx\| \le c\|x\|$$

T is already linear and with this value of c we can say matrices is a linear bounded operator.in last four examples three are linear operator but differential was not linear operator.

MODULE NO. 71

LINEAR FUNCTION (EXAMPLES):

- Space $C[a \;\; \text{b}]$
- Space l^2

Space $C[a \;\; \text{b}]$**:**

We have define a linear function on space $C[a \;\; \text{b}]$ that we have fixed an element t_o from the set J as $t_o \in J$. Now define a functional operator f(x) which is operating on x which is element from $C[a \;\; \text{b}]$. $x \in C[a \;\; \text{b}]$

This x is not a variable, it is a function. So f_1 which is defined on $C[a \;\; \text{b}]$ linear as it is a linear operator. f_1 is bounded. To find the norm

$$|f_1| = |x(b)| \le \|x\|$$

$$\|x\| = 1 \quad \Rightarrow \quad \|f_1\| \le 1..........(i)$$

If we take $x_0 = 1$ and substitute in this equation we get

$$|f_1(x_o)| \le \|f_1\|.\|x\|$$

$$1 \le \|f_1\|.1 \quad \Rightarrow \quad \|f_1\| \ge 1...........(ii$$

From i) and ii)

$$\|f_1\| = 1$$

So the function defined on C is linear, bounded and Norm is 1.

Space l^2

We choose a fix say $a=(a_j)\in l^2$

$$f(x)=\sum_{j=1}^{\infty}\xi_j a_j \qquad x\in l^2,\ x=(\xi_j)$$

This sequence is linear, converging and bounded.

For boundedness

$$|f(x)|=\left|\sum_{j=1}^{\infty}\xi_j a_j\right|\le\sum_{j=1}^{\infty}\left|\xi_j a_j\right|\le\sqrt{\sum_{j=1}^{\infty}\left|\xi_j\right|^2}\sqrt{\sum_{j=1}^{\infty}\left|a_j\right|^2}=\|x\|.\|a\|$$

It is the same definition of bounded.

M of a complete metric space *X* is itself complete if and only if the set *M* is closed in *X*.

Module No. 72

Linear Function:

- Algebraic Dual Space
- Second Algebraic Dual Space
- Canonical Mapping

Algebraic Dual Space

Set of all linear function defined on a vector space X is itself a vector space and called Algebraic Dual Space and denoted by X^*

Operation on this vector space are

1st Operation Sum

$$f_1+f_2 \qquad f_1, f_2 \text{ linear functional}$$

$$\left(f_1+f_2\right)x=f_1(x)+f_2(x) \quad x\in X$$

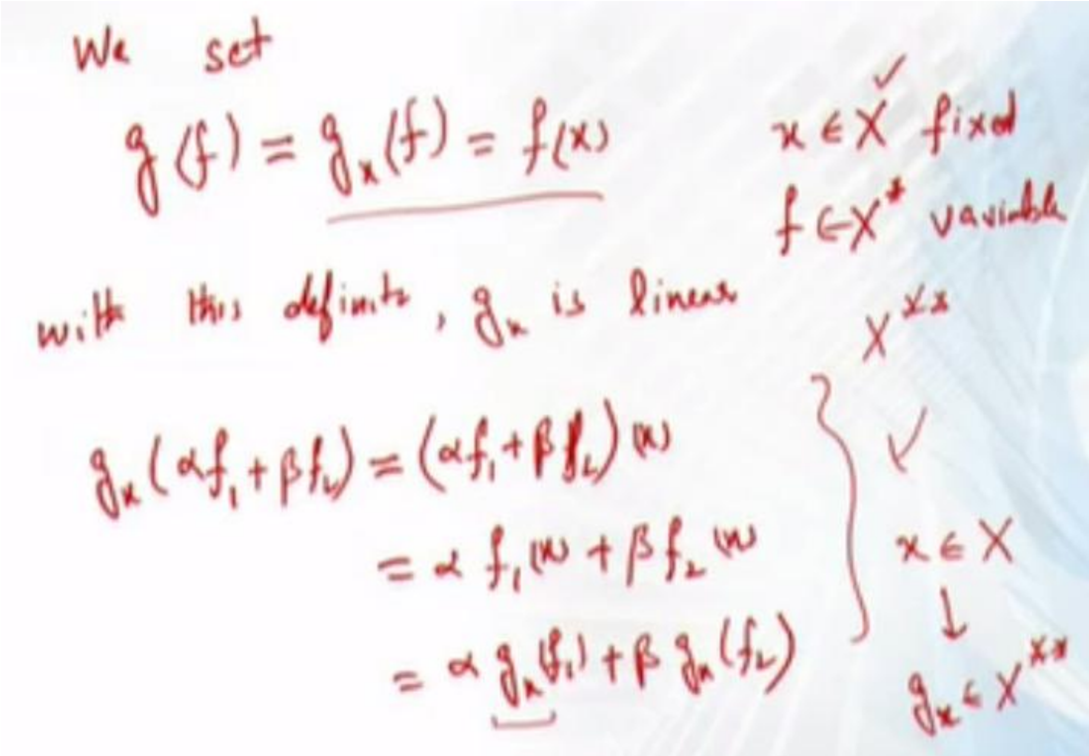
2nd Operation Scalar Multiplication

$$\left(af\right)x=af(x)$$

Second Algebraic Dual Space X^{**}

Space	element	Vector at a point
X	$x\in X$	
X^*	g	f (x)
X^{**}	G	g(x)

For each $x,g\in X^{**}$



Conical Mapping:

$C: X \to X^{**}$ this mapping is called canonical mapping of X into X^{**} defined as

$x \mapsto g_x$.

$C(\alpha x + \beta y)(f) = g_{\alpha x + \beta y}(f)$

$= f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha g_x(f) + \beta g_{y\partial}(f)$

$= \alpha(Cx)(f) + \beta(Cy)(f)$

So, this is a linear function as well. Canonical mapping is a relation between X and X^{**}.

Module No. 73

Linear Function:

- Algebraically Reflexive
- Second Algebraic Dual Space
- Canonical Mapping

Isomorphism:

It is one-one and onto map.

Algebraically Reflexive:

$T: (X, d) \to (\tilde{X}, \tilde{d})$ bijective

$\tilde{d}(T_x, T_y) = d(x, y)$

$C: X \to X^{**}$ $x \mapsto g_x$.

If C is surjective (on b) bijection. $\Re(C) = X^{**}$

We call X to be algebraically reflexive.

Set of all linear function defined on a vector space X is itself a vector space and called

Module No. 74

Linear Operators and functional on finite Dimensional Spaces:

Finite dimensions mean basis which have finite many elements.

Let X and Y bef.gfinite dimension vector spaces over the same field.

