MTH646 – PARTIAL DIFFERENTIAL EQUATIONS

LECTURE NO. 01

Introduction:

In Mathematics, a partial differential equation is one of the types of differential equations, in which the equation contains unknown multi variables with their partial derivatives. It is a special case of an ordinary differential equation.

Many world related problems in applied sciences, physics and engineering are modeled mathematically with partial differential equations.

In numerical simulation, it is important to find out the exact solutions of partial differential equations but unfortunately we do not have appropriate methods to find the exact analytical solution of many types of partial differential equations. In this situation we utilize different approximations and other techniques to solve the problem numerically. There are several numerical methods are available in literature that help us to understand the mechanism and complexity of the differential problems.

Definition Partial Derivative:

Let u be a function of independent variables x, y, z and t i.e. $u = u(x, y, z, t)$. The partial derivative of u w.r.t x is denoted by $\frac{6u}{6x}$ or u_x and defined as

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y, z, t) - u(x, y, z, t)}{\Delta x}$$

Provided that above limit exists.

Similarly, partial derivative of u w.r.t to y and z can be defined,

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y, z, t) - u(x, y, z, t)}{\Delta y}$$

$$\frac{\partial u}{\partial z} = \lim_{\Delta z \to 0} \frac{u(x, y, z+\Delta z, t) - u(x, y, z, t)}{\Delta z}$$

Example:

Suppose u is a function of more than one variable such that,

$$u(x, y, z) = x\cos z + x^2 y^2 e^z$$

$$\frac{\partial u}{\partial x} = \cos z + 2xy^2 e^z$$

$$\frac{\partial u}{\partial y} = 0 + x^2 2ye^z = 2yx^2 e^z$$

$$\frac{\partial u}{\partial z} = x(-sinz) + x^2 y^2 e^z$$

Partial Differential Equations:

A partial differential equation (PDE) is a relationship between an unknown function of several variables and its partial derivatives. Let $u(x,\ t)$ is unknown function and x and t are independent variables, then we usually write, $u = u(x, t)$ and we say that u is dependent variable.

Examples:

Heat Equation $$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2},$$

Wave Equation $$\frac{\partial^2 u(x,t)}{\partial^2 t} = \frac{\partial^2 u(x,t)}{\partial x^2},$$

Laplace Equation $$\frac{\partial^2 u(x,y)}{\partial x^2} = \frac{\partial^2 u(x,y)}{\partial y^2}$$

PDE involves two or more independent variables. In this example, x and t are independent variables.

Order of PDE:

The order of a PDE is the order of the highest derivative that occurs in the equation.

Examples:

1st order — 1) $\frac{\partial u}{\partial t}-\frac{\partial u}{\partial x}=0,$ $order=$ order PDE

2nd order — 2) $\frac{\partial^2 u}{\partial x^2}=(1+\frac{\partial u}{\partial y})^{1/2}$, $order=$ order PDE

4th order — 3) $\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial t^2}-u=0,$ $order=$ order PDE

Degree of a PDE:

A PDE is the degree of the highest order derivative which occurs in it after the equation has been rationalized. Examples:

1) $Degree = 1$ — 1) $\frac{\partial u}{\partial t}=c(\frac{\partial^2 u}{\partial x^2}),$

2) $Degree = 2$ — 2) $(\frac{\partial^2 u}{\partial x^2})^3=(\frac{\partial u}{\partial x})+u=0,$

3) $\sqrt{1+(\frac{\partial u}{\partial x})^2}=y\frac{\partial^3 u}{\partial x^3},$

Squaring,

$1+(\frac{\partial u}{\partial x})^2=y^2(\frac{\partial^3 u}{\partial x^3})^2,$

$Degree = 2$

Dimension of PDE:

The dimension of a PDE is the number of independent variables taken in space direction in the partial differential equation. Examples:

1) $\frac{\partial u}{\partial t}=c\frac{\partial^2 u}{\partial x^2},$ $\dim=$ or $dim = 1$ or $1D$

2) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,$ $\dim=$ or $dim = 2$ or $2D$

3) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0,$ $\dim=$ or $dim = 3$ or $3D$

Linear and non-linear partial differential equation:

If the dependent variable and all its partial derivatives occur linearly in any PDE then such an equation is called linear PDE otherwise a nonlinear PDE.

Examples:

1) $(x^2+y^2)\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x\partial y}-3u=0$

2) $ux\frac{\partial^2 u}{\partial x^2}+u^2xy\frac{\partial^2 u}{\partial x\partial y}+uy\frac{\partial^2 u}{\partial y^2}+(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2+u^3=0$

3) $\frac{\partial^2 u}{\partial x^2}+(\frac{\partial^2 u}{\partial x\partial y})^2+\frac{\partial^2 u}{\partial y^2}=x^2+y^2$

Eq. (1) is linear but Eqs. (2) and (3) are non-linear partial differential equations.

LECTURE NO. 02

Homogeneous and non-homogeneous PDEs:

If all the terms of a PDE contain the dependent variable or its partial derivatives then such a PDE is called homogeneous or non-homogeneous otherwise. Examples:

1) $(x^2+y^2)\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x\partial y}-3u=0$

2) $ux\frac{\partial^2 u}{\partial x^2}+u^2xy\frac{\partial^2 u}{\partial x\partial y}+uy\frac{\partial^2 u}{\partial y^2}+(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2+u^3=0$

3) $\frac{\partial^2 u}{\partial x^2}+(\frac{\partial^2 u}{\partial x\partial y})^2+\frac{\partial^2 u}{\partial y^2}=x^2+y^2$

Eqs. (1) and (2) are homogeneous while Eq.(3) is non-homogeneous partial differential equations.

General Solutions of PDE:

A solution of a PDE is any function which satisfies the equation. A general solution of a PDE is a solution which contains the number of arbitrary independent functions equal to the order of the equation.

Example: The solution of the second order PDE

$$\frac{\partial^2 u}{\partial x \partial x} = 2x - y \text{ is}$$
$$u = x^2 y + \frac{1}{2} xy^2 + F(x) + G(y)$$

Here $F(x)$ and $G(y)$ are arbitrary independent functions, it is a general solution.

Particular Solution of PDE:

A particular solution is one which can be obtained from general solution by particular choice of arbitrary function. Example:

$$\text{In particular, } F(x) = 2\sin x,\ G(x) = 3y^4 - 5, \text{ then we have}$$
$$u = x^2 y + \frac{1}{2} xy^2 + F(x) + G(x)$$
$$u = x^2 y + \frac{1}{2} xy^2 + 2\sin x + 3y^4 - 5$$
$$\text{Which is particular solution of the PDE.}$$

Auxiliary Conditions:

The PDEs that represent physical systems usually have infinite number of solutions. For example:

$$\text{The functions } u = x^2 - y^2,\ u = e^x \cos y,\ u = \log(x^2 + y^2),$$
$$u = \sin x . \sinh y, \ldots$$
$$\text{are entirely different from each other are solutions of PDE}$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

To obtain a unique (i.e. single) solution of the PDE corresponding to a given physical problem, one must use the additional information (i.e. auxiliary condition) arising from the physical situation. They fall in two categories:

1) Boundary conditions
2) Initial conditions

Boundary Conditions:

These are the conditions that must be satisfied at the points on the boundary S of the region R in which the partial differential equation hold. (or a condition that is required to be satisfied at all or part of the boundary S of a region R in which a set of differential conditions is to be solved).These are three types of boundary condition;

- Dirichlet condition: $u(x, y) = g(x, y) \text{ on } S$
- Neumann (or flax) condition: $\frac{\partial u(x, y)}{\partial \bar{n}} = g(x, y) \text{ on } S$

Where $\frac{6u}{6n}$ normal derivative i.e. a directional derivative taken in the direction normal to some surface.

- Cauchy (or mixed) condition: $\alpha(x, y)u(x, y) + \beta(x, y)\frac{\partial u(x, y)}{\partial \bar{n}} = g(x, y) \text{ on } S$

Initial Conditions:

These are the conditions that must be satisfied throughout the region R at the instant when consideration of the physical system begins. When time t is one of the independent variable and we specify a condition at $t = 0$, we refer to it an initial condition.

A typical initial condition is said to be of Cauchy type if the values of both u and $\frac{6u}{6t}$ on the boundary at $t = 0$ i.e. the initial values of u and $\frac{6u}{6t}$ are given.

The physical problems associated with PDE may be classified as:

- Boundary-valued problems (B.V.P)
- Initial-valued problems (I.V.P)
- Initial-Boundary-valued problems (I.B.V.P)

Classification of second order Linear PDEs:

Generally, the PDE for a dependent u and the independent variables x and t, are described in the following form,

$$F(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial x\partial t},\frac{\partial^2 u}{\partial t^2},\dots)=0$$

The most common form of second-order linear PDE in two independent variables x and t is given by,

$$A\frac{\partial^2 u(x,t)}{\partial x^2}+B\frac{\partial^2 u(x,t)}{\partial x\partial y}+C\frac{\partial^2 u(x,t)}{\partial y^2}+D\frac{\partial u(x,t)}{\partial x}+E\frac{\partial u(x,t)}{\partial y}+Fu(x,t)=G$$

Where A, B, C, D, E, F and G are functions of x and t.

Rewrite the above equation in the following form,

$$A(x,t)\frac{\partial^2 u}{\partial x^2}+B(x,t)\frac{\partial^2 u}{\partial x\partial y}+C(x,t)\frac{\partial^2 u}{\partial y^2}=\Phi(x,y,u,\frac{\partial}{\partial x},\frac{\partial}{\partial t})$$

By utilizing the short notations for partial derivatives, we have the following,

$$A(x,t)u_{xx}+B(x,t)u_{xy}+A(x,t)u_{yy}=\Phi(x,y,u,u_x,u_y)$$

The classification of second order PDEs are based on discriminant Δ by reducing the above equation by coordinate transformation to standard format a point (x_o, t_o)

$$\Delta(x_0,t_0)=B^2(x_0,t_0)-4A(x_0,t_0)C(x_0,t_0)\dots \qquad Eq(1)$$

$If\ \Delta(x_0,t_0)>0,$ Then PDE is called hyperbolic, for example, wave equation.

$If\ \Delta(x_0,t_0)=0,$ Then PDE is called parabolic, for example, heat equation.

$If\ \Delta(x_0,t_0)<0,$ Then PDE is called elliptical, for example, Laplace equation.

Examples:

Classify the following PDEs,

$$1)\ \frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+4\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial x}+2\frac{\partial u}{\partial y}=0$$

Solution:

Compare the above equation with Eq.(1), We have

$A=1, B=4, C=4,$ *therefore,*

$B^2-4AC=(4)^2-4\times1\times4=0,$ *So the equation is parabolic.*

$$2)\ (1+x^2)\frac{\partial^2 u}{\partial x^2}+(5+2x)\frac{\partial^2 u}{\partial x\partial t}+(4+x^2)\frac{\partial^2 u}{\partial t^2}=0$$

Solution: Here $A=(1+x^2), B=(5+2x^2), C=(4+x^2),$ *therefore,*

$B^2-4AC=(5+2x)^2-4(1+x^2)(4+x^2)=9>0,$

So the equation is hyperbolic.

$$3)\ x^2\frac{\partial^2 u}{\partial x^2}+(1-y^2)\frac{\partial^2 u}{\partial y^2}=0, -\infty<x<\infty, -1<y<1$$

Solution:

Here $A=x^2, B=0, C=(1-y^2),$

$B^2-4AC=0-4\times x^2\times(1-y^2)=4x^2(y^2-1)$

Since for all x between $-\infty$ *and* $+\infty$, x^2 *is positive.*

Also for all y between -1 *and* $+1$, $y^2<1.$

Therefore, $B^2-4AC<0.$

So the equation is elliptic.

LECTURE NO. 03

Finite Difference Method (FDM):

Finite difference method (FDM) is utmost common, efficient, frequent and universally applicable method for the solution of various types of PDEs. The numerical solutions obtained from FDM are actually the values of discrete points in the solution domain which we are called them grid points as shown in Figure 1.

We usually prefer the space between the grid points in both x and y directions should be uniform and grid spacing between the points in x-direction is denoted by Δx or h_x.

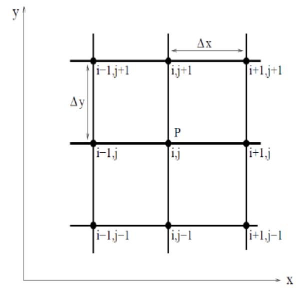

Figure 1: Discretization of discrete grid points

Likewise space between the grid points in y-direction is denoted by Δy or h_y.

One can also be utilized the unequal (non-uniform) grid spacing in both coordinate directions but the difference between successive pairs of grid points in each direction should be the same.

If (i, j) represents the coordinates of the grid point P in solution domain as shown in the Figure. Then the grid point $(\mathrm{i}+1, \mathrm{j})$ will show its position is immediately to the right of the grid point (i, j) in positive x direction and likewise the grid point $(\mathrm{i}-1, \mathrm{j})$ will show its position is immediately to the left of the point (i, j) in negative x-direction. Similarly, the grid point $(\mathrm{i}, \mathrm{j}+1)$ will move immediately one step up in positive y-direction and grid point $(\mathrm{i}, \mathrm{j}-1)$ will move immediately one step down in negative y-direction.

Finite difference approximation techniques are basically applied on as an alternative source of the derivatives to find out the approximate solution by converting the desire research problem in the form of PDEs into the easily solvable algebraic difference equations.

