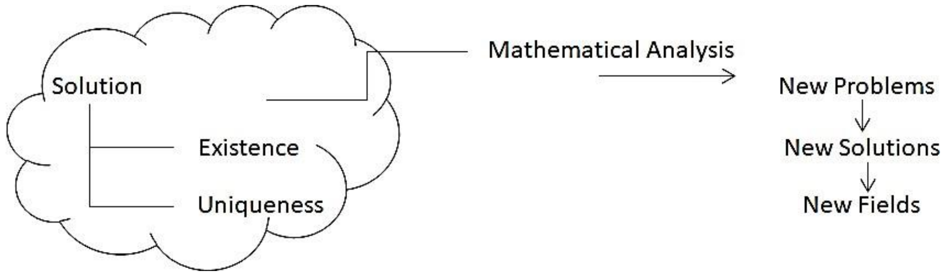
# LECTURE NO. 01

**Mathematical Modelling:** It is the approximation of objects with geometrical objects.
**Mathematical Formulation:** In this step we drive the equation corresponds to the given phenomenon. It is based on existing law i.e. Newton's law of motion, laws of thermodynamics. If laws are not available for given existing problem, then we go for experiments.

Example: $F_e = G\frac{m_e m_s}{(r_{es})^2}$ ① $\frac{d^2 r}{dt^2} = G\frac{m}{r}$

# LECTURE NO. 02




Example: Heat flow problem → Formulation → PDEs → Solutions → Expansion of function terms of 'sine' and 'cosine' (Fourier analysis).

**Physical Interpretation:** E.g. $x = \frac{dx}{dt}$ ① $x(t) = Ae^t$ $\therefore A \in R$

# LECTURE NO. 03

**Vibrating String Equation:**

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Here we'll try to derive the above given equation. So,
Let $'\Delta s'$ represents the elements of arc of string and $'T'$ is tension at $'x'$ and $'x+\Delta x'$ (constant).

$$Net\ upward\ force = T\ sin\theta_2 - T\ sin\theta_1$$

For small angles; $sin\theta \approx tan\theta$ $\therefore sin0 = tan0 = 0$

So $Net\ force = T\ tan\theta_2 - T\ tan\theta_1$ ...... (1)

By newton second law of motion

$$Force = ma = (mass\ of\ \Delta s)(acceleration\ of\ \Delta s)$$

$$Force = (\mu\ \Delta s)\ (\frac{6^2 y}{6t^2} + s)$$ ...... (2) $\therefore \mu$ Is linear mass density

Note: Here acceleration is in derivative form and while discussing the acceleration for a segment (large number of particles), there will be some approximation/error. That's why we are using $'c'$ as approximation symbol.

By comparing equation 1 and 2, we have

$$T\ tan\theta_2 - T\ tan\theta_1 = (\mu\ \Delta s)\ (\frac{\partial^2 y}{\partial t^2} + s)$$

$$T[tan\theta_2 - tan\theta_1] = (\mu\ \Delta s)\ (\frac{\partial^2 y}{\partial t^2} + s)$$

As $tan\theta = \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$, so above equation become

$$T\ [(\frac{dy}{dx})_{x+\Delta x} - (\frac{dy}{dx})_x] = (\mu\ \Delta s)\ (\frac{\partial^2 y}{\partial t^2} + s)$$

Note: Here $'x + \Delta x\ and\ x'$ are $'final\ and\ initial\ points'$ respectively.