Let $T: X \to Y$ be a linear operator. let $E = \{e_1, ..., e_n\}$ be the basis for X and $B = \{b_1, ..., b_n\}$ be the basis for Y.

$x \in X$, x=$\xi_1 e_1 + \xi_2 e_2 + ... + \xi_n e_n$

$$y = Tx = T\left(\sum_{k=1}^{n} \xi_k e_k\right) = \sum_{k=1}^{n} T(\xi_k e_k) = \sum_{k=1}^{n} \xi_k T(e_k)$$

T is uniquely determinal if the image $y_k = Te_k$ of n basis vectors $e_1, ..., e_n$ are prescribed.

y=Tx ; $y \in Y$ $\{b_1, ..., b_n\}$

y=$\eta_1 b_1 + \eta_2 b_2 + ... + \eta_r b_r$

$Te_k \in Y$, Te$_1$=$\tau_{11} b_1 + \tau_{12} b_2 + ... + \tau_{1r} b_r$

$$Te_k = \sum_{j=1}^{r} \tau_{kj} b_j$$

$$y = \sum_{j=1}^{r} \eta_j b_j = \sum_{k=1}^{n} \xi_k Te_k = \sum_{k=1}^{n} \xi_k \sum_{j=1}^{r} \tau_{kj} b_j$$

Comibinig these two summation

$$y = \sum_{j=1}^{r}\left(\sum_{k=1}^{n} \tau_{kj} \xi_k\right) b_j$$

$$\eta_j = \sum_{k=1}^{n} \tau_{kj} \xi_k$$

The image y=Tx=$\sum \eta_j b_j$ of $x = \sum \xi_k Te_k$ can be obtained from

$$\eta_j = \sum_{k=1}^{n} \tau_{kj} \xi_k$$

Module No. 75

Operators on finite Dimensional Spaces:

Remarks:

As in the case of linear operators on a finite dimensional normed space, every linear functional defined on a finite dimensional normed space is bounded and hence continuous. Since for linear funcionals range is either $\mathbb{R}$ or $\mathbb{C}$, which are complete. So X^* as the space of all bounded linear functionals defined on X, is also complete and hence is Banach space. This is true even if X is not a Banach space.

"Algebraic Dual Space of X": set of all linear funcionals defined on X.

"Dual or Conjugate Space of X": X^* set of all continuous or bounded linear functionals defined on X.

We take algebraic dual when there is no condition of continuous or bounded linear functions.

Theorem:

Let X be an n-dimensional vector space and X^* be its dual space. Then

$$\dim X^* = \dim X = n.$$

X^* is collection of linear functions or linear operator while X may be any space.

Proof:

Let $\dim X = n.$

Let basis of X be $B = \{e_1, ..., e_2\}$

We define a function.

$$f_j(e_1) = \begin{cases} 1 & if \ i = j \\ 0 & if \ i \neq j : \ i, j = 1, ..., n \end{cases}$$

e.g. j=1 , $f(e_1) = 1, f(e_2) = 0, f(e_3) = 0, ..., f(e_n) = 0$

j=2 , $f(e_1) = 0, f(e_2) = 1, f(e_3) = 0, ..., f(e_n) = 0$

but each n-tuples f_j in this case can be extended as linear functions on X.

Module No. 76

Operators on Finite Dimensional Spaces:

Lemma(Zero Vector):

Let X be a finite deimensional vector space. If $x_0 \in X$ has the property that $f(x_0) = 0$ for all $f \in X^*$ then $x_0 = 0$.

B^* is the basis of X^*

$\{f_1, f_2, ..., f_n\}$

$$\Rightarrow f_j(e_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_{ij}$$

Proof:

For all $x_0 = 0$,

$$x_0 = \sum_{i=1}^{n} x_i e_i \quad ; \quad f \in X^* \quad ,$$

$$f(x_0) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} f\left(\sum_{i=1}^{n} x_i e_i\right) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n} x_i f(e_i) = 0 \quad , \quad j=1,......,n$$

$$\Rightarrow \quad x_j = 0 \quad , \quad \forall \ j=1,........,n$$

$$x_0 = \sum_{i=1}^{n} x_i e_i = 0 \quad \Rightarrow \quad x_0 = \delta$$

Module No. 77

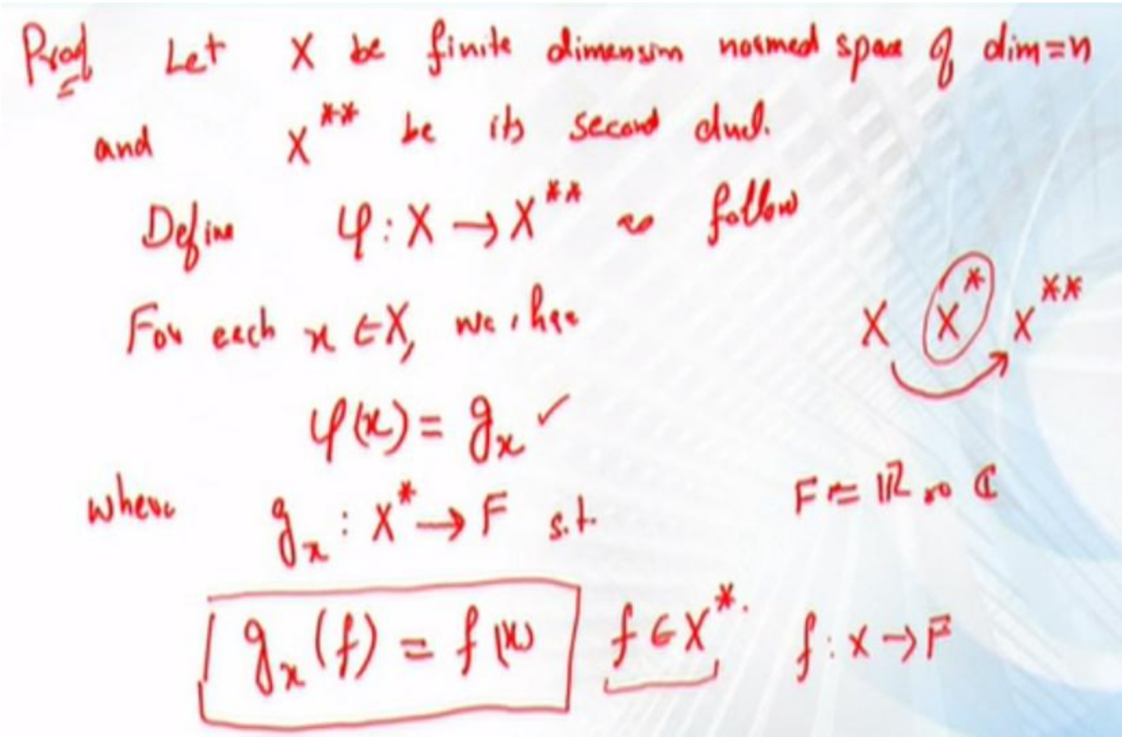
Operators on Finite Dimensional Spaces:

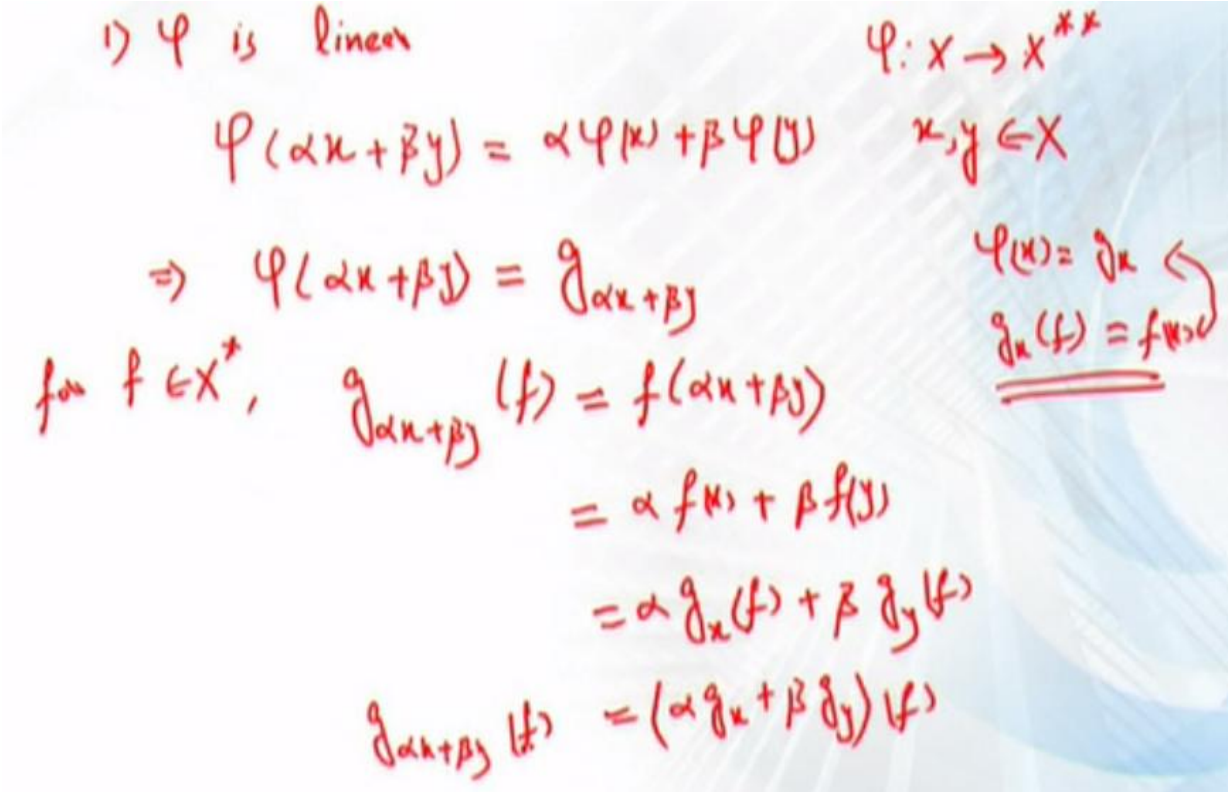
Theorem(Reflexivity):

A normed space X is said to algebraically reflexive if there is an isometric isomorphism between X and X^{**}.
Ordinarily a normed spacer may not be reflexive.
If X is an incomplete normed space even then X^{*} and X^{**} are Banach spaces. So in this case X cannot be a reflexive space.
However there are Banach spaces which are not reflexive.

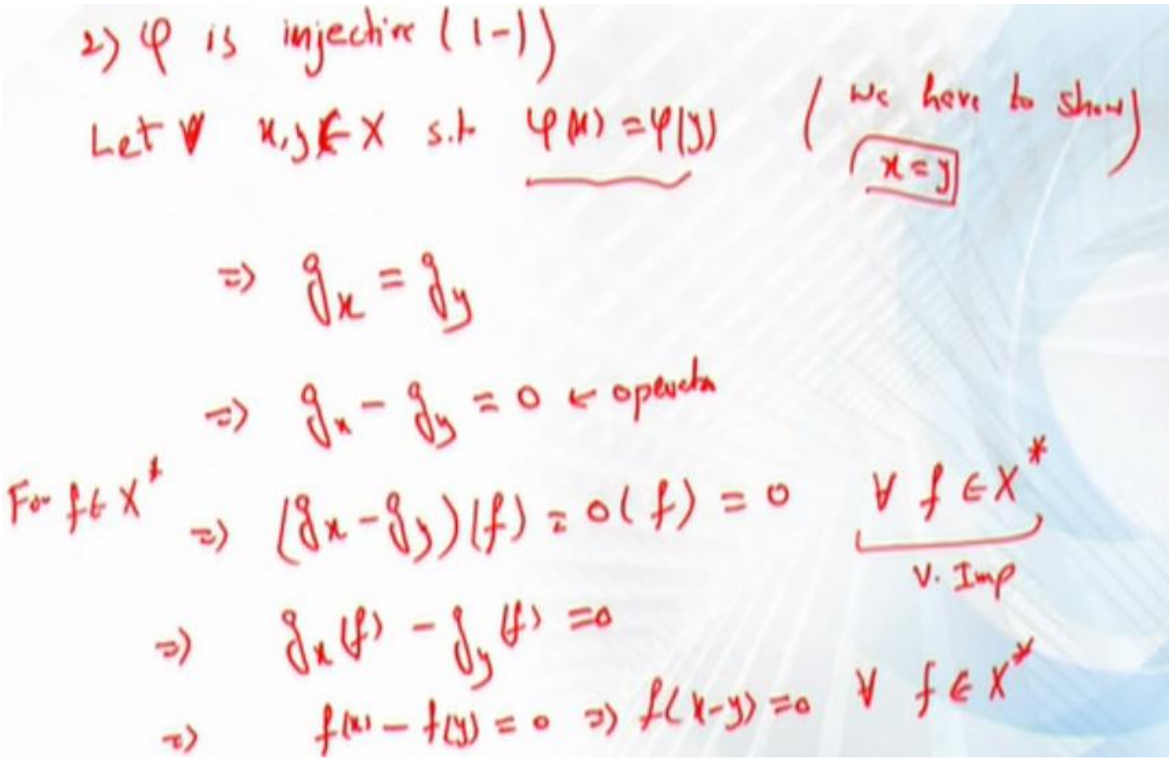
Theorem:

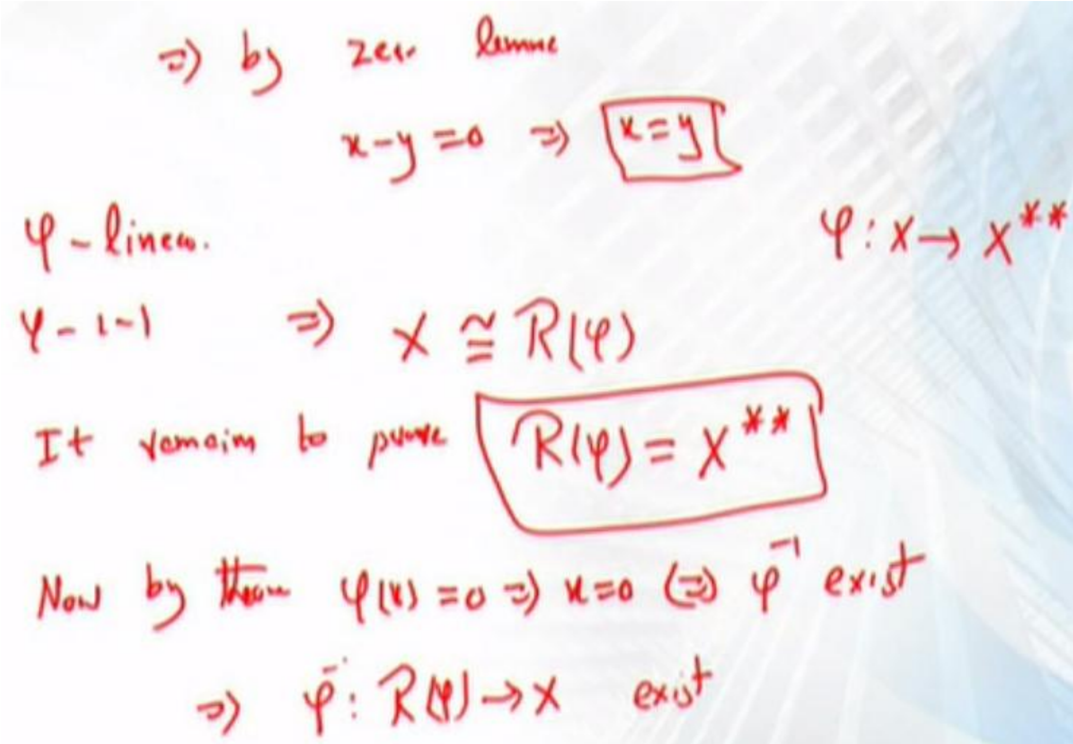
A finite dimensional vector space is reflexive.
Equivalently, A finite dimensional normed space is isomorphic space is isomorphic to its second dual.

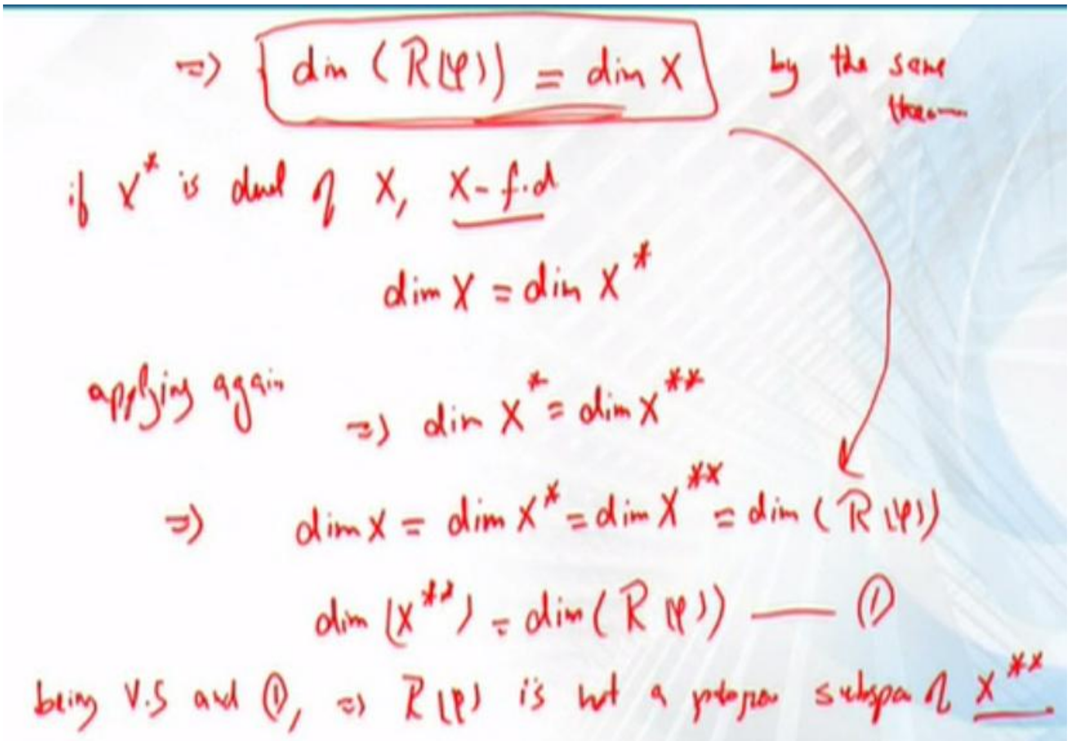




$$g_{\alpha x+\beta y} = \alpha g_x + \beta g_y$$
$$\varphi(\alpha x+\beta y) = \alpha\varphi(x) + \beta\varphi(y)$$







$R(\varphi) = X^{**}$ φ *is onto*

$X \cong X^{**}$ X *reflexive*

Module No. 78

Linear Transformation:

Q No.1:

Find the null space of $T : \mathbb{R}^3 \to \mathbb{R}^2$ represented by

$$\begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ -2x_1 + x_2 \end{bmatrix}$$

$2\times 3 \quad 3\times 1 \qquad 2\times 1$

What is meant by null space, it means we have to find those values of $x \in \mathbb{R}^3$ say $x = (x_1, x_2, x_3)$ such that we operate T the answer is zeros as

All those x are element of null space.

$$\begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ -2x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Also we can also say that

$$x_1 + 3x_2 + 2x_3 = 0$$
$$-2x_1 + x_2 = 0$$

We can solve it by using any linear algebra method that will give us solution like echelon form or reduced echelon form and the base of that solution is called basis of null space. Basis mean when apply the element of $\mathbb{R}^3$ the answer should be zero and get a system of linear equation. Find the solution of this system of linear equation. And after finding the solution find the basis that basis are basis of null space.

Example.

Q.NO2

Find the null space of $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $(\xi_1, \xi_2, \xi_3) \leftrightarrow (\xi_1, \xi_2, -\xi_1 - \xi_2)$

1) Basis of $\mathbb{R}(T)$
2) Basis of N(T)
3) Matrix representation.