Taylor Series Expansion Applied to Finite Difference Method:

The partial derivatives in PDEs are replaced by the finite difference approximations at each grid point which are approximated by then neighboring values utilizing the Taylor's series expansion. The general interpretation of Taylor's series expansion says that if we know the value of a function and its derivatives at some particular point, say $(x_{\mathrm{i}}, y_{\mathrm{j}}, t_k)$ then we can easily find the value s of function at its nearby points

$$(x_i+h_x, y_j, t_k) \; and \; (x_i-h_x, y_j, t_k).$$

By Taylor's series expansions about the point $(x_{\mathrm{i}}, y_{\mathrm{j}}, t_k)$, the exact expression for

$$f(x_i+h_x, y_j, t_k) \; and \;\; f(x_i-h_x, y_j, t_k).$$

Is given as

$$f(x_i+h_x,y_j,t_k) = f(x_i,y_j,t_k) + \frac{(h_x)}{1!} f_x(x_i,y_j,t_k) + \frac{(h_x)^2}{2!} f_{xx}(x_i,y_j,t_k) + \frac{(h_x)^3}{3!} f_{xxx}(x_i,y_j,t_k) + \ldots,$$

And

$$f(x_i-h_x,y_j,t_k) = f(x_i,y_j,t_k) - \frac{(h_x)}{1!} f_x(x_i,y_j,t_k) + \frac{(h_x)^2}{2!} f_{xx}(x_i,y_j,t_k) - \frac{(h_x)^3}{3!} f_{xxx}(x_i,y_j,t_k) + \cdots.$$

In particular, h_x if is very small, then $(h_x)^3$ and its higher power can be neglected.

Rewrite the above two questions as;

$$f(x_i+h_x,y_j,t_k) \approx f(x_i,y_j,t_k) + \frac{(h_x)}{1!} f_x(x_i,y_j,t_k) + \frac{(h_x)^2}{2!} f_{xx}(x_i,y_j,t_k),$$

And

$$f(x_i-h_x,y_j,t_k) \approx f(x_i,y_j,t_k) - \frac{(h_x)}{1!} f_x(x_i,y_j,t_k) + \frac{(h_x)^2}{2!} f_{xx}(x_i,y_j,t_k).$$

The above two equations are second order accurate. If terms of order $(h_x)^2$ and higher order terms are neglected, than the above terms reduced to the following expressions;

$$f(x_i+h_x,y_j,t_k) \approx f(x_i,y_j,t_k) + \frac{h_x}{1!} f_x(x_i,y_j,t_k),$$

And

$$f(x_i-h_x,y_j,t_k) \approx f(x_i,y_j,t_k) - \frac{h_x}{1!} f_x(x_i,y_j,t_k).$$

The above two equations are first order accurate. The truncation error is the amount of quantity by which the solution of a PDE fails to satisfy the approximate solution at some grid point. The truncation error can be reduced by retaining more terms in the Taylor's series expansion.

LECTURE NO. 04

Simple Finite Difference Approximation to a Derivative:

We will derive some simple finite difference approximation to first and second order derivatives in both x and y directions. Solve equations form previous lecture for $f_x(x_i, y_j, z_k)$ we have,

$$f_x(x_i, y_j, t_k) = \frac{f(x_i + h_x, y_j, t_k) - f_x(x_i, y_j, t_k)}{h_x} + O(h_x) \quad \text{... (1) (FWD)}$$

And

$$f_x(x_i, y_j, t_k) = \frac{f(x_i, y_j, t_k) - f(x_i - h_x, y_j, t_k)}{h_x} + O(h_x) \quad \text{... (2) (BWD)}$$

Where $o(h_x)$ is the 'terms of order h_x indicating the order of magnitude of the truncation error. Two above equations are identified as first order forward and backward difference.

Now subtracting eq. 2 from eq. 1, we have the following expression,

$$f(x_i + h_x, y_j, t_k) - f(x_i - h_x, y_j, t_k) = 2(h_x) f_x(x_i, y_j, t_k) + \frac{(h_x)^3}{3} f_{xxx}(x_i, y_j, t_k) + \dots$$

Solving for $f_x(x_i, y_j, z_k)$

$$f_x(x_i, y_j, t_k) = \frac{f(x_i + h_x, y_j, t_k) - f(x_i - h_x, y_j, t_k)}{2(h_x)} + O(h_x)^2$$

The above expression is the second order central difference approximation. In order to obtain a finite difference for the second order partial derivative add Eqs 1 and 2, we get

$$f(x_i + h_x, y_j, t_k) + f(x_i - h_x, y_j, t_k) = 2f(x_i, y_j, t_k) + \frac{(h_x)^2}{1} f_{xx}(x_i, y_j, t_k) + \dots$$

Solving for $f_{xx}(x_i, y_j, z_k)$

$$f_{xx}(x_i, y_j, t_k) = \frac{f(x_i + h_x, y_j, t_k) - 2f(x_i, y_j, t_k) + f(x_i - h_x, y_j, t_k)}{(h_x)^2} + O(h_x)^2$$

The above equation is second order central difference form for the derivative $f_{xx}(x_i, y_j, z_k)$ at some point (x_i, y_j, z_k). Finite difference approximations for the y-derivatives are obtained in exactly the similar way as the results are analogous to the expressions for the x-derivatives.

$$f_y(x_i, y_j, t_k) = \frac{f(x_i, y_j + h_y, t_k) - f(x_i, y_j, t_k)}{h_y} + O(h_y) \quad \text{Forward Difference}$$

$$f_y(x_i, y_j, t_k) = \frac{f(x_i, y_j, t_k) - f(x_i, y_j - h_y, t_k)}{h_y} + O(h_y) \quad \text{Backward Difference}$$

$$f_y(x_i, y_j, t_k) = \frac{f(x_i, y_j + h_y, t_k) - f(x_i, y_j - h_y, t_k)}{2(h_y)} + O(h_y)^2 \quad \text{Central Difference of first derivative}$$

$$f_{yy}(x_i, y_j, t_k) = \frac{f(x_i, y_j + h_y, t_k) - 2f(x_i, y_j, t_k) + f(x_i, y_j - h_y, t_k)}{(h_y)^2} + O(h_y)^2 \quad \text{Central Difference of second order}$$

The second order central difference can also be obtained by utilizing the both forward and backward differences as follows:

$$f_{xx}(x_i, y_j, t_k) = f_x(f_x(x_i, y_j, t_k)) = \frac{f_x(x_i + h_x, y_j, t_k) - f_x(x_i, y_j, t_k)}{h_x}$$

$$f_{xx}(x_i, y_j, t_k) = [(\frac{f((x_i + h_x, y_j, t_k) - f((x_i, y_j, t_k))}{(h_x)}) - (\frac{f((x_i, y_j, t_k) - f((x_i - h_x, y_j, t_k))}{(h_x)})]\frac{1}{(h_x)}.$$

Therefore

$$f_{xx}(x_i, y_j, t_k) \simeq \frac{f(x_i + h_x, y_j, t_k) - 2f(x_i, y_j, t_k) + f(x_i - h_x, y_j, t_k)}{(h_x)^2}.$$

By using this technique we can also generate a formula for difference approximation for the mixed partial derivatives $f_{xx}(x_i, y_j, z_k)$ at some point (x_i, y_j, z_k). To find difference approximation for $f_{xy}(x_i, y_j, z_k)$, first we apply x-derivative as central difference and after that y-derivative as central difference as follows

$$f_{xy}(x_i,y_j,t_k) = f_x(f_y(x_i,y_j,t_k)) = \frac{f_y(x_i+h_x,y_j,t_k) - f_y(x_i-h_x,y_j,t_k)}{2(h_x)}$$

$$= [(\frac{f(x_i+h_x,y_j+h_y,t_k) - f(x_i+h_x,y_j-h_y,t_k)}{4(h_x)(h_y)}) - (\frac{f(x_i-h_x,y_j+h_y,t_k) - f(x_i-h_x,y_j-h_y,t_k)}{4(h_x)(h_y)})].$$

Therefore

$$f_{xy}(x_i,y_j,t_k) = \frac{1}{4(h_x)(h_y)}(f(x_i+h_x,y_j+h_y,t_k) - f(x_i+h_x,y_j-h_y,t_k)$$

$$- f(x_i-h_x,y_j+h_y,t_k) + f(x_i-h_x,y_j-h_y,t_k)] + O[(h_x)^2,(h_y)^2].$$

To determine such type of finite difference approximation for the mixed partial derivatives are highly effective in solving the many types of PDEs in which mixed partial derivatives are involved. The other difference approximations for higher order derivative as well as for mixed partial order derivatives can also be derived by utilizing the same procedure.

LECTURE NO. 05

Stability of FDM:

The stability of finite difference is connected to the deterioration or growth of the error with the passage of time throughout any phase of computing. The schemes will be stable if the computing error does not rise with time.

To find the out the approximation solution of the research problem that governing the PDEs to the exact analytical solution it is necessary that our computational domain should be discretized to a finite number of grid points (x_i, y_j, t_k), where h_x and h_y are the constant gird spacing in x and y directions and Δt is the time spacing between the time levels.

Suppose that $u_{i,j}^k$ is the approximate solution of the finite difference applied to PDE and exact analytical solution of PDE is donated by $U_{i,j}^k$, and then $e_{i,j}^k$ is defined by the following relation

$$e_{i,j}^k = |u_{i,j}^k - U_{i,j}^k|$$

The choice of Δt is very important while we are studying the stability of the finite difference method. The finite difference scheme applied to PDE can produce acceptable results that grow boundedly for the choice of small Δt as compared to the results that grow unboundedly. If too large Δt is selected, in this case, scheme is said to be unstable.

A finite difference scheme is said to be stable if error do not grown unboundedly with the passage of time on each time level k

$$|e^{k+1}| \leq |e^k|, \qquad k \in N$$

The two most commonly used techniques for analyzing the stability of the method are the matrix and the Fourier Stability analysis.

Matrix Stability Analysis:

The matrix stability analysis applied to finite difference scheme on each grid point of the computational domain will result of the following two time levels system of linear equation

$$A\underline{u}^{k+1} = B\underline{u}^k \qquad \text{Where}$$

$$u^{k+1} = [u_2^{k+1}, u_2^{k+1}, \dots, u_{(N-1)}^{k+1}]^T \quad \text{And} \quad u^k = [u_1^k, u_1^k, \dots, u_{(N-1)}^k]^T$$

Are the solution vectors at $(k+1)$ and k time level respectively while A and B are the $(N-1) \times (N-1)$ matrices of known values. If $\underline{e}$ is the error vector of eq. 1 then it must satisfy the equation, therefore

$$A\underline{e}^{k+1} = B\underline{e}^k \quad \text{Or} \quad |\underline{e}^{k+1}| = |A^{-1}B||\underline{e}^k|$$

Therefore, the finite difference scheme described in eq. 1 is stable if $|A^{-1}B| \leq 1$. The condition described in eq. 2 is sufficient condition for a finite difference scheme to be stable.

Fourier (Von Neumann) Stability Analysis:

This model is applicable to the linear finite difference PDEs and when spatial domain is periodic. To test whether finite difference scheme is stable, then it is sufficient to look its round-off errors or simple say 'error' that should not amplified in the calculation with time. This is the numerical error introduced for a repetitive number of calculations in which the computer is constantly rounding the number to some decimal points. Round-off error $\varepsilon^k_{i,j}(x, y)$ is actually the difference between the numerical solutions from a real computer with finite accuracy, we donate it by $u^k_{i,j}$ and for exact solution of the finite-difference equation we donate it by $U^k_{i,j}$.

$$\varepsilon^k_{i,j}(x,y) = U^k_{i,j}(x,y) - u^k_{i,j}(x,y) \qquad \text{... (3)}$$

The eq. 3 satisfy the finite difference approximation equation and round-off error can be expressed in the Fourier series expansion as

$$\varepsilon^k(x,y) = \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \lambda^k(l_1, l_2) e^{\frac{2\pi\omega}{L(l_1x+l_2y)}}$$

$$Where \quad \lambda^k(l_1, l_2) = \frac{1}{L^2}\int_0^L \int_0^L \varepsilon^k(x, y) e^{-2\pi\omega/L(l_1x+l_2y)} dxdy$$

And $\omega = \sqrt{-1}$ and L is the interval of the function. Suppose that the solution error has the following form,

$$\varepsilon^k_{i,j} = \lambda^k e^{2\pi\omega/L(l_1x+l_2y)}$$

Define:
$$G(l_1, l_2) = \frac{\lambda^{k+1}(l_1, l_2)}{\lambda^k(l_1, l_2)}$$

Where $G(l_1, l_2)$ is the amplification factor and $\lambda(l_1, l_2)$ is the Fourier component. The finite difference scheme will be stable if every Fourier component is stable.

The Von Neumann stability is given by

$$|G(l_1, l_2)| \le 1 \quad \text{... (4)}$$

If the finite difference scheme satisfied the condition defined in eq. 4 then amplification factor $G(l_1, l_2)$ will not grow as we march forward in time t. It has been observed that more and more time steps (Δt) are required for calculation over a given interval of time. When we utilize explicit finite difference scheme in which we are forced to choose Δt that should be less than to a specific limit imposed by stability constrains. Whereas in case of implicit and Crank-Nicolson finite difference schemes, fewer time steps (Δt) are required for our calculation over a given interval of time in which we are free to choose even larger values of time steps Δt.

LECTURE NO. 06

Consistency and Convergence of FDM:

If the magnitude of truncation error approaches zero as the grid sizes h_x and h_y in both directions of coordinate axis along with Δt approaches zero, then the approximate solution of the PDE is said to be consistent with the exact numerical solution. In other words, truncation error vanishes as we utilize small mesh sizes and time steps that tend to zero i.e. h_x, h_y and $\Delta t \to 0$.

Note that consistency is a necessary condition but not a sufficient condition for convergence. A finite difference scheme is said to be convergent if approximate solution, $u^k_{i,j}$ of PDE approaches to zero, as grid sizes h_x and h_y as well as Δt approaches zero.

$$\lim_{h_x, h_y, \Delta t \to 0} |U^k_{i,j} - u^k_{i,j}| = 0$$

The stability and consistency of linear PDEs with constant coefficients implies convergence.

Explicit, Implicit and Crank-Nicolson Schemes:

A finite difference scheme is said to be fully explicit scheme if we can find the value of the function at the next time level on each grid points of the computational domain with the help of an explicit formula which contains gird point values in the previous time level. The fully explicit scheme leads us to impose the restriction on choosing the maximum acceptable time steps Δt for stability. So to attain the stability of the finite difference scheme, we have to utilize so many time steps if we are going to choose Δt such that it approaches to zero.