For small vibrations; $\Delta s = \Delta x$ ① $\mu\ \Delta s = \mu\ \Delta x$

By using this relation, we get

$$T\,[(\frac{dy}{dx})_{x+\Delta x} - (\frac{dy}{dx})_x] = \mu\,\Delta x\,(\frac{\partial^2 y}{\partial t^2} + s)$$

Dividing by $\Delta x$ on both sides

$$T\,[\frac{(\frac{dy}{dx})_{x+\Delta x} - (\frac{dy}{dx})_x}{\Delta x}] = \frac{\mu\,\Delta x}{\Delta x}\,(\frac{\partial^2 y}{\partial t^2} + s)$$

$$\frac{T}{\mu}[\frac{(\frac{dy}{dx})_{x+\Delta x} - (\frac{dy}{dx})_x}{\Delta x}] = (\frac{\partial^2 y}{\partial t^2} + s)$$

For $\Delta x \to 0$ and $s \to 0$, it becomes

$$\frac{T}{\mu}[\frac{\partial}{\partial x}(\frac{\partial y}{\partial x})] = \frac{\partial^2 y}{\partial t^2}$$

$$\frac{T}{\mu}(\frac{\partial^2 y}{\partial x^2}) = \frac{\partial^2 y}{\partial t^2}$$

Here $\frac{T}{\mu} = a^2$, so equation becomes

$$a^2\,(\frac{\partial^2 y}{\partial x^2}) = \frac{\partial^2 y}{\partial t^2}$$

Or it can be written as

$$\frac{\partial^2 y}{\partial t^2} = a^2\,(\frac{\partial^2 y}{\partial x^2})$$

This relation is known as Vibrating string equation.

LECTURE NO. 04

**Boundary Conditions for Vibrating String Equation:**

Statement: Write boundary conditions for a vibrating string of length $'L'$ for which i) Ends $x = 0$ and $x = L$ are fixed. ii) Initial shape of string is $f(x)$. iii) Initial velocity distribution is $g(x)$. iv) Displacement at any instant $'t'$ is bounded.

**Solution:**

i) String is fixed at $x = 0$ and $x = L$, then its displacement at

$$starting\ point: y(0,t) = 0 = end\ point: y(L,t) \qquad ✯\ t > 0$$

Note: above relation is in "$function\ (displacement, time) = y(x,t)$" form. At starting point "$x = 0$" and at final point "$x = L$".

ii) Initial shape at $t = 0$ is given as

$$y(x,0) = f(x)$$

Note: Just put $t = 0$ in "$y(x,t)$" relation and here $f(x)$ is just an expression.

iii) Initial velocity at time $'t'$ is given as

$$\frac{\partial y}{\partial t} = y_t(x,t)$$

At $t = 0$

$$y_t(x,0) = g(x) \qquad ✯ \quad 0 < x < L$$

Note: Here velocity is in derivative form and $g(x)$ is just an expression.

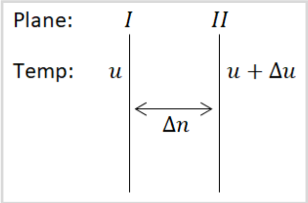
iv) Displacement is bounded: It implies that their exist some finite real number $(M < \infty)$ such that

$$|y(x,t)| < M \qquad ✯ \quad 0 < x < L\ \ and\ \ t > 0$$

These are some examples of boundary conditions applied on vibrating string equation.

LECTURE NO. 05

**Heat Flux across the Plane:**




Suppose planes $'I'$ and $'II'$ are parallel and at distance $'\Delta n'$ apart. Let

$u = temp\ of\ plane\ I$ And $u + \Delta u = temp\ of\ plane\ II$

Heat flows from higher temp plane to lower temp plane.

Heat flux is defined as; Heat per unit area per unit time. Mathematically

$$Heat\ flux \propto \frac{difference\ of\ temperature}{distance\ between\ planes}$$

$$Heat\ flux\ form\ I\ to\ II = k\frac{(u) - (u + \Delta u)}{\Delta n} \Rightarrow -k\ \frac{\Delta u}{\Delta n}$$

Where $'k'$ is constant of proportionality and is called thermal conductivity.

Thermal conductivity is defined as; Measure of how well a material can conduct or transfer heat.