Module No.79

Exercises

Dual Basis

Example 1:

a): Find the dual basis of X when basis of X are $B = \{(1,-1,3),(0,1,-1),(0,3,-2)\}$,

Find $B^* = ?, X^* = ?$ do it yourself

b): let $\{f_1, f_2, f_3\}$ be basis of dual space for X and if X is given by

$e_1 = (1,1,1), \ \ e_2 = (1,1,-1), \ \ e_3 = (1,-1,1)$

Find $f_1(x), f_2(x), f_3(x) \text{ when } x = (0,1,0)$

Module No.80

Normed Spaces of Operators

- Examples of Dual Spaces
- $\mathbb{R}^n$

Isometric Isomorphism

A linear operator $\phi : X \to Y$. X, Y normed spaces, is said to be Isometric Isomorphism if

ϕ isbijective.

ϕ preserve norms.

That is for any

$x \in X$, $\|\phi(x)\| = \|x\|$ is

MODULE NO.81

EXAMPLES SPACES OF OPERATORS

- Examples of Dual Spaces
- l^1

Space l^1

The dual space of l^n is l^∞ means that it is bijective, it is linear and it preserve norm.
After defining the map we shall prove these properties one by one.

Proof:

MODULE NO.82

BOUNDED LINEAR OPERATORS

Quiz: Complete norm spaces are called Banach spaces.

Theroem

Let*B(X, Y)* be the set of all bounded linear operators form a normed space *X* to a normed space *Y*.

If ***Y*** is a Banach space, then ***B(X, Y)*** is also a Banach.

Proof:

Let $\{T_n\}$ be an arbitrary Cauchy seq. in *B(X,Y)*.

We will show that $\{T_n\}$ converges to an operator *T* in *B(X, Y)*.Since $\{T_n\}$ is Cauchy for every

$\varepsilon > 0 \quad \exists$ N such that $\|T_n - T_m\| < \varepsilon$ (m,n>N)

For all $x \in X$ and (m,n>N) we have

$$\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\|$$
$$\leq \|T_n - T_m\|\|x\| < \varepsilon\|x\|$$

Thus for a fixed *x* and given $\overline{\varepsilon}$

Thus for a fixed x and given $\bar{\varepsilon}$ we may choose $\varepsilon = \varepsilon_x$ so that

$$\varepsilon_x \|x\| < \bar{\varepsilon}$$

$\Rightarrow \quad \|T_n(x) - T_m(x)\| < \varepsilon\|x\| = \varepsilon_x \|x\| < \bar{\varepsilon}$

$\Rightarrow \{T_n(x)\}$ is a Cauchy seq. in Y.

Since Y is complete

Let $T_n(x) \xrightarrow{\text{converg}} y$ for some $y \in Y$

We can define a map

$$T : X \to Y$$

$$T(x) = y$$

We will show that T is the required bounded linear operator.

to show ① T is bounded ② $T_n \to T$

T is linear $\quad T(\alpha x + \beta z) \; : \; x, z \in X$

$T(x) = y, \; T(z) = w$

$$T(\alpha x + \beta z) = \lim_{n\to\infty} T_n(\alpha x + \beta y)$$

$$= \lim_{n\to\infty} T_n(\alpha x) + \lim_{n\to\infty} T_n(\beta y)$$

$$= \alpha \lim_{n\to\infty} T_n(x) + \beta \lim_{n\to\infty} T_n(y)$$

$$= \alpha y + \beta w$$

$$= \alpha T(x) + \beta T(z) \Rightarrow T \text{ is linear.}$$

1) T is bounded

$$\|T_n(x) - T_m(x)\| < \varepsilon \|x\|$$

$T_n(x) \to y$; $T : X \to Y$ $y = T(x)$

$Tx = T(x)$

$T_m(x) \to y = T(x)$

Thus by continuity of norm we have

$$\|T_n(x) - T(x)\| = \|T_n(x) - \lim_{m\to\infty} T_m(x)\|$$

$$= \lim_{m\to\infty} \|T_n(x) - T_m(x)\| \leq \varepsilon \|x\|$$

$\Rightarrow$ $(T_n - T)$ with $n > N$ is bounded

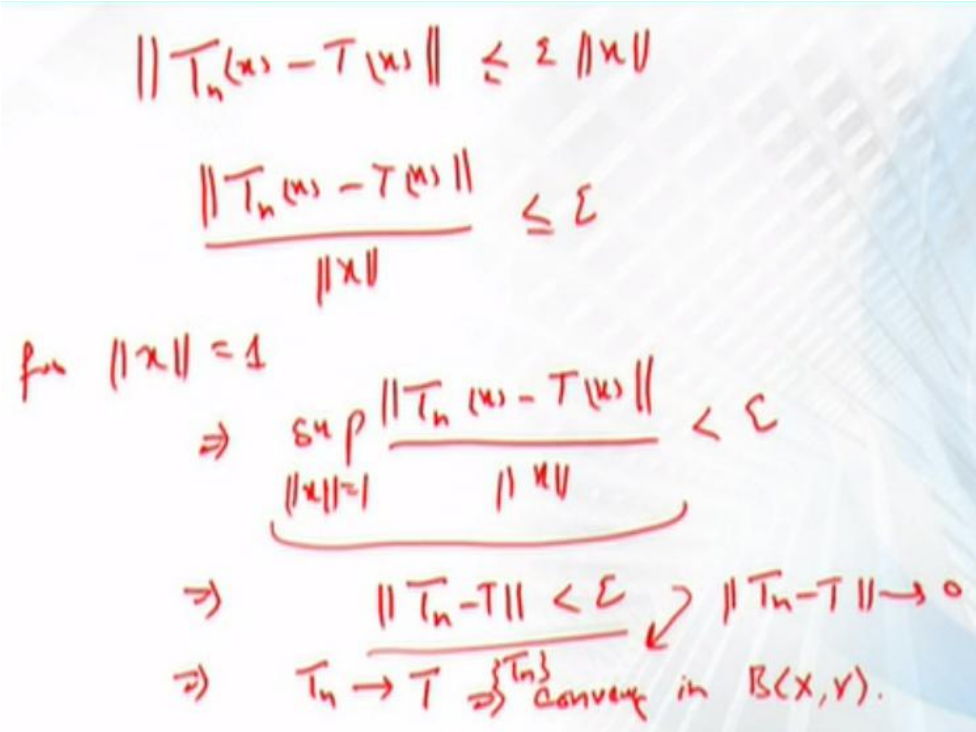
Also T_n is bounded.

$\Rightarrow$ $T = T_n - (T_n - T)$

$\Rightarrow$ T is also bounded

$\Rightarrow$ $T \in B(X, Y)$ ← $T : X \to Y$

$\{T_n\}$; $T_n \to T$



Hence B(X, Y) is complete and Banach space.