On the other hand, in fully implicit schemes there are no such explicit formula exists to find the values at the grid points directly from the previous time level because the values on each the grid points are scattered between two time levels on both sides of the difference equation. The resultant finite difference equation generates system of linear algebraic equation that can be solved by utilizing matrices. In fully implicit schemes, there is no restriction on choosing the maximum acceptable time steps Δt for stability such as fully explicit scheme. Therefore, fewer number of time steps can be utilized with a large time steps Δt for stability.

Mathematically, to understand the concept of fully explicit and implicit schemes, we need to consider the following 2D head conduction PDE at some grid point (x_i, y_j, t_k), of the solution domain.

$$\frac{\partial u(x_i, y_j, t_k)}{\partial t} = \alpha\left[\frac{\partial^2 u(x_i, y_j, t_k)}{\partial x^2} + \frac{\partial^2 u(x_i, y_j, t_k)}{\partial y^2}\right] \quad \text{Or} \quad \frac{\partial u^k_{i,j}}{\partial t} = \alpha\left[\frac{\partial^2 u^k_{i,j}}{\partial x^2} + \frac{\partial^2 u^k_{i,j}}{\partial y^2}\right] \quad \dots (1)$$

Where α is the thermal diffusivity and dependent variable $u(x_i, y_j, t_k)$ is a temperature function of space (x_i, y_j) and time (t_k). Replace the space derivatives by the second order central difference at time level k and time derivative by first order forward difference in eq. and writing the expression $u(x_i, y_j, t_k)$ as $u^k_{i,j}$, we obtain the following expression

$$\frac{u^{k+1}_{i,j} - u^k_{i,j}}{\Delta t} = \alpha\left[\frac{u^k_{i+1,j} - 2u^k_{i,j} + u^k_{i-1,j}}{(\Delta x)^2} + \frac{u^k_{i,j+1} - 2u^k_{i,j} + u^k_{i,j-1}}{(\Delta y)^2}\right] \quad \dots (2)$$

The eq. 2 contains only unknown dependent variable $u^{k+1}_{i,j}$ at time $(k+1)$ that can explicitly be solved from the unknown values of $u^k_{i,j}$, $u^k_{i-1,j}$, $u^k_{i+1,j}$, $u^k_{i,j-1}$ and $u^k_{i,j+1}$ at time k. This is a typical example of fully explicit infinite difference method. Now replace the space derivatives by the second order central difference at time level $(k+1)$ and time derivative by the first order forward difference in equation.

We obtain form equation 1:

$$\frac{\partial u^{k+1}_{i,j}}{\partial t} = \alpha\left[\frac{\partial^2 u^{k+1}_{i,j}}{\partial x^2} + \frac{\partial^2 u^{k+1}_{i,j}}{\partial y^2}\right] \quad \dots (3)$$

Similarly form equation 2:

$$\frac{u^{k+1}_{i,j} - u^k_{i,j}}{\Delta t} = \alpha\left[\frac{u^{k+1}_{i+1,j} - 2u^{k+1}_{i,j} + u^{k+1}_{i-1,j}}{(\Delta x)^2} + \frac{u^{k+1}_{i,j+1} - 2u^{k+1}_{i,j} + u^{k+1}_{i,j-1}}{(\Delta y)^2}\right] \quad \dots (4)$$

The schemes defined in eq. 4s is called fully implicit scheme. In contrast to the fully explicit scheme, the temperature variable $u^{k+1}_{i,j}$ cannot be solved purely in terms of function values at time step k. By replacing the space derivative on the right side of the eq. 1 by average between two times levels k and$(k+1)$, we meet with the following expression of the form

$$\frac{u^{k+1}_{i,j} - u^k_{i,j}}{\Delta t} = \frac{\alpha}{2}\Big[\Big(\frac{u^{k+1}_{i+1,j} - 2u^{k+1}_{i,j} + u^{k+1}_{i-1,j}}{(\Delta x)^2} + \frac{u^k_{i+1,j} - 2u^k_{i,j} + u^k_{i-1,j}}{(\Delta x)^2}\Big) + \Big(\frac{u^{k+1}_{i,j+1} - 2u^{k+1}_{i,j} + u^{k+1}_{i,j-1}}{(\Delta y)^2} + \frac{u^k_{i,j+1} - 2u^k_{i,j} + u^k_{i,j-1}}{(\Delta y)^2}\Big)\Big]. \quad \dots (5)$$

The eq. 5 the unknown dependent temperature variable $u^{k+1}_{i,j}$ at time level $k+1$ cannot explicitly be evaluated or expressed in term of the known values at time level k. The unknown quantity $u^{k+1}_{i,j}$ can also be not solvable for some particular grid point $u(x_i, y_j, t_k)$. Therefore eq. 5 can only be solved over all the grid points of the computational domain which will be the result of system of large simultaneous linear equations that can be solved with the help of matrices.

The expression in eq. 5 is typically example of implicit scheme and this is known as implicit Crank-Nicolson scheme. In the above section, we'll briefly summaries the advantages and disadvantages of fully explicit, implicit and Crank-Nicolson finite difference schemes.

LECTURE NO. 07

Fully Explicit Scheme:

- The unknown value of function in fully explicit finite difference scheme can be expressed in terms of known values of functions which can be directly be evaluated.
- A fully explicit finite difference method give values of unknown function at the next time level on each gird points of the computational domain.
- In fully explicit finite difference method, the maximum acceptable time step is restricted by the stability constrains.
- Fully explicit finite difference method is often difficult to perform the stability analysis.

Fully Implicit Scheme:

- The known and unknown value of function on each grid points of fully implicit finite difference scheme are scattered between two time levels on both side of the difference equation.
- The values of the unknown function can be determined on the all gird points of computational domain at a time.
- There is no limitation of choosing the maximum size of time step to attain the stability.
- Since a fully implicit finite difference schemes solves system of linear algebraic equations therefore it suffers a large computational effort on each time level.

Crank-Nicolson Scheme:

- In Crank-Nicolson Scheme we need to solve the coupled linear system at (k) and $(k+1)$ time level separately on both side of the equation.
- Since Crank-Nicolson Scheme combines the fully implicit and explicit schemes. Therefore, spatial and time derivative are both countered around $(k+1/2)$.
- There is no limitation of choosing the maximum size of time step to attain the stability like full implicit scheme.
- Crank-Nicolson Scheme has unconditionality stability and second order accuracy in both time and space.

Iterative Methods:

To solve the system of linear equations which are in the form of sparse matrices, the iterative methods are very efficient. In all types of iterative methods we first need the initial guess to start the iterative process and this process continuously repeated until satisfactory converged solutions are achieved by applying a certain predefined convergence criteria. The system of linear equations can be represented by the following equation

$$Ax = b \qquad \text{...... (1)}$$

Where $'A'$ is a non-singular co-efficient matrix and $'b'$ refers to the known column vector and $'x'$ is the column vector that to be determined. The co-efficient matrix A in equation 1 can be partitioned as follows,

$$A = D + L + U$$

Where matrix $'D'$ is refer to the diagonal matrix, $'L \text{ and } U'$ are lower and upper triangular elements of matrix $'A'$ respectively.

Iterative Methods:

Some well-known methods to solve iterative problems are;

1. Jacobi's Method
2. Gauss-Seidel Method
3. Relaxation Method
   i) SOR (Successive Over Relaxation Method) ii) SUR (Successive Under Relaxation Method)

Jacobi's Method: Consider

$$\begin{array}{ccccccccc} a_{11}x_1^{(k+1)} & + & a_{12}x_2^{(k)} & + & \cdots & + & a_{1n}x_n^{(k)} & = & b_1 \\ a_{21}x_1^{(k)} & + & a_{22}x_2^{(k+1)} & + & \cdots & + & a_{2n}x_n^{(k)} & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{n1}x_1^{(k)} & + & a_{n2}x_2^{(k)} & + & \cdots & + & a_{nn}x_n^{(k+1)} & = & b_n \end{array}$$

This system can also be written as

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1\,n} \\ a_{n1} & \cdots & a_{n\,n-1} & 0 \end{bmatrix}\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

$$D=\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix},\ L=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n1} & \cdots & a_{nn-1} & 0 \end{bmatrix} \text{ and } U=\begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1\,n} \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

Then Jacobi's method can be written in matrix-vector notation as

$$D\boldsymbol{x}^{(k+1)} + (L+U)\boldsymbol{x}^{(k)} = \boldsymbol{b}$$

$$\boldsymbol{x}^{(k+1)} = D^{-1}[(-L-U)\boldsymbol{x}^{(k)} + \boldsymbol{b}].$$

Gauss-Seidel Method: Consider

$$\begin{array}{ccccccccc} a_{11}x_1^{(k+1)} & + & a_{12}x_2^{(k)} & + & \cdots & + & a_{1n}x_n^{(k)} & = & b_1 \\ a_{21}x_1^{(k+1)} & + & a_{22}x_2^{(k+1)} & + & \cdots & + & a_{2n}x_n^{(k)} & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{n1}x_1^{(k+1)} & + & a_{n2}x_2^{(k+1)} & + & \cdots & + & a_{nn}x_n^{(k+1)} & = & b_n \end{array}$$

This system can also be written as

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n1} & \cdots & a_{n\,n-1} & a_{nn} \end{bmatrix}\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1\,n} \\ 0 & \cdots & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

That is

$$(L+D)\boldsymbol{x}^{(k+1)} + U\boldsymbol{x}^{(k)} = \boldsymbol{b},$$

$$\boldsymbol{x}^{(k+1)} = (L+D)^{-1}[-U\boldsymbol{x}^{(k)} + \boldsymbol{b}].$$

Relaxation Method:

Consider recurrence relation

$$(D + \omega L)x_{n+1} = -((1-\omega)L + U)x_{n+1} + b \quad \text{...... (2)}$$

Here ω is defined in the equation 2 as the acceleration parameter which is used to accelerate convergence rate. When $\omega = 0$ is chosen, we come across with Jacobi iterative method and when $\omega = 1$ is selected, Gauss-Seidel iterative method is obtained. The value of acceleration parameter ω lies between 0 and 2, if we want to select value of $\omega = 0.5$ (say) i.e. the values between 0 and 1, then the method will between Jacobi and Gauss-Seidel. We could even select the value of $\omega > 1$, resulting that we are using a method beyond the Gauss-Seidel method (known as SOR – Successive Over Relaxation Method).

Convergence of Iterative Method:

If we apply the standard form of the iterative scheme, we have the following expression,

$$x_{n+1} = Px_n + q \qquad \text{...... (1)}$$

Here n represents the number of iterations. Suppose that iterative scheme described in equation 1 has the exact solution $x = r$, then we have the following equation,

$$r = Pr + q \qquad \text{...... (2)}$$

Subtract eq. 2 from eq. 1 we have

$$x_{n+1} - r = P(x_n - r) \qquad \text{..... (3)}$$

Or equivalently

$$e_{n+1} = Pe_n \qquad \text{...... (4)}$$

Where $e_n = x_n - r$, is defined to donate the error on the nth iteration e_n w.r.to the exact solution. For convergence, we need to check the magnitude of the error vector e (can be measured by some vector norm) that approaches to zero as the number of iteration n increases to infinity (i.e. $\|e_n\| \to 0 \ as \ n \to \infty$ which in turn implies that $x_n \to 0 \ as \ n \to \infty$). Apply the vector norm on both sides of the equation 4,

$$\|e_{n+1}\| = \|Pe_n\|$$

Apply the compatibility inequality we obtained,

$$\|e_{n+1}\| \le \|P\| \, \|e_n\|$$

The iterative scheme will converge if the iteration matrix $'P'$ satisfies the following property,

$$\|P\| \le 1 \qquad \text{...... (5)}$$

If the inequality described in equation 5 holds, then

$$\|e_{n+1}\| \le \|e_n\| \qquad \text{Or} \qquad \frac{\|e_{n+1}\|}{\|e_n\|} \le 1 \qquad \forall n.$$

LECTURE NO. 08

Some Useful Approximations to a Derivative:

The following are the useful approximations to a derivative that should be kept in mind for the solutions of various types of PDEs.

1. $\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h} + o(h)$ Forward Difference
2. $\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{h} + o(h)$ Backward Difference
3. $\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{h} + o(h)$ Centre Difference of 1st order derivative
4. $\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + o(h^2)$ Centre Difference of 2nd order derivative

Similarly approximations can be defined for "$time \ t$" as follows;

5. $\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{k} + o(k)$ Forward Difference
6. $\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j} - u_{i,j-1}}{k} + o(k)$ Backward Difference
7. $\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{k} + o(k)$ Centre Difference of 1st order derivative
8. $\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} + o(k^2)$ Centre Difference of 2nd order derivative

Example: 1

Find the numerical solution of following heat equation by forward difference method (Explicit Method) by taking $h = 0.25$;

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{...... 1D Heat equation}$$

Subjected boundary conditions

$$\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}$$

Similarly

$$u(x,0) = \sin\pi x$$

The exact solution is given by the equation,

$$u(x,t) = e^{-\alpha^2\pi^2 t}\sin\pi x$$

Solution: Use the forward difference approximation for $\frac{\partial u}{\partial t}$ and central difference approximation for $\frac{\partial^2 u}{\partial x^2}$ on given equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \alpha^2 \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right)$$

$$u_{i,j+1} - u_{i,j} = \frac{\alpha^2 k}{h^2}\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right)$$

$$u_{i,j+1} = u_{i,j} + \left(\frac{\alpha^2 k}{h^2}\right)(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

Let $\lambda = \frac{\alpha^2 k}{h^2}$, so by solving we have

$$u_{i,j+1} = \lambda u_{i-1,j} + (1 - 2\lambda)u_{i,j} + \lambda u_{i+1,j} \qquad \text{...... (i)}$$

Since $\lambda = \frac{\alpha^2 k}{h^2}$

Let $\alpha^2 = 1$ and given $h = 0.25$, also suppose $k = 0.0625$. By putting values,

$$\lambda = \frac{1(0.0625)}{(0.25)^2} = 1$$

So above equation (i) becomes by putting value of λ (and by replacing $\mu \; by \; \omega$)

$$\omega_{i,j+1} = \omega_{i-1,j} - \omega_{i,j} + \omega_{i+1,j} \qquad \text{...... (1)}$$