Note: As in this case heat flows from plane $'I\ to\ II'$ (lower temp to higher temp), that's why we have the negative value of heat flux. $\therefore \Delta u > 0$ (Heat value will be positive) if heat flows from plane $'II\ to\ I'$ (higher temp to low temp) and vice versa.

For limiting case: If $\Delta n \to 0\ \ and\ \ \Delta u \to 0$ then

$$Heat\ flow\ across\ plane\ I = -k\frac{\partial u}{\partial n}$$

In vector form

$$Heat\ flow\ across\ plane\ I = \{-k\frac{\nabla u}{\nabla n}\}$$

Note: As $\Delta n \to 0$ (distance between planes approaches to zero), so we can write it as $plane\ 'I'$ only.

LECTURE NO. 06

**Partial Differential Equations (Definition and Related Terms):**

Definition: It is an equation containing unknown functions of two or more variables and partial derivatives w.r.to these variables. Example

$$\frac{\partial^2 u}{\partial x dy} = 2x - y \qquad order = 2$$

**Order:** It is the order of highest derivative involved in PDE.

**Solution:** It is a function which satisfy the given DE. Example:

$$u = x^2 y - \frac{1}{2}xy^2$$

$$\frac{\partial u}{\partial y} = x^2 - xy \qquad 1st\ derivative\ w.r.to\ 'y'$$

$$\frac{\partial^2 u}{\partial x dy} = \frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) = 2x - y \qquad 2nd\ derivative\ w.r.to\ 'x'$$

Similarly; $u = x^2 y - \frac{1}{2}xy^2 + F(x) + G(y)$ is also a solution. Here $'F(x) and\ G(x)'$ are arbitrary functions.

**Particular Solution:** It is obtained from the general solution by particular choice of arbitrary functions. Example:

$$u = x^2 y - \frac{1}{2}xy^2 + 2sinx + 3y^4 - 5$$

**Singular Solution:** It cannot be obtained from the general solution by choosing arbitrary functions.

LECTURE NO. 07

Example: $xy' = \sqrt{y} \Rightarrow y = \frac{1}{4}(lncx)^2$ (General Solution—Here $'c'$ is arbitrary constant).

But $'y = 0'$ is also a solution of given DE.As it is not obtained from general solution, that's why it is called singular solution. ($\nexists\ c \in R\ such\ that\ y = 0$)

LECTURE NO. 08

**Linear PDEs and Their Classification:**

General form of linear partial differential equation of order 2in two independent variables is of the form

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial xdy}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+Fu=G$$

Here $A, B, C, D, E, F\ and\ G$ may depend on $'x'$ or $'y'$ only but not on both $'x'$and $'y'$.

If at least one of these$(A, B, C, D, E, F\ and\ G)$ is function of $'x'$ and $'y'$, then it is non-linear.

If $G = 0$, then it is homogeneous.

Example:

$$x^2\frac{\partial^3 u}{\partial y^3}=y^3\frac{\partial^2 u}{\partial x^2} \qquad (order = 3, linear)$$

Example:

$$(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2=1 \qquad (order = 1, non-linear)$$

LECTURE NO. 09

**Topic Continue…**

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial xdy}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+Fu=G$$

It is defined as

i) Elliptical if $B^2 - 4AC < 0$

ii) Hyperbolic if $B^2 - 4AC > 0$

iii) Parabolic if $B^2 - 4AC = 0$

Example:

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial y^2}+3y^2\frac{\partial u}{\partial x}=0$$

Here $A = x, b = 0\ and\ c = y$. Now

$$B^2 - 4AC = -4xy \begin{cases} xy > 0, then\ it\ is\ elliptic \\ xy < 0, then\ it\ is\ hyperbolic \\ xy = 0, then\ it\ is\ parabolic \end{cases}$$

LECTURE NO. 10

**Show that** $u(x,t) = e^{-8t} sin2x,$ is a solution to BVP (Boundary Value Problem):

$$I)\ \frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, II)\ u(0,t) = u(\pi,t) = 0, III)\ u(x,0) = sin2x$$