Module No.83

Finite Hilbert Spaces

Functional analysis course consist of three major parts parts

1. Metric space is set and we define a space on it that has a certain properties. If it is completer then it is complete space means it should converge within the space
2. Normed Spaces: Norm is a vector space and we define a norm on vector space. Norm is a generalization of distance function.
3. Finite Hilbert Spaces (Inner Product Space)

Hilbert Space

Quiz: Complete inner product space is called a Hilbert Space.

In inner product the generalization is dot product.

Inner product Space

Let V be a vector space over a field F where $F$is $\mathbb{R}$ or $\mathbb{C}$.

An inner product in V is a function $<\bullet,\bullet>: V \times V \to F$ satisfying the following conditions:

Quiz:

Let $x, y, z \in V$; $\alpha \in$ F where α may be real or complex.

i. $<x,x>\geq 0; \quad <x,x>=0 \quad \Leftrightarrow x=0$

ii. $<\alpha x, y>=\alpha<x,y>$; but not true for second value as $<x,\alpha y> \neq \alpha<x,y>$

iii. $<x+y,z>=<x,z>+<y,z>$

iv. $<x,y>=\overline{<y,x>}$

$<\bullet,\bullet>: V \times V \to F$ inner product.

Inner Product Space

The pair $(V,<\bullet,\bullet>)$ is called an inner product space.

a): $<ax+by,z>$ *where* $x,y,z \in V$, $a,b \in F$

Using (iii) property $<ax+by,z>=<ax,z>+<by,z>$

Using (ii) property $a<x,z>+b<y,z>$

$$<0,z>=<0.x,z>=0<x,z>=0$$

b): ***Quiz:***

for all $x,y\in V$, $a\in F$

$$<x,ay>=\overline{<ay,x>}=\overline{a<y,x>}$$

$$=\bar{a}\overline{<y,x>}=\bar{a}<x,y>$$

Module No.84

Cauchy Schwarz Inequality

Theorem:

For any two elements *x,y* is an inner product space *V*,

$|<x,y>|\le\|x\|.\|y\|$, the define norm is $\|x\|=\sqrt{<x,x>}$, x,y $\in$ V

Proof:

If x=y=0 then 0=0

Let at least one of x and y is not equal to zero

Let $|<x+\lambda y,x+\lambda y>|\ge 0$ by definition

$$<x,x+\lambda y>+<\lambda y,x+\lambda y>$$

$$<x,x+\lambda y>+y<y,x+\lambda y>$$

Module No.85

Norm on Inner Product space

Theorem:

In an inner product space V, the function $\|\ .\ \|:V\to\mathbb{R}^+$ given by

$\|x\|=\sqrt{<x,y>}$, $x\in V$ defines a norm in V.

Proof:

N1: $\|x\|\ge 0$

For a $x\in V$, $\|x\|=\sqrt{<x,x>}\ge 0$ as $<x,x>\ge 0$

N2:

$$\|x\|=0 \Leftrightarrow \sqrt{<x,x>}=0 \Leftrightarrow <x,x>=0 \Leftrightarrow x=0$$

N3: $\|\alpha x\|=|\alpha|\|x\|$

now $\|\alpha x\|=\sqrt{<\alpha x,\alpha x>} \Rightarrow \|\alpha x\|^2=<\alpha x,\alpha x>$

$\Rightarrow$ $\|\alpha x\|^2=\alpha\bar{\alpha}<x,x>=|\alpha|^2\|x\|^2$

N4: $\|x+y\|\le\|x\|+\|y\|$ $\forall$ x, y $\in$ V

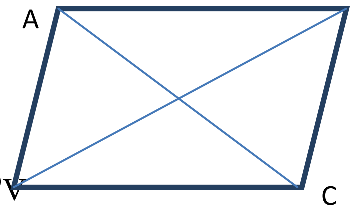
$$\|x+y\|^2 = <x+y, x+y>$$
$$= <x, x+y> + <y, x+y>$$
$$= \overline{<x+y, x>} + \overline{<x+y, y>}$$
$$= \overline{<x,x> + <y,x>} + \overline{<x,y> + <y,y>}$$
$$= \overline{<x,x>} + \overline{<y,x>} + \overline{<x,y>} + \overline{<y,y>}$$

Now $= <x,x> + <x,y> + \overline{<x,y>} + <y,y>$

$$= \|x\|^2 + 2\operatorname{Re} <x,y> + \|y\|^2 \qquad \because \operatorname{Re}(z) \le |z|$$
$$\le \|x\|^2 + 2|<x,y>| + \|y\|^2$$
$$\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \qquad \because \ |<x,y>| \le \|x\|\|y\|$$
$$= (\|x\| + \|y\|)^2$$
$$\|x+y\|^2 \le \|x\| + \|y\|$$

MODULE NO.86

PARALLELOGRAM LAW



$$\overline{AC}^2 + \overline{BD}^2 = 2\left(\overline{AB}^2 + \overline{BC}^2\right)$$ **<u>Quiz</u>**

Theorem:

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$ for all x,y ∈V

Proof:

$$\|x+y\|^2 = <x+y, x+y>$$
$$= <x,x> + <x,y> + \overline{<x,y>} + <y,y>$$
$$= \|x\|^2 + 2\operatorname{Re} <x,y> + \|y\|^2 \qquad ....(i$$

Replace y=-y

$$\|x-y\|^2 = <x+y, x+y>$$
$$= <x,x> - <x,y> - \overline{<x,y>} + <y,y>$$
$$= \|x\|^2 - 2\operatorname{Re} <x,y> + \|y\|^2 \qquad .....(ii$$

Adding (i and (ii

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

That we have to prove.

Special Case:

Another result from above equations is

Subtracting (ii from (i

$$\|x+y\|^2 - \|x-y\|^2 = 4\operatorname{Re} <x,y>$$

If V is a real inner product space

Re(z)=z or Re<x,y>=<x,y>

$$<x,y> = \frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2\right\}$$

The above form is when V is a real inner product space not complex space.

Module No.87

- **Polarization Identity**
- **Appolonius Identity**

Polarization Identity

For any x, y in complex inner product space

$$< x, y >= \frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right\}$$

We have to prove this complex inner product space.