Boundary conditions and initial conditions becomes accordingly (by replacing $\mu \; by \; \omega$)

$$\begin{cases} \omega(0,t) = \omega_{0,j} = 0 \\ \omega(1,t) = \omega_{4,j} = 0 \end{cases}$$

And

$$\omega(x,0) = \omega_{i,0} = sin\pi x_i$$

$\underline{Fix; \, j = 0}$

Now, for $i = 1$ in equation 1, we have

$$\omega_{1,1} = \omega_{0,0} - \omega_{1,0} + \omega_{2,0}$$
$$\omega_{1,1} = 0 - \sin(0.25\pi) + \sin(0.5\pi)$$
$$\omega_{1,1} = 0 - 0.707 + 1 \;=>\; 0.293$$

For $i = 2$ in equation 1, we have

$$\omega_{2,1} = \omega_{1,0} - \omega_{2,0} + \omega_{3,0}$$
$$\omega_{2,1} = 0.707 - 1 + 0.707 => 0.414$$

For $i = 3$ in equation 1, we have

$$\omega_{3,1} = \omega_{2,0} - \omega_{3,0} + \omega_{4,0}$$
$$\omega_{3,1} = 1 - 0.707 + 0 => 0.293$$

$\underline{Fix; \, j = 1}$

Now for $i = 1$ in equation 1, we have

$$\omega_{1,2} = \omega_{0,1} - \omega_{1,1} + \omega_{2,1}$$
$$\omega_{1,2} = 0 - 0.293 + 0.414 => 0.129$$

For $i = 2$ in equation 1, we have

$$\omega_{2,2} = \omega_{1,1} - \omega_{2,1} + \omega_{3,1}$$
$$\omega_{2,2} = 0.293 - 0.414 + 0.293 => 0.172$$

For $i = 3$ in equation 1, we have

$$\omega_{3,2} = \omega_{2,1} - \omega_{3,1} + \omega_{4,1}$$
$$\omega_{3,2} = 0.414 - 0.293 + 0 => 0.121$$

$\underline{Fix; \, j = 2}$

Similarly for $i = 1$ in equation 1, we have $\quad \omega_{1,3} = 0.043$

For $i = 2$ in equation 1, we have $\quad \omega_{2,3} = 0.078$

For $i = 3$ in equation 1, we have $\quad \omega_{3,3} = 0.051$

LECTURE NO. 09

Example: 2

Find the numerical solution of following heat equation by forward difference method (Explicit Method) by taking $h = 0.1$;

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Subjected to boundary condition

$$\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}$$

Similarly $$u(x,0) = sin\pi x$$

Exact solution

$$u(x,\, t) \;=\; e^{-\alpha^2\pi^2 t} sin\pi x$$

Since

$$n = \frac{b-a}{h} = \frac{1-0}{0.1} = 10$$

Use the forward difference approximation for $\frac{6u}{6t}$ and central difference approximation for $\frac{6^2u}{6x^2}$ on given equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \alpha^2 \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right)$$

$$u_{i,j+1} - u_{i,j} = \frac{\alpha^2 k}{h^2}\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right)$$

$$u_{i,j+1} \;=\; u_{i,j} \;+\; \left(\frac{\alpha^2 k}{h^2}\right)\left(u_{i+1,j} \;-\; 2u_{i,j} \;+\; u_{i-1,j}\right)$$

Let $\lambda = \frac{\alpha^2 k}{h^2}$, so by solving we have

$$u_{i,j+1} \;=\; \lambda u_{i-1,j} \;+\; (1-2\lambda)u_{i,j} \;+\; \lambda u_{i+1,j} \qquad \text{...... (i)}$$

Since $$\lambda = \frac{\alpha^2 k}{h^2}$$

Let $\alpha^2 = 1$ and given $h = 0.1$, also suppose $k = 0.01$. By putting values,

$$\lambda = \frac{1(0.01)}{(0.1)^2} = 1$$

So above equation (i) becomes by putting value of λ (also by replacing $\mu\ by\ \omega$)

$$\omega_{i,j+1} \;=\; \omega_{i-1,j} \;-\; \omega_{i,j} \;+\; \omega_{i+1,j} \qquad \text{...... (1)}$$

Boundary conditions and initial conditions becomes accordingly (by replacing $\mu\ by\ \omega$)

$$\begin{cases} \omega(0,t) = \omega_{0,j} = 0 \\ \omega(1,t) = \omega_{10,j} = 0 \end{cases}$$

And

$$\omega(x,0) \;=\; \omega_{i,0} \;=\; sin\pi x_i$$

$\underline{Fix;\ j = 0}$

Now, for $i = 1$ in equation 1, we have

$$\omega_{1,1} \;=\; \omega_{0,0} \;-\; \omega_{1,0} \;+\; \omega_{2,0}$$

$$\omega_{1,1} = 0 - \sin(0.1\pi) + \sin(0.2\pi) => 0.2788$$

For $i = 2$ in equation 1, we have

$$\omega_{2,1} \;=\; \omega_{1,0} \;-\; \omega_{2,0} \;+\; \omega_{3,0}$$

$$\omega_{2,1} = \sin(0.1\pi) - \sin(0.2\pi) + \sin(0.3\pi) => 0.5302$$

Similarly

For $i = 3$ in equation 1, we have

$$\omega_{3,1} \;=\; \omega_{2,0} \;-\; \omega_{3,0} \;+\; \omega_{4,0} \;=>\; 0.7298$$

For $i = 4$ in equation 1, we have

$$\omega_{4,1} \;=\; \omega_{3,0} \;-\; \omega_{4,0} \;+\; \omega_{5,0} \;=>\; 0.8580$$

For $i = 5$ in equation 1, we have

$$\omega_{5,1} \;=\; \omega_{4,0} \;-\; \omega_{5,0} \;+\; \omega_{6,0} \;=>\; 0.9021$$

For $i = 6$ in equation 1, we have

$$\omega_{6,1} = \omega_{5,0} - \omega_{6,0} + \omega_{7,0} => 0.8580$$

For $i = 7$ in equation 1, we have

$$\omega_{7,1} = \omega_{6,0} - \omega_{7,0} + \omega_{8,0} => 0.7298$$

For $i = 8$ in equation 1, we have

$$\omega_{8,1} = \omega_{7,0} - \omega_{8,0} + \omega_{9,0} => 0.5302$$

For $i = 9$ in equation 1, we have

$$\omega_{9,1} = \omega_{8,0} - \omega_{9,0} + \omega_{10,0} => 0.2788$$

Exact Solution:

$$u(x,t) = e^{-\pi^2 t}\sin\pi x \qquad \therefore\ \alpha^2 = 1 \qquad Take\ t = 0.01$$

$$\omega_{1,1} = \omega(x_1,t_1) = \omega(0.1\,,0.01) = e^{-\pi^2(0.01)}\sin(0.1\pi) => 0.2799$$

Similarly

$$\omega_{2,1} = \omega(x_2,t_1) = \omega(0.2\,,0.01) = e^{-\pi^2(0.01)}\sin(0.2\pi) => \_\_\_\_\_\_$$

$$\omega_{3,1} = \omega(x_3,t_1) = \omega(0.3\,,0.01) = e^{-\pi^2(0.01)}\sin(0.3\pi) => \_\_\_\_\_\_$$

$$\omega_{4,1} = \omega(x_4,t_1) = \omega(0.4\,,0.01) = e^{-\pi^2(0.01)}\sin(0.4\pi) => \_\_\_\_\_\_$$

$$\omega_{5,1} = \omega(x_5,t_1) = \omega(0.5\,,0.01) = e^{-\pi^2(0.01)}\sin(0.5\pi) => \_\_\_\_\_\_$$

$$\omega_{6,1} = \omega(x_6,t_1) = \omega(0.6\,,0.01) = e^{-\pi^2(0.01)}\sin(0.6\pi) => \_\_\_\_\_\_$$

$$\omega_{7,1} = \omega(x_7,t_1) = \omega(0.7\,,0.01) = e^{-\pi^2(0.01)}\sin(0.7\pi) => \_\_\_\_\_\_$$

$$\omega_{8,1} = \omega(x_8,t_1) = \omega(0.8\,,0.01) = e^{-\pi^2(0.01)}\sin(0.8\pi) => \_\_\_\_\_\_$$

$$\omega_{9,1} = \omega(x_9,t_1) = \omega(0.9\,,0.01) = e^{-\pi^2(0.01)}\sin(0.9\pi) => \_\_\_\_\_\_$$

Error:

Since, the calculated value of $\omega_{1,1} = 0.2788$ and the calculated value from exact solution is $\omega_{1,1} = 0.2799$, so the error can be calculated as;

$$E_1 = |\omega_{1,1}(approx.) - \omega_{1,1}(exact)| = |0.2788 - 0.2799| => 0.0011$$

Similarly

$$E_2 = |\omega_{2,1}(approx.) - \omega_{2,1}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| => \_\_\_\_\_\_$$

$$E_3 = |\omega_{3,1}(approx.) - \omega_{3,1}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| => \_\_\_\_\_\_$$

$$E_4 = |\omega_{4,1}(approx.) - \omega_{4,1}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| => \_\_\_\_\_\_$$

$$E_5 = |\omega_{5,1}(approx.) - \omega_{5,1}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| => \_\_\_\_\_\_$$

$$E_6 = |\omega_{6,1}(approx.) - \omega_{6,1}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| => \_\_\_\_\_\_$$

$$E_7 = |\omega_{7,1}(approx.) - \omega_{7,1}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| => \_\_\_\_\_\_$$

$$E_8 = |\omega_{8,1}(approx.) - \omega_{8,1}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| => \_\_\_\_\_\_$$

$$E_9 = |\omega_{9,1}(approx.) - \omega_{9,1}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| => \_\_\_\_\_\_$$

(Do Your Self; Take values from above calculations.)

(Understand carefully, first we take j = 0, then calculate exact solution and error. Now we'll perform the same calculations with next step values i.e. j = 1, then calculate exact solution and error; and Done.)

Fix; $j = 1$

$$\omega_{i,j+1} = \omega_{i-1,j} - \omega_{i,j} + \omega_{i+1,j} \qquad \text{...... (1)}$$

(This equation given above, here typed again just for help.)

Now, for $i = 1$ in equation 1, we have

$$\omega_{1,2} = \omega_{0,1} - \omega_{1,1} + \omega_{2,1}$$

$$\omega_{1,2} = 0 - 0.2788 + 0.5302 => 0.2514$$

For $i = 2$ in equation 1, we have

$$\omega_{2,2} = \omega_{1,1} - \omega_{2,1} + \omega_{3,1}$$

$$\omega_{2,2} = 0.2788 - 0.5302 + 0.7298 => 0.4784$$

Similarly

For i = 3 in equation 1, we have

$$\omega_{3,2} = \omega_{2,1} - \omega_{3,1} + \omega_{4,1} \Rightarrow 0.6584$$

For i = 4 in equation 1, we have

$$\omega_{4,2} = \omega_{3,1} - \omega_{4,1} + \omega_{5,1} \Rightarrow \_\_\_\_\_\_\_$$

For i = 5 in equation 1, we have

$$\omega_{5,2} = \omega_{4,1} - \omega_{5,1} + \omega_{6,1} \Rightarrow \_\_\_\_\_\_\_$$

For i = 6 in equation 1, we have

$$\omega_{6,2} = \omega_{5,1} - \omega_{6,1} + \omega_{7,1} \Rightarrow \_\_\_\_\_\_\_$$

For i = 7 in equation 1, we have

$$\omega_{7,2} = \omega_{6,1} - \omega_{7,1} + \omega_{8,1} \Rightarrow \_\_\_\_\_\_\_$$

For i = 8 in equation 1, we have

$$\omega_{8,2} = \omega_{7,1} - \omega_{8,1} + \omega_{9,1} \Rightarrow \_\_\_\_\_\_\_$$

For i = 9 in equation 1, we have

$$\omega_{9,2} = \omega_{8,1} - \omega_{9,1} + \omega_{10,1} \Rightarrow \_\_\_\_\_\_\_$$

(Do Your Self; Take values from above calculations.)

Exact Solution:

$$u(x,t) = e^{-\pi^2 t} \sin\pi x \qquad \therefore\ \alpha^2 = 1 \qquad Take\ t = 0.02$$

$$\omega_{1,2} = \omega(x_1, t_2) = \omega(0.1\ , 0.02) = e^{-\pi^2(0.02)} \sin(0.1\pi) \Rightarrow 0.2799$$

Similarly

$$\omega_{2,2} = \omega(x_2, t_2) = \omega(0.2\ , 0.02) = e^{-\pi^2(0.02)} \sin(0.2\pi) \Rightarrow \_\_\_\_\_\_\_$$

$$\omega_{3,2} = \omega(x_3, t_2) = \omega(0.3\ , 0.02) = e^{-\pi^2(0.02)} \sin(0.3\pi) \Rightarrow \_\_\_\_\_\_\_$$

$$\omega_{4,2} = \omega(x_4, t_2) = \omega(0.4\ , 0.02) = e^{-\pi^2(0.02)} \sin(0.4\pi) \Rightarrow \_\_\_\_\_\_\_$$

$$\omega_{5,2} = \omega(x_5, t_2) = \omega(0.5\ , 0.02) = e^{-\pi^2(0.02)} \sin(0.5\pi) \Rightarrow \_\_\_\_\_\_\_$$

$$\omega_{6,2} = \omega(x_6, t_2) = \omega(0.6\ , 0.02) = e^{-\pi^2(0.02)} \sin(0.6\pi) \Rightarrow \_\_\_\_\_\_\_$$

$$\omega_{7,2} = \omega(x_7, t_2) = \omega(0.7\ , 0.02) = e^{-\pi^2(0.02)} \sin(0.7\pi) \Rightarrow \_\_\_\_\_\_\_$$

$$\omega_{8,2} = \omega(x_8, t_2) = \omega(0.8\ , 0.02) = e^{-\pi^2(0.02)} \sin(0.8\pi) \Rightarrow \_\_\_\_\_\_\_$$

$$\omega_{9,2} = \omega(x_9, t_2) = \omega(0.9\ , 0.02) = e^{-\pi^2(0.02)} \sin(0.9\pi) \Rightarrow \_\_\_\_\_\_\_$$

Error:

Error can be calculated as;

$$E_1 = |\omega_{1,2}(approx.) - \omega_{1,2}(exact)| = |0.2514 - 0.2566| \Rightarrow 0.0052$$

Similarly

$$E_2 = |\omega_{2,2}(approx.) - \omega_{2,2}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| \Rightarrow \_\_\_\_\_\_\_$$

$$E_3 = |\omega_{3,2}(approx.) - \omega_{3,2}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| \Rightarrow \_\_\_\_\_\_\_$$

$$E_4 = |\omega_{4,2}(approx.) - \omega_{4,2}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| \Rightarrow \_\_\_\_\_\_\_$$

$$E_5 = |\omega_{5,2}(approx.) - \omega_{5,2}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| \Rightarrow \_\_\_\_\_\_\_$$

$$E_6 = |\omega_{6,2}(approx.) - \omega_{6,2}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| \Rightarrow \_\_\_\_\_\_\_$$

$$E_7 = |\omega_{7,2}(approx.) - \omega_{7,2}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| \Rightarrow \_\_\_\_\_\_\_$$

$$E_8 = |\omega_{8,2}(approx.) - \omega_{8,2}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| \Rightarrow \_\_\_\_\_\_\_$$

$$E_9 = |\omega_{9,2}(approx.) - \omega_{9,2}(exact)| = |\_\_\_\_\_\_ - \_\_\_\_\_\_| \Rightarrow \_\_\_\_\_\_\_$$

(Do Your Self; Take values from above calculations.)