Solution:

$II)$ $u(x,t) = e^{-8t} sin2x => u(0,t) = e^{-8t}\sin(0) = 0$

$II)$ $u(x,t) = e^{-8t} sin2x => u(\pi,t) = e^{-8t}\sin(2\pi) = 0$

$III)$ $u(x,t) = e^{-8t} sin2x => u(x,0) = e^{-8(0)}\sin 2x = sin2x$

Now, try to calculate the values for condition $'I'$

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \qquad \ldots\ldots(1)$$

Taking derivative of equation 1 w.r.to $'t'$

$$\frac{\partial u}{\partial t} = -8e^{-8t} sin2x$$

Taking derivative of equation 1 w.r.to $'x'$

$$\frac{\partial u}{\partial x} = 2e^{-8t}cos2x$$

Taking 2nd derivative, we have

$$\frac{\partial^2 u}{\partial x^2} = -4e^{-8t}sin2x$$

Now, by putting the calculated values in equation 1 and checking, we get

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = -8e^{-8t}sin2x \Rightarrow 2(-4e^{-8t}sin2x) = 2\frac{\partial^2 u}{\partial x^2}$$

Or simply

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$$

Hence proved$('I')$ that the given function "$u(x,t) = e^{-8t}sin2x$" is solution to given BVP.

LECTURE NO. 11

**Show that** $v = F(y - 3x)$, where $'F'$ is arbitrary differentiable function.

$I$) The general solution of equation is $\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = 0.$

$II$) Also find its particular solution if $v(0, y) = 4siny$

Solution:

$I$) Given that $v = F(y - 3x)$ ...... (1)

Let / say $v(x, y) = F(u) Where\ u = y - 3x$

$$v = F(u) and\ u = u(x, y)$$

By applying chain rule w.r.to $'x'$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial u}.\frac{\partial u}{\partial x} = F'(u)(-3) \Rightarrow -3F'(u)$$

By applying chain rule w.r.to $'y'$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial u}.\frac{\partial u}{\partial y} = F'(u)(1) \Rightarrow F'(u)$$

Now by putting value in equation

$$\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = -3F'(u) + 3F'(u)$$

By solving, we have

$$\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = 0$$

Hence, the general solution of equation proved.

$II$) Now, also given that

$$v(x, y) = F(y - 3x)$$

Given condition

$$v(0, y) = F(y) = 4siny$$

It implies

$$v(x, y) = F(y - 3x) = 4\sin(y - 3x)$$

This is our required particular solution.

LECTURE NO. 12

**Solving PDEs by the Method of ODEs:**

Question: $t\frac{\partial^2 u}{\partial x\partial t}+2\frac{\partial u}{\partial x}=2tx$

Solution: Given that

$$t\frac{\partial^2 u}{\partial x\partial t}+2\frac{\partial u}{\partial x}=2tx$$
$$t\frac{\partial}{\partial x}(\frac{\partial u}{\partial t})+2\frac{\partial}{\partial x}u=2tx$$
$$\frac{\partial}{\partial x}\{t\frac{\partial u}{\partial t}+2u\}=2tx$$

Integrating w.r.to $'x'$

$$\int\frac{\partial}{\partial x}\{t\frac{\partial u}{\partial t}+2u\}\,dx=\int 2tx\,dx$$
$$t\frac{\partial u}{\partial t}+2u=x^2t+G(t)$$

Dividing by $'t'$ on both sides

$$\frac{\partial u}{\partial t}+\frac{2}{t}u=x^2+\frac{G(t)}{t}$$
$$Integrating\ Factor=I.F=e^{\int\frac{2}{t}dt}$$
$$I.F=e^{2\ln t}=e^{\ln t^2}=t^2$$