Proof:

Let $x, y \in V$

$$\|x+y\|^2 =< x+y, x+y >$$
$$= \|x\|^2 + 2\operatorname{Re} < x, y > + \|y\|^2$$
$$= \|x\|^2 + < x, y > + \overline{< x, y >} + \|y\|^2$$
$$= \|x\|^2 + < x, y > + < y, x > + \|y\|^2 \qquad ....(i$$

If we replace y=-y

$$\|x+y\|^2 = \|x\|^2 + < x, -y > + < -y, x > + \|-y\|^2$$
$$= \|x\|^2 - < x, y > - < y, x > + \|y\|^2 \qquad ....(ii$$

Replace $y = iy$ in eq(i

$$\|x+iy\|^2 = \|x\|^2 + < x, iy > + < iy, x > + \|iy\|^2$$
$$= \|x\|^2 + \bar{i} < x, y > + i < y, x > + \|y\|^2 \qquad \because \|iy\|^2 =< iy, iy >= i\bar{i} < y, y >= -i^2 < y, y >$$
$$= \|x\|^2 - i < x, y > + i < y, x > + \|y\|^2 \qquad ......(iii$$

Replace $y = -iy$ in eq(i

$$\|x-iy\|^2 = \|x\|^2 + < x, -iy > + < -iy, x > + \|-iy\|^2$$
$$= \|x\|^2 + i < x, y > - i < y, x > + \|y\|^2 \qquad ......(iv$$

Subtracting (ii from (i

$$\|x+y\|^2 - \|x-y\|^2 = 4\operatorname{Re} < x, y > \qquad ....(v$$

Subtracting (iv from (iii

$$\|x+iy\|^2 - \|x-iy\|^2 = 2\{i < y, x > - i < x, y >\}$$
$$= -2i\{< x, y > - < y, x >\} = -2i\left\{< x, y > - \overline{< x, y >}\right\}$$
$$= -2i(2i)\operatorname{Im} < x, y >= 4\operatorname{Im} < x, y > \qquad .....(vi$$

Now we solve $4\operatorname{Re} < x, y > + 4\operatorname{Im} < x, y >$

$$\|x+y\|^2 - \|x-y\|^2 + i\|x+y\|^2 - i\|x-y\|^2 = 4\{< x, y >\}$$

Appolonius Identity

$$\|z-x\|^2+\|z-y\|^2=\frac{1}{2}\|x-y\|^2+2\left\|z-\frac{1}{2}(x+y)\right\|^2 \quad , \quad x,y,z\in V$$

Using parallelogram law

$$\|x'+y'\|^2+\|x'-y'\|^2=2\|x'\|^2+2\|y'\|^2 \quad \text{put} \quad x'= z\text{-}x,\ y'=z\text{-}y$$

Self-assignment

MODULE NO.88

- **SPACE** $C\left[0,\frac{\pi}{2}\right]$
- **SPACE** l^p

Counter example 1: Space $C\left[0,\frac{\pi}{2}\right]$

Inner product define a norm and under this norm
Every inner product space is a norm space.
Every norm space is not an inner product space. This is not true always.
If a space is inner product then it satisfied the parallelogram law otherwise it is not an inner product space.
We take a norm and built an inner product space and then prove that this inner product space does not satisfy the parallelogram law.

The given set is $C\left[0,\frac{\pi}{2}\right]$ real valued continuous function defined on C[a, b].

The norm of function $f\in C\left[0,\frac{\pi}{2}\right]$, is

$$\|f\|=\sup_{x\in\left[0,\frac{\pi}{2}\right]}|f(x)| \quad ,$$

Let $f,g\in C\left[0,\frac{\pi}{2}\right]$; $f(t)=\sin t,\quad g(t)=\cos t$

We know that sin and cos are continuous functions. Let $C\left[0,\frac{\pi}{2}\right]$ is an inner product space

where the inner product $<\bullet,\bullet>$ define by

$$\|f\|=\sqrt{<f,f>} \quad \Rightarrow \quad <f,f>=\|f\|^2$$

$$\|f\|=\sup_{x\in\left[0,\frac{\pi}{2}\right]}|f(x)|$$

$$\|f+g\|^2+\|f-g\|^2=2\|f\|^2+2\|g\|^2$$

As $f(t)=\sin t,\quad g(t)=\cos t$

$$\|f\|=\sup_{x\in\left[0,\frac{\pi}{2}\right]}|\sin(x)|=1=\|g\|$$

$$\|f+g\| = \sup_{x\in\left[0,\frac{\pi}{2}\right]} |f(x)+g(x)|$$

$$= \sup_{x\in\left[0,\frac{\pi}{2}\right]} |\sin x+\cos x| = \sqrt{2}$$

$$\|f-g\| = 1$$

Now

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

$$\left(\sqrt{2}\right)^2 + (1)^2 = 2\times 1^2 + 2\times 1^2$$

$$2+1 = 2+2$$

$$3 = 4$$

But $3 \neq 4$ so our supposition is wrong. This inner product space does not satisfied parallelogram law. Hence every norm space is not inner product space.

Counter example2: Space l^p

l^p Collection of all bounded sequences,
$P>1, P\neq 2$ if p=2 then it will give inner product space

$$\{x_i\},\quad \|x\| = \sqrt[p]{\sum_{i=1}^{\infty}|x_i|^p}$$

We will see that $<x,x>=\|x\|^2$ is an inner product space or not. We will check this if it satisfied the parallelogram or not.

Let

$$x=(1,1,0,0,.........)\ ;\ y=(1,-1,0,0,...........)$$

$$\|x\| = \sqrt[p]{1^p+1^p+0+0+....} = \sqrt[p]{2} = 2^{\frac{1}{p}}$$

$$\|y\| = \sqrt[p]{1^p+(-1)^p+0+0+....} = \sqrt[p]{2} = 2^{\frac{1}{p}}$$

$$x+y=(2,0,0,0,.....) \quad\Rightarrow\quad \|x+y\| = \sqrt[p]{2^p} = 2^{p\times\frac{1}{p}} = 2$$

$$x-y=(0,2,0,0,.....) \quad\Rightarrow\quad \|x-y\| = \sqrt[p]{2^p} = 2^{p\times\frac{1}{p}} = 2$$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

$$2^2+2^2 = 2\times 2^{\frac{1}{p}} + 2\times 2^{\frac{1}{p}}$$

$$8 = 4\times 2^{\frac{2}{p}} \quad \text{as } p>1,\ p\neq 2$$

The values on both sides are also not equal so this does not satisfied the parallelogram law. Contradict to our supposition. So norm space is not an inner product space.

MODULE NO.90

➢ THEOREM (CONTINUITY OF INNER PRODUCT)

Theorem:

Let V be any inner product space. For any sequences $\{x_n\}$ and $\{y_n\}$ in V

$$x_n \to x \ , \quad y_n \to y \quad \text{implies} < x_n, y_n > \ \to \ < x, y >$$

Proof:

$$|< x_n, y_n > - < x, y >|$$
$$= |< x_n, y_n > - < x_n, y > + < x_n, y > - < x, y >|$$
$$= |< x_n, y_n - y > + < x_n - x, y >|$$
$$\le |< x_n, y_n - y >| + |< x_n - x, y >|$$

Now from Cauchy Swarzinequality

$$|x, y| \le \|x\|\|y\|$$
$$\le \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\|$$

Given that $x_n \to x$, $y_n \to y$ so,

$$\|y_n - y\| = \|y - y\| = 0 \quad , \quad \|x_n - x\| = \|x - x\| = 0 \quad \text{as } n \to \infty$$

As $n \to \infty$

$$|< x_n, y_n > - < x, y >| \le 0$$
$$< x_n, y_n > \to < x, y > \quad \text{as } n \to \infty$$

Theorem:

If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in V, then the inner product $< x_n, y_n >$ is a Cauchy sequence in F.