DEDICATED TO UNKNOWN STUDENTS WHO ARE OUR FUTURE HEROES.

Regards: Virtual Alerts

LECTURE NO. 10

Example: 3

Find the numerical solution of following heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad or \quad u_t = \alpha^2 u_{xx}$$

By forward difference method at $T = 0.005, h = 0.1 \; and \; k = 0.001$

$$Boundary\ Condition;\ \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}$$

$$Initial\ Condition;\ u(x,0) = \begin{cases} 2x & , \quad 0 \le x \le \frac{1}{2} \\ 2(1-x) & , \quad \frac{1}{2} < x \le 1 \end{cases}$$

Solution: It is given that

$$h = 0.1\,, k = 0.001 \quad and \quad T = 0.005$$

Since

$$n = \left(\frac{b-a}{h}\right) = \frac{1-0}{0.1} = 10 \qquad i = 1,2,3 \dots n-1$$

Also

$$m = \left(\frac{T-t_0}{k}\right) = \frac{0.005-0}{0.001} = 5 \qquad j = 0,1,2,3 \dots m-1$$

Therefore

$$n = 10, \quad so \quad i = 1,2,\dots,9$$
$$m = 5, \quad so \quad j = 0,1,2,3,4$$

Apply forward difference (FDW) on time-derivative and central difference (CD) on space derivative of the given equation and choose $\alpha = 1$, we get

$$\omega_{i,j+1} = \lambda\omega_{i-1,j} + (1-2\lambda)\omega_{i,j} + \lambda\omega_{i+1,j} \quad \dots (1)$$

Where

$$\lambda = \frac{k}{h^2} = \frac{0.001}{(0.1)^2} = 0.1$$

Therefore $\qquad 1 - 2\lambda = 1 - 2(0.1) = 0.8$

By putting value of λ , equation 1 becomes

$$\omega_{i,j+1} = 0.1\,\omega_{i-1,j} + 0.8\,\omega_{i,j} + 0.1\,\omega_{i+1,j} \quad \dots (2)$$

First we'll calculate values for ω using initial conditions as follows,

$(Condition: u(x,0) = 2x \; if \;\; 0 \le x \le \frac{1}{2} \quad \therefore h = 0.1)$

$$\omega_{1,0} = \omega(x_1, t_0) = \omega(0.1\,,0) = 2x_1 = 2 \times 0.1 = 0.2$$
$$\omega_{2,0} = \omega(x_2, t_0) = \omega(0.2\,,0) = 2x_2 = 2 \times 0.2 = 0.4$$
$$\omega_{3,0} = \omega(x_3, t_0) = \omega(0.3\,,0) = 2x_3 = 2 \times 0.3 = 0.6$$
$$\omega_{4,0} = \omega(x_4, t_0) = \omega(0.4\,,0) = 2x_4 = 2 \times 0.4 = 0.8$$
$$\omega_{5,0} = \omega(x_5, t_0) = \omega(0.5\,,0) = 2x_5 = 2 \times 0.5 = 1.0$$

$(Condition: u(x,0) = 2(1-x) \; if \;\; \frac{1}{2} < x \le 1 \quad \therefore h = 0.1)$

$$\omega_{6,0} = \omega(x_6, t_0) = \omega(0.6\,,0) = 2(1-x_6) = 2(1-0.6) = 0.8$$
$$\omega_{7,0} = \omega(x_7, t_0) = \omega(0.7\,,0) = 2(1-x_7) = 2(1-0.7) = 0.6$$
$$\omega_{8,0} = \omega(x_8, t_0) = \omega(0.8\,,0) = 2(1-x_8) = 2(1-0.8) = 0.4$$
$$\omega_{9,0} = \omega(x_9, t_0) = \omega(0.9\,,0) = 2(1-x_9) = 2(1-0.9) = 0.2$$
$$\omega_{10,0} = \omega(x_{10}, t_0) = \omega(1.0\,,0) = 2(1-x_{10}) = 2(1-1) = 0.0$$

$\underline{Fix\ j = 0}$

$And\ put\ i = 1,2,3,4,5,6,7,8,9\ in\ equation\ 2$

$$\omega_{i,j+1} = 0.1\, \omega_{i-1,j} + 0.8\, \omega_{i,j} + 0.1\, \omega_{i+1,j} \quad \dots (2)$$

$i = 1;\quad \omega_{1,1} = 0.1\, \omega_{0,0} + 0.8\, \omega_{1,0} + 0.1\, \omega_{2,0} = 0.1(0.0) + 0.8(0.2) + 0.1(0.4) = 0.20$

$i = 2;\quad \omega_{2,1} = 0.1\, \omega_{1,0} + 0.8\, \omega_{2,0} + 0.1\, \omega_{3,0} = 0.1(0.2) + 0.8(0.4) + 0.1(0.6) = 0.40$

$i = 3;\quad \omega_{3,1} = 0.1\, \omega_{2,0} + 0.8\, \omega_{3,0} + 0.1\, \omega_{4,0} = 0.1(0.4) + 0.8(0.6) + 0.1(0.8) = 0.60$

$i = 4;\quad \omega_{4,1} = 0.1\, \omega_{3,0} + 0.8\, \omega_{4,0} + 0.1\, \omega_{5,0} = 0.1(0.6) + 0.8(0.8) + 0.1(1.0) = 0.80$

$i = 5;\quad \omega_{5,1} = 0.1\, \omega_{4,0} + 0.8\, \omega_{5,0} + 0.1\, \omega_{6,0} = 0.1(0.8) + 0.8(1.0) + 0.1(0.8) = 0.96$

$i = 6;\quad \omega_{6,1} = 0.1\, \omega_{5,0} + 0.8\, \omega_{6,0} + 0.1\, \omega_{7,0} = 0.1(1.0) + 0.8(0.8) + 0.1(0.6) = 0.80$

$i = 7;\quad \omega_{7,1} = 0.1\, \omega_{6,0} + 0.8\, \omega_{7,0} + 0.1\, \omega_{8,0} = 0.1(0.8) + 0.8(0.6) + 0.1(0.4) = 0.60$

$i = 8;\quad \omega_{8,1} = 0.1\, \omega_{7,0} + 0.8\, \omega_{8,0} + 0.1\, \omega_{9,0} = 0.1(0.6) + 0.8(0.4) + 0.1(0.2) = 0.40$

$i = 9;\quad \omega_{9,1} = 0.1\, \omega_{8,0} + 0.8\, \omega_{9,0} + 0.1\, \omega_{10,0} = 0.1(0.4) + 0.8(0.2) + 0.1(0.0) = 0.20$

In the same way you can find the values of next time levels fixing j=1 and put x=1, 2, 3...9.

Also find the other values for fixing $j = 2, j = 3$ and $j = 4$.

Example: 4

Find the numerical solution of following heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad , \quad 0 < x < 2 \ , \ t > 0$$

By forward difference method at $T = 0.01$ and $n = 2\ , m = 2$

$$Boundary\ condition;\ \begin{cases} u(0,t) = 0 \\ u(2,t) = 0 \end{cases}$$

$$Initial\ Condition; u(x,0) = \sin\left(\frac{\pi}{2}x\right)$$

Solution: Since

$$h = \frac{b-a}{n} = \frac{2-0}{2} = 1$$

$$k = \frac{T-t_0}{m} = \frac{0.1-0}{2} = 0.05$$

As $\quad n = 2\ ,\ m = 2\ ,\ i = 1\ ,\ j = 0,1$

$$\omega_{i,j+1} = \lambda\omega_{i-1,j} + (1-2\lambda)\omega_{i,j} + \lambda\omega_{i+1,j} \quad \dots (1)$$

Where

$$\lambda = \frac{\alpha^2 k}{h^2} = \frac{1(0.05)}{(1)^2} = 0.05$$

By putting value of λ , equation 1 becomes

$$\omega_{i,j+1} = 0.05\, \omega_{i-1,j} + 0.9\, \omega_{i,j} + 0.05\, \omega_{i+1,j} \quad \dots (2)$$

$\underline{Fix\ j = 0}$

$And\ put\ i = 1\ in\ equation\ 2$

$$\omega_{1,1} = 0.05\, \omega_{0,0} + 0.9\, \omega_{1,0} + 0.05\, \omega_{2,0} = 0.05(0) + 0.9(0.1) + 0.05(0) = 0.9$$

$\underline{Fix\ j = 1}$

$And\ put\ i = 1\ in\ equation\ 2$

$$\omega_{1,2} = 0.05\, \omega_{0,1} + 0.9\, \omega_{1,1} + 0.05\, \omega_{2,1} = 0.05(0) + 0.9(0.9) + 0.05(0) = 0.81$$

LECTURE NO. 11

Example: 5

Find the numerical solution of following heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad , \quad 0 \le x \le 2 \ , \ t > 0$$

By forward difference method at $T = 0.05, h = 0.1$ and $k = 0.01$

$$Boundary\ Condition;\ \{u(0,t) = u(2,t) = 0 \qquad \forall\, t$$

$$Initial\ Condition; \{u(x,0) = sin(2\pi x) \qquad 0 \le x \le 2$$

Exact solution is given as

$$u(x,t) = e^{-4\pi^2 t}\, sin(2\pi x)$$

Solution: It is given that

$$h = 0.1\,, k = 0.01 \;\; and \;\; T = 0.05$$

Since

$$n = \left(\frac{b-a}{h}\right) = \frac{2-0}{0.1} = 20$$

Also

$$m = \left(\frac{T-t_0}{k}\right) = \frac{0.05-0}{0.01} = 50$$

Therefore

$$i = 1,2,\dots,19 \qquad \& \qquad j = 0,1,2,\dots,49$$

Apply forward difference (FDW) on $\left(\frac{\partial u}{\partial t}\right)$ and central difference (CD) on $\left(\frac{\partial^2 u}{\partial x^2}\right)$ and $\lambda = \frac{k}{h^2}$. We get the following equation $(By\ replacing\ "\omega"\ by\ "u")$

$$\omega_{i,j+1} = \lambda\omega_{i-1,j} + (1-2\lambda)\omega_{i,j} + \lambda\omega_{i+1,j} \quad \dots (A)$$

Now calculate

$$\lambda = \frac{k}{h^2} = \frac{0.01}{(0.1)^2} = 1$$

So equation (A) becomes,

$$\omega_{i,j+1} = \omega_{i-1,j} - \omega_{i,j} + \omega_{i+1,j} \quad \dots (B)$$

Given conditions in "ω" form can be written as

$$Boundary\ Conditions;\ \begin{cases} \omega_{0,j} = 0 & \forall\, j = 0,1,2,\dots,49 \\ \omega_{20,j} = 0 & \forall\, j = 0,1,2,\dots,49 \end{cases}$$

$$Initial\ Condition;\ \{\omega_{i,0} = sin(2\pi x) \qquad \forall\, i = 0,1,2,\dots,19$$

$\underline{Fix\ j = 0}$

$And\ put \quad i = 1,2,\dots,19 \quad in\ equation\ (B)$

$$\omega_{i,j+1} = \omega_{i-1,j} - \omega_{i,j} + \omega_{i+1,j} \quad \dots (B)$$

$$i=1; \qquad \omega_{1,1} = \omega_{0,0} - \omega_{1,0} + \omega_{2,0} = sin(2\pi\times 0.0) - sin(2\pi\times 0.1) + sin(2\pi\times 0.2) = 0.3633$$

$$i=2; \qquad \omega_{2,1} = \omega_{1,0} - \omega_{2,0} + \omega_{3,0} = sin(2\pi\times 0.1) - sin(2\pi\times 0.2) + sin(2\pi\times 0.3) = 0.5878$$

Similarly

$$i=3; \qquad \omega_{3,1} = \omega_{2,0} - \omega_{3,0} + \omega_{4,0} = 0.5878$$

$$i=4; \qquad \omega_{4,1} = \omega_{3,0} - \omega_{4,0} + \omega_{5,0} = 0.3633$$

$$i=5; \qquad \omega_{5,1} = \omega_{4,0} - \omega_{5,0} + \omega_{6,0} = 0.0$$

$$i=6; \qquad \omega_{6,1} = \omega_{5,0} - \omega_{6,0} + \omega_{7,0} = -0.3633$$

$$i=7; \qquad \omega_{7,1} = \omega_{6,0} - \omega_{7,0} + \omega_{8,0} = -0.5878$$

Similarly

$i = 8;$	$\omega_{8,1} = -0.5878$	$i = 9;$	$\omega_{9,1} = -0.3633$
$i = 10;$	$\omega_{10,1} = 0.0$	$i = 11;$	$\omega_{11,1} = 0.3633$
$i = 12;$	$\omega_{12,1} = 0.5878$	$i = 13;$	$\omega_{13,1} = 0.5878$
$i = 14;$	$\omega_{14,1} = 0.3633$	$i = 15;$	$\omega_{15,1} = 0.0$
$i = 16;$	$\omega_{16,1} = -0.3633$	$i = 17;$	$\omega_{17,1} = -0.5878$
$i = 18;$	$\omega_{18,1} = -0.5878$	$i = 19;$	$\omega_{19,1} = -0.3633$

Now

$\underline{Fix\ j = 1}$

$And\ put \quad i = 1,2,\dots,19 \quad in\ equation\ (B)$

Similarly; perform calculations till fixing $j =$ 49 and Find Exact Solution.

Lengthy calculations; so you can skip the above part. Watch Lecture.

LECTURE NO. 12

Example: 6

Find the numerical solution of following heat equation

$$\frac{\partial u(x,t)}{\partial t} - \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \quad , \quad 0 \le x \le 2 \ , \ t > 0$$

By forward difference method at $T = 0.5, h = 0.1 \ and \ k = 0.05$

$$Boundary\ Condition; \quad \{u(0,t) = u(2,t) = 0 \qquad \forall\ t$$

$$Initial\ Condition \quad \{u(x,0) = sin(2\pi x) \qquad 0 \le x \le 2$$

Exact solution is given as

$$u(x,t) = e^{-4\pi^2 t}\ sin(2\pi x)$$

Solution: It is given that

$$h = 0.1\ , k = 0.05 \ \ and \ \ T = 0.5$$

Since

$$n = (\frac{b-a}{h}) = \frac{2-0}{0.1} = 20$$

Also

$$m = (\frac{T-t_0}{k}) = \frac{0.5-0}{0.05} = 10$$

Therefore

$$i = 1,2,\dots,19 \qquad \& \qquad j = 0,1,2,\dots,9$$

Given conditions in "ω" form can be written as

$$Boundary\ Conditions; \begin{cases} \omega_{0,j} = 0 & \forall\ j = 0,1,2,\dots,9 \\ \omega_{20,j} = 0 & \forall\ j = 0,1,2,\dots,9 \end{cases}$$

$$Initial\ Condition; \{\omega_{i,0} = sin(2\pi x) \qquad \forall\ i = 0,1,2,\dots,19$$

Since

$$\lambda = \frac{\alpha^2 k}{h^2} = \frac{(1)^2(0.05)}{(0.1)^2} = 5$$

Apply forward difference (FDW) on $(\frac{\partial u}{\partial t})$ and central difference (CD) on $(\frac{\partial^2 u}{\partial x^2})$. We get the following equation$(By\ replacing\ "\omega"\ by\ "u")$

$$\omega_{i,j+1} = \lambda\omega_{i-1,j} + (1-2\lambda)\omega_{i,j} + \lambda\omega_{i+1,j} \quad \dots(A)$$

By Putting $\lambda = 5$ value equation (A) becomes,

$$\omega_{i,j+1} = 5\ \omega_{i-1,j} - 9\ \omega_{i,j} + 5\ \omega_{i+1,j} \quad \dots(1)$$

$\underline{Fix\ j = 0}$

$$And\ put \quad i = 1,2,\dots,19 \quad in\ equation\ (1)$$

$$\omega_{i,j+1} = 5\ \omega_{i-1,j} - 9\ \omega_{i,j} + 5\ \omega_{i+1,j} \quad \dots(1)$$

$i = 1; \quad \omega_{1,1} = 5\omega_{0,0} - 9\omega_{1,0} + 5\omega_{2,0} = 5sin(2\pi \times 0.0) - 9sin(2\pi \times 0.1) + 5sin(2\pi \times 0.2) = -0.5348$

$i = 2; \quad \omega_{2,1} = 5\omega_{1,0} - 9\omega_{2,0} + 5\omega_{3,0} = 5sin(2\pi \times 0.1) - 9sin(2\pi \times 0.2) + 5sin(2\pi \times 0.3) = -0.8653$

Similarly

$$i = 3; \quad \omega_{3,1} = 5\omega_{2,0} - 9\omega_{3,0} + 5\omega_{4,0} = -0.8653$$

$$i = 4; \quad \omega_{4,1} = 5\omega_{3,0} - 9\omega_{4,0} + 5\omega_{5,0} = -0.5348$$

$$i = 5; \quad \omega_{5,1} = 5\omega_{4,0} - 9\omega_{5,0} + 5\omega_{6,0} = 0.0$$

$$i = 6; \quad \omega_{6,1} = 5\omega_{5,0} - 9\omega_{6,0} + 5\omega_{7,0} = 0.5348$$

$$i = 7; \quad \omega_{7,1} = 5\omega_{6,0} - 9\omega_{7,0} + 5\omega_{8,0} = 0.8653$$

$$i = 8; \quad \omega_{8,1} = 5\omega_{7,0} - 9\omega_{8,0} + 5\omega_{9,0} = 0.8653$$

$$i = 9; \quad \omega_{9,1} = 5\omega_{8,0} - 9\omega_{9,0} + 5\omega_{10,0} = 0.5348$$

$$i = 10; \quad \omega_{10,1} = 5\omega_{9,0} - 9\omega_{10,0} + 5\omega_{11,0} = 0.0$$

Similarly

$i = 11;$	$\omega_{11,1} = -0.5348$	$i = 12;$	$\omega_{12,1} = -0.8653$
$i = 13;$	$\omega_{13,1} = -0.8653$	$i = 14;$	$\omega_{14,1} = -0.5348$
$i = 15;$	$\omega_{15,1} = 0.0$	$i = 16;$	$\omega_{16,1} = 0.5348$
$i = 17;$	$\omega_{17,1} = 0.8653$	$i = 18;$	$\omega_{18,1} = 0.8653$
$i = 19;$	$\omega_{19,1} = 0.5348$	$i = 20;$	$\omega_{20,1} =$ ______?

Now

$\underline{Fix\ j = 1}$

$And\ put \quad i = 1,2,\dots,19 \quad in\ equation\ (1)$

Similarly; perform calculations till fixing $j = 9$.

Lengthy calculations; so you can skip the above part.

$\underline{Exact\ Solution}$:

$$u(x,t) = e^{-4\pi^2 t} \sin 2\pi x$$

$\omega_{1,1} = \omega(x_1, t_1) = \omega(0.1\ , 0.05) =$ ______ $\quad \omega_{2,1} = \omega(x_2, t_1) = \omega(0.2\ , 0.05) =$ ______

$\omega_{3,1} = \omega(x_3, t_1) = \omega(0.3\ , 0.05) =$ ______ $\quad \omega_{4,1} = \omega(x_4, t_1) = \omega(0.4\ , 0.05) =$ ______

$\omega_{5,1} = \omega(x_5, t_1) = \omega(0.5\ , 0.05) =$ ______ $\quad \omega_{6,1} = \omega(x_6, t_1) = \omega(0.6\ , 0.05) =$ ______

$\omega_{7,1} = \omega(x_7, t_1) = \omega(0.7\ , 0.05) =$ ______ $\quad \omega_{8,1} = \omega(x_8, t_1) = \omega(0.8\ , 0.05) =$ ______

$\omega_{9,1} = \omega(x_9, t_1) = \omega(0.9\ , 0.05) =$ ______ $\quad \omega_{10,1} = \omega(x_{10}, t_1) = \omega(1.0\ , 0.05) =$ ______

$\omega_{11,1} = \omega(x_{11}, t_1) = \omega(1.1\ , 0.05) =$ ______ $\quad \omega_{12,1} = \omega(x_{12}, t_1) = \omega(1.2\ , 0.05) =$ ______

$\omega_{13,1} = \omega(x_{13}, t_1) = \omega(1.3\ , 0.05) =$ ______ $\quad \omega_{14,1} = \omega(x_{14}, t_1) = \omega(1.4\ , 0.05) =$ ______

$\omega_{15,1} = \omega(x_{15}, t_1) = \omega(1.5\ , 0.05) =$ ______ $\quad \omega_{16,1} = \omega(x_{16}, t_1) = \omega(1.6\ , 0.05) =$ ______

$\omega_{17,1} = \omega(x_{17}, t_1) = \omega(1.7, 0.05) =$ ______ $\quad \omega_{18,1} = \omega(x_{18}, t_1) = \omega(1.8\ , 0.05) =$ ______

$\omega_{19,1} = \omega(x_{19}, t_1) = \omega(1.9\ , 0.05) =$ ______ $\quad \omega_{20,1} = \omega(x_{20}, t_1) = \omega(2.0\ , 0.05) =$ ______

Example: 7

Find the numerical solution of following heat equation

$$\frac{\partial u(x,t)}{\partial t} - \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \quad , \quad 0 \le x \le 1\ ,\ t > 0$$

By forward difference method at $T = 0.1, n = 3\ and\ m = 2$

$$Boundary\ Condition; \quad \{u(0,t) = u(1,t) = 0 \quad \forall\, t > 0$$

$$Initial\ Condition \quad \{u(x,0) = 2\sin(2\pi x) \quad 0 \le x \le 1$$

Exact solution is given as

$$u(x,t) = 2e^{-\frac{\pi^2}{4}t} \sin(2\pi x)$$

Solution: It is given that

$$T = 0.1\ , \quad n = 3\ and\ \ m = 2$$

Since

$$h = \left(\frac{b-a}{n}\right) = \frac{1-0}{3} = 0.33$$

Also

$$k = \left(\frac{T-t_0}{m}\right) = \frac{0.1-0}{2} = 0.05$$

Since

$$n = 3 \quad so \quad i = 1,2 \qquad \& \qquad m = 2 \quad so \quad j = 1$$

Apply forward difference (FDW) on $\left(\frac{\partial u}{\partial t}\right)$ and central difference (CD) on $\left(\frac{\partial^2 u}{\partial x^2}\right)$ and $\lambda = \frac{k}{h^2}$. We get the following equation $(By\ replacing\ "\omega"\ by\ "u")$

$$\omega_{i,j+1} = \lambda\omega_{i-1,j} + (1-2\lambda)\omega_{i,j} + \lambda\omega_{i+1,j} \quad \dots (1)$$

Where

$$\lambda = \frac{\alpha^2 k}{h^2} = \frac{(0.25)^2(0.05)}{(0.33)^2} = 0.029$$

So, equation (1) becomes,

$$\omega_{i,j+1} = 0.029\ \omega_{i-1,j} + 0.942\ \omega_{i,j} + 0.029\ \omega_{i+1,j} \quad \dots (2)$$

$\underline{Fix\ j = 0}$

$$And\ put \quad i = 1,2 \quad in\ equation\ (2)$$

$$i = 1; \qquad \omega_{1,1} = 0.029\ \omega_{0,0} + 0.029\ \omega_{1,0} + 0.029\ \omega_{2,0} = 1.6001$$

$$i = 2; \qquad \omega_{2,1} = 0.029\ \omega_{1,0} + 0.029\omega_{2,0} + 0.029\ \omega_{3,0} = 1.6001$$

Now

$\underline{Fix\ j = 1}$

$$And\ put \quad i = 1,2 \quad in\ equation\ (2)$$

$$i = 1; \qquad \omega_{1,} = 0.029\ \omega_{0,1} + 0.029\ \omega_{1,1} + 0.029\ \omega_{2,1} = \_\_\_\_\_\_\_\_$$

$$i = 2; \qquad \omega_{2,2} = 0.029\ \omega_{1,1} + 0.029\omega_{2,1} + 0.029\ \omega_{3,1} = \_\_\_\_\_\_\_\_$$

Do Your Self. Also find the Exact Solution.

LECTURE NO. 13

Example: 8

Find the numerical solution of following heat equation

$$\frac{\partial u(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad , \quad 0 \le x \le 1\ ,\ t > 0$$

By backward difference method (implicit scheme) at $T = 1$, ⍰ $= 0.25\ and\ k = 0.0625$

$$Boundary\ Condition; \qquad \{u(0,t) = u(1,t) = 0 \qquad \forall\ t$$

$$Initial\ Condition \qquad \{u(x,0) = \ sin\pi x \qquad 0 \le x \le 1$$

Solution: It is given that

$$T = 1\ ,\ ⍰ = 0.25 \ \ and\ \ k = 0.0625$$

Since

$$n = (\frac{b-a}{⍰}) = \frac{1-0}{0.25} = 4$$

Also

$$k = (\frac{T-t_0}{m}) = \frac{1-0}{0.0625} = 16$$

Since

$$i = 1,2,3 \qquad \& \qquad j = 1,2,\dots,15$$

Apply backward difference/implicit scheme (IS) on $(\frac{\partial u}{\partial t})$ and central difference (CD) on $(\frac{\partial^2 u}{\partial x^2})$. We get the following equation $(By\ replacing\ "\omega"\ by\ "u")$

$$\omega_{i,j} - \omega_{i,j-1} = \frac{\alpha^2 k}{⍰^2}(\omega_{i-1,j} - 2\ \omega_{i,j} + \lambda\omega_{i+1,j})$$

Where

$$\frac{\alpha^2 k}{⍰^2} = \lambda$$

By solving above equation, we have

$$\omega_{i,j-1} = -\lambda\omega_{i-1,j} + (1+2\lambda)\omega_{i,j} - \lambda\omega_{i+1,j} \quad \dots (1)$$

$\underline{Fix\ j = 1}$

$$And\ put \quad i = 1,2,3 \quad in\ equation\ (1)$$

$$i = 1; \qquad \omega_{1,0} = -\lambda\omega_{0,1} + (1+2\lambda)\ \omega_{1,1} - \lambda\ \omega_{2,1}$$

$$i = 2; \qquad \omega_{2,0} = -\lambda\ \omega_{1,1} + (1+2\lambda)\ \omega_{2,1} - \lambda\ \omega_{3,1}\} \dots (A)$$

$$i = 3; \qquad \omega_{3,0} = -\lambda\ \omega_{2,1} + (1+2\lambda)\ \omega_{3,1} - \lambda\ \omega_{4,1}$$

Since given conditions in "ω" form can be written as

$$Boundary\ Conditions; \begin{cases} u(0,t) = 0 \ \ => \ \ \omega_{0,j} = 0 \\ u(1,t) = 0 \ \ => \ \ \omega_{4,j} = 0 \end{cases}$$

$$Initial\ Condition;\ \{u(x,0) = sin\pi x$$
$$\omega_{i,o} = sin\pi x_i \qquad \forall\ i = 1,2,3$$

By Solving, we have

$$\omega_{1,0} = sin(0.25\pi) = 0.707$$
$$\omega_{2,0} = sin(0.50\pi) = 1.0$$
$$\omega_{3,0} = sin(0.75\pi) = 0.707$$

Now, from equation (A) in matrix form

$$\begin{bmatrix} 1+2\lambda & -\lambda & 0 \\ -\lambda & 1+2\lambda & \lambda \\ 0 & -\lambda & 1+2\lambda \end{bmatrix} \begin{bmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{bmatrix} = \begin{bmatrix} \omega_{1,0} \\ \omega_{2,0} \\ \omega_{3,0} \end{bmatrix} \quad \dots (B)$$

Since

$$\lambda = \frac{\alpha^2 k}{h^2} = \frac{(1)^2(0.0625)}{(0.25)^2} = 1$$

By Putting $\lambda = 1$ value equation (B) becomes

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{bmatrix} = \begin{bmatrix} 0.707 \\ 1 \\ 0.707 \end{bmatrix}$$

As we know matrix relation,

$$A\omega = b \quad => \quad \omega = A^{-1}b$$

By calculating, we get

$$\begin{bmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{bmatrix} = \begin{bmatrix} 0.4458 \\ 0.6305 \\ 0.4458 \end{bmatrix}$$

Now

$\underline{Fix\ j = 1}$

$And\ put \quad i = 1,2,3 \quad in\ equation\ (1)$

Similarly; perform calculations till fixing $j = 15$.