Proof:

$\{x_n\}, \{y_n\}$ are Cauchy sequence

To show $< x_n, y_n >$ is also Cauchy Sequence.

$$\Rightarrow \quad \|x_n - x_m\| \to 0 \quad ; \quad \|y_n - y_m\| \to 0, \ m,n \to \infty$$
$$|< x_n, y_n > - < x_m, y_m >| = |< x_n, y_n > - < x_n, y_m > + < x_n, y_m > - < x_m, y_m >|$$
$$= |< x_n, y_n - y_m > + < x_n - x_m, y_m >|$$
$$\le |< x_n, y_n - y_m >| + |< x_n - x_m, y_m >|$$
$$\le \|x_n\|\|y_n - y_m\| + \|x_n - x_m\|\|y_m\|$$
$$\Rightarrow \quad |< x_n, y_n > - < x_m, y_m >| \to 0, \ \text{as} \quad n,m \to \infty$$
$$\Rightarrow \quad < x_n, y_n > \text{is a Cauchy Sequence}$$

MODULE NO.91

Examples of Inner product spaces

➢ SPACE $\mathbb{R}^n$

➢ SPACE $\mathbb{C}^n$

- **SPACE** $\mathbb{C}[a,b]$
- **SPACE** l^n
- **SPACE** P_n (Collection of all polynomials of degree n)

Proof:

1. $\mathbb{R}^n$, the elements are of the form

$$x=(x_1,x_2,.........x_n)\ \ ;\ y=(y_1,y_2,.........y_n)$$

The inner product form is $<x,y>=\sum_{i=1}^{n} x_i y_i$ (Note: check all axiom self-assignment)

The Norm is $\|x\|=\sqrt{<x,x>}=\sqrt{\sum_{i=1}^{n} x_i x_i}=\sqrt{\sum_{i=1}^{n} x_i^{\,2}}$

2. $\mathbb{C}^n$

The elements are $z=(z_1,z_2,........,z_n)\ \ ;\ z'=(z_1',z_2',.........,z_n')$ if conjugate does not define then it does not satisfied the second or third axiom of inner product space.

The inner product form is $<z,z'>=\sum_{i=1}^{n} z_i \overline{z}'_i$ (Note: check all axiom self-assignment)

3. $\mathbb{C}[a,b]$ be the space of all continuous function defined on [a, b].

$$<f,g>=\int_a^b f(t).\overline{g(t)}dt$$ define an inner product on C[a, b]

(Note: complex function can also be including. In previous example the C[a, b] was not inner product space with define function definition).

$$<\bullet,\bullet>:V\times V\to F$$

We will check all four properties of inner product as

i): $<f,f>=0 \quad \Leftrightarrow \quad f=0$

ii): $<f+g,h>=<f,h>+<g,h>$

iii): $<\alpha f,g>=\alpha<f,g>$

iv): $<g,f>=\overline{<f,g>}$

it define inner product and is define inner product space.

4. l^n is a space of sequences.

$$l^2:x\{x_i\}$$

The condition or norm is

$$\sum_{i=1}^{\infty}|x_i|^2<\infty$$

Let defined the inner product of $y=\{y_i\}$ is

$$<x,y>=\sum_{i=1}^{\infty} x_i\overline{y_i}$$

Checd all four axioms as exercise for inner product.

5. P_n

Let P_n be the collection of all polynomial of degree n(or less than n).

We can write this as $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a$ e.g $3x^2 - 2x + 1$ of degree two.

Let $u(x), v(x) \in P_n$

The inner product is

$$< u(x), v(x) >= \int_a^b u(x)v(x)dx \quad , \quad x \in [a,b]$$

with this define P_n is an inner product space.

We have not defined conjugate of v(x) as the interval defined is a real valued so its conjugate is also real valued.

Module No.92

Orthogonal Systems

➢ Pythagorean Theorem

The dot product of two vectors when they are perpendicular is zero. Similarly in inner product if two vectors are perpendicular then their inner product is zero.

Theorem:

In an inner product space V and x, y in V if $x \perp y$ then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

Proof:

$$\|x+y\|^2 =< x+y, x+y >$$

$$=< x,x > + < x,y > + < y,x > + < y,y >$$

As x and y are perpendicular so $< x,y >= 0, < y,x >= 0$

$$\|x+y\|^2 =< x,x > + < y,y >= \|x\|^2 + \|y\|^2$$

Generalized form:

$\{x_1, x_2, ...x_n\}$ be nonzero vectors in V inner product space such that

$< x_i, x_j >= 0 \quad , \quad i \neq j$

This system $\{x_1, x_2, ...x_n\}$ is called orthogonal system as all vectors inside it are perpendicular to each other.

The generalized statement is $\|x_1 + x_2 + ...+ x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + ...+ \|x_n\|^2$

The idea of proof is

$$\left\|\sum_{i=1}^n x_i\right\|^2 = \langle \left\|\sum_{i=1}^n x_i, \sum_{i=1}^n x_i\right\| \rangle$$

$$=< x_1 + ...+ x_n, x_1 + ...+ x_n >$$

$$=< x_1, x_1 + ...+ x_n > + ...+ < x_n, x_1 + ...+ x_n >$$

$$= \sum_{i=1}^n \sum_{j=1}^n < x_i, x_j >$$

$< x_i, x_j >= \|x_i\|^2 \quad$, if $i \neq j \quad < x_i, x_j >= 0$ and for $i = j$ then $< x_i, x_j >= \|x_i\|^2$

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n}\|x_i\|^2$$

Module No.93

Orthogonal Systems

➢ Theorem (Linearly Independence)

Any sequence $\{x_n\}$ of non-zero mutually orthogonal vectors in an inner product space V is linearly independent.

Proof: do it yourself

Let $x = (x_1, x_2, ...x_n)$ be the orthogonal sequence.

Remark:

If $< x, x_1 >= 0$, $\forall$ i=1,2,.......,n $\Rightarrow$ $\left\langle \sum_{i=0}^{n} \alpha_i x_i, x \right\rangle = 0$

$$\left\langle \sum_{i=0}^{n} a_i x_i, x \right\rangle = \left\langle a_1 x_1 + a_2 x_2 + ... + a_n x_n, x \right\rangle = a_1 \left\langle x_1, x \right\rangle + ... + a_n \left\langle x_n, x \right\rangle = 0$$