Lengthy calculations; so you can skip the above part.

LECTURE NO. 14

Example: 9

Find the numerical solution of following heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad , \qquad 0 \le x \le 1\ ,\ t > 0$$

By backward difference method (implicit scheme) at $T = 1, h = 0.2\ and\ k = 0.04$

$$Boundary\ Condition; \qquad \{u(0,t) = u(1,t) = 0 \qquad \forall\ t$$
$$Initial\ Condition \qquad \{u(x,0) = \ sin\pi x \qquad 0 \le x \le 1$$

Solution: It is given that

$$T = 1\ ,\ \ h = 0.2\ ,\ \ k = 0.04\ ,\ \ a = 0 \ \ and \ \ b = 1$$

Since

$$n = (\frac{b-a}{h}) = \frac{1-0}{0.2} = 5$$

Also

$$m = (\frac{T - t_0}{k}) = \frac{1-0}{0.04} = 25$$

Since

$$i = 1,2,3,4 \qquad \& \qquad j = 1,2,\dots,24$$

Apply backward difference/implicit scheme (IS) on $(\frac{\partial u}{\partial t})$ and central difference (CD) on $(\frac{\partial^2 u}{\partial x^2})$. We get the following equation $(By\ replacing\ "\omega"\ by\ "u")$

$$\omega_{i,j-1} = -\lambda\omega_{i-1,j} + (1+2\lambda)\omega_{i,j} - \lambda\omega_{i+1,j} \quad \ldots (1)$$

Where

$$\frac{\alpha^2 k}{h^2} = \lambda$$

<u>$Fix\ j = 1$</u>

$And\ put \quad i = 1,2,3,4 \quad in\ equation\ (1)$

$$\left.\begin{array}{ll} i = 1; & \omega_{1,0} = -\lambda\omega_{0,1} + (1+2\lambda)\,\omega_{1,1} - \lambda\,\omega_{2,1} \\ i = 2; & \omega_{2,0} = -\lambda\,\omega_{1,1} + (1+2\lambda)\,\omega_{2,1} - \lambda\,\omega_{3,1} \\ i = 3; & \omega_{3,0} = -\lambda\,\omega_{2,1} + (1+2\lambda)\,\omega_{3,1} - \lambda\,\omega_{4,1} \\ i = 4; & \omega_{4,0} = -\lambda\,\omega_{3,1} + (1+2\lambda)\,\omega_{4,1} - \lambda\,\omega_{5,1} \end{array}\right\} \ldots (A)$$

Since given conditions in "ω" form can be written as

$$Boundary\ Conditions;\ \begin{cases} u(0,t) = \omega_{0,j} = 0 & \forall\, j = 1,2,\ldots,24 \\ u(1,t) = \omega_{4,j} = 0 & \forall\, j = 1,2,\ldots,24 \end{cases}$$

$$Initial\ Condition;\ \{u(x,0) = sin\pi x \qquad \forall\, i = 1,2,3,4$$

By solving, we have

$$\begin{aligned} \omega_{1,0} &= \omega(x_1,t_0) = \omega(0.2\,,0) = sin(0.2\pi) = 0.5877 \\ \omega_{2,0} &= \omega(x_2,t_0) = \omega(0.4\,,0) = sin(0.4\pi) = 0.9510 \\ \omega_{3,0} &= \omega(x_3,t_0) = \omega(0.6\,,0) = sin(0.6\pi) = 0.9510 \\ \omega_{4,0} &= \omega(x_4,t_0) = \omega(0.8\,,0) = sin(0.8\pi) = 0.5877 \end{aligned}$$

Equation (A) becomes (in Matrix form),

$$\begin{bmatrix} 1+2\lambda & -\lambda & 0 & 0 \\ -\lambda & 1+2\lambda & -\lambda & 0 \\ 0 & -\lambda & 1+2\lambda & -\lambda \\ 0 & 0 & -\lambda & 1+2\lambda \end{bmatrix} \begin{bmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \\ \omega_{4,1} \end{bmatrix} = \begin{bmatrix} 0.5877 \\ 0.9540 \\ 0.9510 \\ 0.5877 \end{bmatrix} \ldots (B)$$

Since

$$\lambda = \frac{\alpha^2 k}{h^2} = \frac{(1)^2(0.04)}{(0.2)^2} = 1$$

By Putting $\lambda = 1$ value equation (B) becomes

$$\begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \\ \omega_{4,1} \end{bmatrix} = \begin{bmatrix} 0.5877 \\ 0.9540 \\ 0.9510 \\ 0.5877 \end{bmatrix} \ldots (C)$$

$$A\ \ \omega = b$$

To find ω values, first we find inverse of $A\ i.e.\ A^{-1}$

$$\begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now apply Row Operations given below

1) $R_1 \leftrightarrow R_2 \ \ \& -1R_1$
2) $-3R_1 + R_2$
3) $R_2 \leftrightarrow R_3 \ \ \& -1R_2$
4) $3R_2 + R_1 \ \ \& -8R_2 + R_3$
5) $R_3 \leftrightarrow R_4$
6) $8R_3 + R_1\ ,\ 3R_3 + R_2 \ \ \& -21R_3 + R_4$
7) $\frac{1}{55}R_4$
8) $3R_4 + R_3\ ,\ 9R_4 + R_2 \ \ \& \ 21R_4 + R_1$

Hence our required value for

$$A^{-1} = \begin{bmatrix} \frac{21}{55} & \frac{8}{55} & \frac{3}{55} & \frac{-1}{55} \\ \frac{9}{55} & \frac{27}{55} & \frac{17}{55} & \frac{-24}{55} \\ \frac{3}{55} & \frac{9}{55} & \frac{24}{55} & \frac{-8}{55} \\ \frac{1}{55} & \frac{3}{55} & \frac{8}{55} & \frac{-21}{55} \end{bmatrix}$$

So equation (C) becomes,

$$\begin{bmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \\ \omega_{4,1} \end{bmatrix} = \frac{1}{55}\begin{bmatrix} 21 & 8 & 3 & -1 \\ 9 & 27 & 17 & -24 \\ 3 & 9 & 24 & -8 \\ 1 & 3 & 8 & -21 \end{bmatrix}\begin{bmatrix} 0.5877 \\ 0.9540 \\ 0.9510 \\ 0.5877 \end{bmatrix}$$

Hence we get

$$\omega_{1,1} = 0.40395 \qquad \omega_{2,1} = 0.60054$$
$$\omega_{3,1} = 0.51720 \qquad \omega_{4,1} = -0.02352$$

Required result.

LECTURE NO. 15

Example: 10

Find the numerical solution of following heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \dots \quad 1D$$

By backward difference method (implicit scheme) at $h = 0.25$ and $k = 0.005$

$$Boundary\ Condition; \qquad \{u(0,t) = u(1,t) = 0 \qquad \forall\, t$$

$$Initial\ Condition;\ u(x,0) = \begin{cases} 2x & , \quad 0 \le x \le \frac{1}{2},\ t = 0 \\ 2(1-x) & , \quad \frac{1}{2} < x \le 1,\ t = 0 \end{cases}$$

Solution:

We can write the given equation as,

$$\omega_{i,j} - \omega_{i,j-1} = \frac{k}{h^2}(\omega_{i-1,j} - 2\omega_{i,j} + \omega_{i+1,j}) \quad \dots (1)$$

Let

$$\frac{k}{h^2} = \lambda = 0.08$$

So, we can write equation (1) as, (without putting λ value)

$$\omega_{i,j} - \omega_{i,j-1} = \lambda\omega_{i-1,j} - 2\lambda\omega_{i,j} + \lambda\omega_{i+1,j}$$

Or by solving we have

$$\omega_{i,j-1} = -\lambda\omega_{i-1,j} + (1+2\lambda)\omega_{i,j} - \lambda\omega_{i+1,j}$$

By putting λ value, it becomes

$$\omega_{i,j-1} = -0.08\,\omega_{i-1,j} + 1.16\,\omega_{i,j} - 0.08\,\omega_{i+1,j} \quad \dots (2)$$

Since

$$n = \left(\frac{b-a}{h}\right) = \frac{1-0}{0.25} = 4 \qquad i = 1,2,3$$

Also

$$m = \left(\frac{T-t_0}{k}\right) = \frac{1-0}{0.005} = 200 \qquad j = 0,1,2,\dots,199$$

Since given conditions in "ω" form can be written as

$$Boundary\ Condition; \qquad \{\omega_{0,j} = \omega_{4,j} = 0$$

$$Initial\ Condition; \quad \begin{cases} \omega_{i,0} = 2x_i & , \quad i = 1,2 \\ \omega_{i,0} = 2(1 - x_i) & , \quad i = 3,4 \end{cases}$$

$\underline{Fix\ j = 1}$

$And\ put \quad i = 1,2,3 \quad in\ equation\ (2)$

$$\left.\begin{aligned} i = 1; \quad & \omega_{1,0} = -0.08\ \omega_{0,1} + 1.16\ \omega_{1,1} - 0.08\ \omega_{2,1} = 0.5 \\ i = 2; \quad & \omega_{2,0} = -0.08\ \omega_{1,1} + 1.16\ \omega_{2,1} - 0.08\ \omega_{3,1} = 1.0 \\ i = 3; \quad & \omega_{3,0} = -0.08\ \omega_{2,1} + 1.16\ \omega_{3,1} - 0.08\ \omega_{4,1} = 0.5 \end{aligned}\right\} \ldots (A)$$

Equation (A) becomes (in Matrix form),

$$\begin{bmatrix} 1.16 & -0.08 & 0 \\ -0.08 & 1.16 & -0.08 \\ 0 & -0.08 & 1.16 \end{bmatrix} \begin{bmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.0 \\ 0.5 \end{bmatrix} \ldots (B)$$

$$A\ \ \omega = b$$

To find ω values, first we find inverse of $A\ i.e.\ A^{-1}$

Note: You can find A^{-1} value by applying row operation, as we did in previous example. So,

$$A^{-1} = \begin{bmatrix} 0.8662 & 0.0600 & 0.0041 \\ 0.0600 & 0.8703 & 0.0600 \\ 0.0041 & 0.0600 & 0.8620 \end{bmatrix}$$

So equation (B) becomes,

$$\omega = A^{-1}\ b$$

$$\begin{bmatrix} \omega_{1,1} \\ \omega_{2,1} \\ \omega_{3,1} \end{bmatrix} = \begin{bmatrix} 0.8662 & 0.0600 & 0.0041 \\ 0.0600 & 0.8703 & 0.0600 \\ 0.0041 & 0.0600 & 0.8620 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1.0 \\ 0.5 \end{bmatrix}$$

Hence we get

$$\omega_{1,1} = 0.49517 \qquad \omega_{2,1} = 0.93032 \qquad \omega_{3,1} = 0.49307$$

Required result.

LECTURE NO. 16

Exercise:

Question#1: Find the numerical solution of following heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,\ t)}{\partial x^2} \quad ,\ 0 \le x \le 1\ , t > 0$$

By backward difference method (implicit scheme) at $T = 1, h = 0.1\ and\ k = 0.005$

$$Boundary\ Condition; \quad \{u(0,t) = u(1,t) = 0 \qquad \forall\ t$$

$$Initial\ Condition;\ u(x,0) = \begin{cases} 2x & , \quad 0 \le x \le \frac{1}{2}\ ,\ t = 0 \\ 2(1-x) & , \quad \frac{1}{2} < x \le 1\ ,\ t = 0 \end{cases}$$

Hint:

$$n = 10 \quad , \quad i = 1,2, \ldots, 9$$
$$m = 200 \quad , \quad j = 1,2, \ldots, 99$$

Question#2: Find the numerical solution of following heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad ,\ 0 \le x \le 1\ , t > 0$$

By backward difference method (implicit scheme) at $T = 1, h = 0.1\ and\ k = 0.01$

$$Boundary\ Condition; \quad \{u(0,t)\ = u(1,t)\ = 0 \qquad \forall\ t > 0$$

$$Initial\ Condition; \quad u(x,0)\ = sin\pi x \qquad 0 \le x \le 1$$

Hint:

$$n = 10 \quad , \quad i = 1,2, \ldots, 9$$
$$m = 200 \quad , \quad j = 1,2, \ldots, 99$$
$$A \rightarrow 9 \times 9 \quad , \quad A^{-1} = ? \quad , \quad A\omega = b \quad , \quad \omega = ?$$

Question#3: Find the numerical solution of following heat equation

$$\frac{\partial u}{\partial t} = \frac{4}{\pi^2}\frac{\partial^2 u}{\partial x^2} \quad , \quad 0 \le x \le 4 \, , t > 0$$

By both forward and backward difference methods at $T = 0.08$, $h = 0.2$ and $k = 0.04$

$$Boundary\ Condition; \qquad \{u(0,t) = u(4,t) = 0 \qquad \forall\, t$$

$$Initial\ condition; \qquad u(x,0) = \sin\frac{\pi x}{4}(1 + 2\cos\frac{\pi x}{4}) \quad , \quad 0 \le x \le 4$$

Exact Solution:

$$u(x,t) = e^{-t}\, sin\,(\frac{\pi x}{2}) + e^{\frac{-t}{4}} sin\,(\frac{\pi}{4})$$

Question#4: Find the numerical solution of following heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2} \quad , \quad 0 \le x \le 1 \, , t > 0$$

By both explicit and implicit methods at $T = 0.08$, $h = 0.1$ and $k = 0.04$

$$Boundary\ Condition; \qquad \{u(0,t) = u(1,t) = 0 \qquad \forall\, t > 0$$

$$Initial\ Condition; \qquad u(x,0) = cos\pi\,(x - \frac{1}{2}) \quad , \quad 0 \le x \le 1$$

Exact Solution:

$$u(x,t) = e^{-t}\, cos\pi\,(x - \frac{1}{2})$$

LECTURE NO. 17

Crout's Method to Solve Tridiagonal System of Equation:

Consider a 4×4 system of linear equations

$$\begin{bmatrix} a_{11}x_1 & a_{12}x_2 & 0 & 0 \\ a_{21}x_1 & a_{22}x_2 & a_{23}x_3 & 0 \\ 0 & a_{32}x_2 & a_{33}x_3 & a_{34}x_4 \\ 0 & 0 & a_{43}x_3 & a_{44}x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \dots(1)$$

Consider $\quad Ax = b \quad \dots(2)$

Let $\quad A = LU$

Then equation (2) becomes

$$LUx = b$$

So $\quad Ux = y \quad \& \quad Ly = b$

Since $\quad LU = A$

Where

$$L = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{bmatrix}, U = \begin{bmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}$$

Since

$$A = LU$$

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & 0 & 0 \\ l_{21} & l_{21}u_{12}+l_{22} & l_{22}u_{23} & 0 \\ 0 & l_{32} & l_{32}u_{23}+l_{33} & l_{33}u_{34} \\ 0 & 0 & l_{43} & l_{43}u_{34}+l_{44} \end{bmatrix}$$

By comparing, we have

$$l_{11} = a_{11} \qquad l_{11}u_{12} = a_{12} \Rightarrow u_{12} = \frac{a_{12}}{l_{11}}$$

$$l_{21} = a_{21} \qquad l_{21}u_{12} + l_{22} = a_{22} \Rightarrow l_{22} = a_{22} - l_{21}u_{12}$$

$$l_{32} = a_{32} \qquad l_{22}u_{23} = a_{23} \Rightarrow u_{23} = \frac{a_{23}}{l_{22}}$$

$$l_{43} = a_{43} \qquad l_{32}u_{23} + l_{33} = a_{33} \Rightarrow l_{33} = a_{33} - l_{32}u_{23}$$

$$l_{33}u_{34} = a_{34} \Rightarrow u_{34} = \frac{a_{34}}{l_{33}} \qquad l_{43}u_{34} + l_{44} = a_{44} \Rightarrow l_{44} = a_{44} - l_{43}u_{34}$$

In general, we can write

$Step-I$: Set $l_{11} = a_{11}$, $u_{12} = \frac{a_{12}}{l_{11}}$

$Step-II$: For $i = 1,2, \dots n-1$

$$l_{i,i-1} = a_{i,i-1}$$

$$l_{ii} = a_{ii} + l_{i,i-1} \,.\, u_{i-1,i} \quad ; \quad i = 2, \dots, n-1$$

$$u_{i,i+1} = \frac{a_{i,i+1}}{l_{ii}} \quad ; \quad i = 2, \dots, n-1$$

$Step-III$:

$$l_{n,n-1} = a_{n,n-1}$$

$$l_{n,n} = a_{n,n} - l_{n,n-1} \,.\, u_{n-1,n}$$

As

$$Ly = b$$

$$\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

So,

$$y_1 = \frac{b_1}{l_{11}} \quad , \quad y_2 = \frac{1}{l_{22}}(b_2 - l_{21}y_1)$$

$$y_3 = \frac{1}{l_{33}}(b_3 - l_{32}y_2) \quad , \quad y_4 = \frac{1}{l_{44}}(b_4 - l_{43}y_3)$$

$Step-IV$: Set $y_1 = \frac{b_1}{l_{11}}$

$Step-V$: For $i = 1,2, \dots n$

$$y_i = \frac{1}{l_{ii}}(b_i - l_{i,i-1} \,.\, y_{i-1})$$

Since

$$Ux = y$$

$$\begin{bmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

So,

$$\begin{cases} x_1 + u_{12}x_2 = y_1 \\ x_2 + u_{23}x_3 = y_2 \\ x_3 + u_{34}x_4 = y_3 \\ x_4 = y_4 \end{cases} \quad This\ implies \Rightarrow \quad \begin{cases} x_1 = y_1 - u_{12}x_2 \\ x_2 = y_2 - u_{23}x_3 \\ x_3 = y_3 - u_{34}x_4 \\ x_4 = y_4 \end{cases}$$

$Step-VI$: Set $x_n = y_n$

$Step-VII$: For $i = 1,2, \dots n-1$

$$x_i = y_i - u_{i,i+1} \,.\, x_{i+1}$$

These are the seven steps to solve the Crout's Method.

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Example: 11

Solve the following system of linear equations by Crout's method.

$$a_1x_1 + c_1x_2 = \alpha_1$$
$$b_2x_1 + a_2x_2 + c_2x_3 = \alpha_2$$
$$b_3x_2 + a_3x_3 + c_3x_4 = \alpha_3$$
$$b_4x_3 + a_4x_4 = \alpha_4$$

Solution: In matrix form, we can write it as

$$\begin{bmatrix} a_1 & c_1 & 0 & 0 \\ b_2 & a_2 & c_2 & 0 \\ 0 & b_3 & a_3 & c_3 \\ 0 & 0 & b_4 & a_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \quad \dots (1)$$

So, $A\,x = b$

Matrix A can be decompose as $L\,U = A \quad \dots (2)$

Where

$$L = \begin{bmatrix} l_1 & 0 & 0 & 0 \\ b_2 & l_2 & 0 & 0 \\ 0 & b_3 & l_3 & 0 \\ 0 & 0 & b_4 & l_4 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & u_1 & 0 & 0 \\ 0 & 1 & u_2 & 0 \\ 0 & 0 & 1 & u_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So equation (2) can written as

$$L\,U = A$$

$$\begin{bmatrix} l_1 & 0 & 0 & 0 \\ b_2 & l_2 & 0 & 0 \\ 0 & b_3 & l_3 & 0 \\ 0 & 0 & b_4 & l_4 \end{bmatrix} \begin{bmatrix} 1 & u_1 & 0 & 0 \\ 0 & 1 & u_2 & 0 \\ 0 & 0 & 1 & u_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & c_1 & 0 & 0 \\ b_2 & a_2 & c_2 & 0 \\ 0 & b_3 & a_3 & c_3 \\ 0 & 0 & b_4 & a_4 \end{bmatrix}$$

$$\begin{bmatrix} l_1 & l_1u_1 & 0 & 0 \\ b_2 & b_2u_1 + l_2 & l_2u_2 & 0 \\ 0 & b_3 & b_2u_2 + l_3 & l_3u_3 \\ 0 & 0 & b_4 & b_4u_3 + l_4 \end{bmatrix} = \begin{bmatrix} a_1 & c_1 & 0 & 0 \\ b_2 & a_2 & c_2 & 0 \\ 0 & b_3 & a_3 & c_3 \\ 0 & 0 & b_4 & a_4 \end{bmatrix}$$

So,

$$l_1 = a_1 \quad , \quad u_1 = {c_1}/{l_1} \quad , \quad u_2 = {c_2}/{l_2} \quad , \quad l_2 = a_2 - b_2u_1$$

$$u_3 = {c_3}/{l_3} \quad , \quad l_3 = a_3 - b_3u_2 \quad , \quad l_4 = a_4 - b_4u_3$$

$Step - I$: Set $l_1 = a_1 \quad , \quad u_1 = {c_1}/{l_1}$

$Step - II$: For $i = 1,2, \dots n-1$

$$l_i = a_i - b_i\,.\,u_{i-1} \quad , \quad u_i = {c_i}/{l_i}$$

$Step - III$: $l_n = a_n - b_n\,.\,u_{n-1}$

So, U becomes,

$$U = \begin{bmatrix} 1 & {c_1}/{l_1} & 0 & 0 \\ 0 & 1 & {c_2}/{l_2} & 0 \\ 0 & 0 & 1 & {c_3}/{l_3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now as, $L\,y = b$ so,

$$\begin{bmatrix} l_1 & 0 & 0 & 0 \\ b_2 & l_2 & 0 & 0 \\ 0 & b_3 & l_3 & 0 \\ 0 & 0 & b_4 & l_4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

So,

$$\begin{cases} y_1 = \dfrac{d_1}{l_1} \\ y_2 = \dfrac{1}{l_2}(d_2 - b_2 y_1) \\ y_3 = \dfrac{1}{l_3}(d_3 - b_3 y_2) \\ y_4 = \dfrac{1}{l_4}(d_4 - b_4 y_3) \end{cases}$$

$Step-IV$: Set $y_1 = \frac{d_1}{l_1}$

$Step-V$: For $i = 1,2, \dots n$

$$y_i = \frac{1}{l_i}(d_i - b_i \,.\, y_{i-1})$$

Now, $U\,x = y$

$$\begin{bmatrix} 1 & u_1 & 0 & 0 \\ 0 & 1 & u_2 & 0 \\ 0 & 0 & 1 & u_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

So,

$$\begin{cases} x_1 = y_1 - u_1 x_2 \\ x_2 = y_2 - u_2 x_3 \\ x_3 = y_3 - u_3 x_4 \\ x_4 = y_4 \end{cases}$$

$Step-VI$: Set $x_n = y_n$

$Step-V$: For $i = 1,2, \dots n-1$

$$x_i = y_i - u_i \,.\, x_{i+1}$$

As,

$$U = \begin{bmatrix} 1 & {c_1}/{l_1} & 0 & 0 \\ 0 & 1 & {c_2}/{l_2} & 0 \\ 0 & 0 & 1 & {c_3}/{l_3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad So, by\ putting\ values \quad \begin{cases} x_1 = y_1 - {c_1}/{l_1}\, x_2 \\ x_2 = y_2 - {c_2}/{l_2}\, x_3 \\ x_3 = y_3 - {c_3}/{l_3}\, x_4 \\ x_4 = y_4 \end{cases}$$

This is our required result.

Example: 12

Solve the following system of linear equations by Crout's method.

$$0.5x_1 + 0.25x_2 = 0.35$$
$$0.3x_1 + 0.80x_2 + 0.40x_3 = 0.77$$
$$0.25x_2 + x_3 + 0.5x_4 = -0.5$$
$$x_3 - 2.0x_4 = -2.25$$

Solution: Since

$$A = \begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0.3 & 0.80 & 0.40 & 0 \\ 0 & 0.25 & 1 & 0.5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

As, $A = L\,U$

$$\begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0.3 & 0.80 & 0.40 & 0 \\ 0 & 0.25 & 1 & 0.5 \\ 0 & 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & u_1 & 0 & 0 \\ 0 & 1 & u_2 & 0 \\ 0 & 0 & 1 & u_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$Step - I$:

$$l_{11} = a_{11} = 0.5 \quad , \quad u_{12} = \frac{a_{12}}{l_{11}} = \frac{0.25}{0.5} = 0.05$$

$Step - II$:

$$l_{i,i-1} = a_{i,i-1} \quad ; \quad i = 2,3,4$$
$$l_{21} = a_{21} = 0.3 \quad , \quad l_{32} = a_{32} = 0.25 \quad , \quad l_{43} = a_{43} = 1$$
$$l_{ii} = a_{ii} - l_{i,i-1} \,.\, u_{i-1,i} \quad ; \quad i = 2,3,4$$
$$l_{22} = a_{22} - l_{21} \,.\, u_{12} = 0.8 - (0.3)(0.05) = 0.7825$$
$$l_{33} = a_{33} - l_{32} \,.\, u_{23} = 1 - (0.25)(0.511) = 0.7822$$
$$l_{44} = a_{44} - l_{43} \,.\, u_{34} = -2 - (1)(-0.5733) = -2.5733$$

$Step - III$:

$$u_{i,i+1} = \frac{a_{i,i+1}}{l_{ii}} \quad ; \quad i = 2,3$$
$$u_{23} = \frac{a_{23}}{l_{22}} = \frac{0.4}{0.7825} = 0.511$$
$$u_{34} = \frac{a_{34}}{l_{33}} = \frac{0.5}{0.8722} = 0.5733$$

Now as, $L\,y = b$

$$\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.3 & 0.7825 & 0 & 0 \\ 0 & 0.25 & 0.872 & 0 \\ 0 & 0 & 1 & -2.5733 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0.35 \\ 0.77 \\ -0.5 \\ -2.25 \end{bmatrix}$$

This implies,

$$y_1 = 0.70 \quad , \quad y_2 = 0.6709 \quad , \quad y_3 = -0.7656 \quad , \quad y_4 = 0.5768$$

Now as, $U\,x = y$

$$\begin{bmatrix} 1 & 0.05 & 0 & 0 \\ 0 & 1 & 0.511 & 0 \\ 0 & 0 & 1 & 0.5733 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.70 \\ 0.6709 \\ -0.7656 \\ 0.5768 \end{bmatrix}$$

This implies,

$$x_1 = 0.6384 \quad , \quad x_2 = 1.2312 \quad , \quad x_3 = -1.9620 \quad , \quad x_4 = 0.5768$$

This is our required result.

"Good Luck For The Mid-Term Exam"