Required solution is

$$I.F\times u=\int(R.H.S\times I.F)dt+F(x)$$

By putting values

$$t^2u=\int((x^2+\frac{G(t)}{t})\times t^2)\,dt+F(x)$$
$$t^2u=\int(x^2t^2+\frac{G(t)}{t}t^2)\,dt+F(x)$$
$$t^2u=x^2\frac{t^3}{3}+\int t.G(t)dt+F(x)$$
$$t^2u=x^2\frac{t^3}{3}+H(t)+F(x)$$

LECTURE NO. 13

**General Solution for Solving PDEs:**

Question $\frac{\partial^2 u}{\partial x^2}+3\frac{\partial^2 u}{\partial x\partial t}+2\frac{\partial^2 u}{\partial y^2}=0$ ...... (1)

Solution: Let

$$u=e^{ax+by}$$

Taking 1st and 2nd derivatives w.r.to $'x'$

$$\frac{\partial u}{\partial x}=e^{ax+by}\frac{\partial}{\partial x}(ax+by)=au$$
$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial u}{\partial x})=\frac{\partial}{\partial x}(au)=a\frac{\partial u}{\partial x}=a^2u$$

Similarly taking derivatives w.r.to $'y'$

$$\frac{\partial^2 u}{\partial y^2}=b^2u$$

And

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}(\frac{\partial u}{\partial y})=\frac{\partial}{\partial x}(bu)=b\frac{\partial u}{\partial x}=abu$$

Putting values in equation 1, we have

$$a^2u + 3abu + 2b^2u = 0$$
$$(a^2 + 3ab + 2b^2)u = 0$$
$$(a + 2b)(a + b) = 0$$

By solving, we get

$$a = -2b \;\; ; \;\; a = -b$$

$\underline{For\ a = -2b}$

$$u_1 = e^{-2bx+by} = e^{b(y-2x)}$$

$\underline{For\ a = -b}$

$$u_2 = e^{-bx+by} = e^{b(y-x)}$$

Given PDE is homogeneous, therefore by superposition principle

$$u = \alpha u_1 + \beta u_2$$

By putting values

$$u = \alpha e^{b(y-2x)} + \beta e^{b(y-x)}$$

Let

$$e^{b(y-2x)} = G(y-2x) \;\; and \;\; e^{b(y-x)} = H(y-x)$$

So, above equation becomes

$$u = G(y-2x) + H(y-x)$$

Required general solution of given PDE.

LECTURE NO. 14

**Solving PDEs by Separation of Variables:**

Question:

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \;\; subject\ to\ boundry\ conditio\ u(0,y) = 8e^{-3y}$$

Solution: Given equation

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \quad \ldots(1)$$

Let $u(x,y) = X(x)\,Y(y) = XY$ Be its solution… (i)

Taking derivative w.r.to $x$ & $y$

$$\frac{\partial u}{\partial x} = X'Y \;\; and \;\; \frac{\partial u}{\partial y} = XY'$$

Putting values in equation 1, we have

$$X'Y = 4\,XY'$$
$$\frac{X'}{4X} = \frac{Y'}{Y} \quad \ldots(2)$$

As

$$X(x) = X\,,Y(y) = Y$$

Here $X\ and\ Y$ are independent variables. Each side of equation 2 must be a constant say $'c'$. So,

| | |
|---|---|
| $\frac{X'}{4X} = c$ | $\frac{Y'}{Y} = c$ |
| $X' = 4cX$ | $Y' = cY$ |
| $\frac{dX}{dx} = 4cX$ | $\frac{dY}{dy} = cy$ |
| $\frac{dX}{X} = 4cdx$ | $\frac{dY}{Y} = cdy$ |

By integrating

| | |
|---|---|
| $\int \frac{dX}{X} = 4c\int dx$ | $\int \frac{dY}{Y} = c\int dy$ |
| $lnX = 4cx + A$ | $lnY = cy + B$ |
| $X = e^{4cx+A}$ | $Y = e^{cy+B}$ |
| $X = e^{A}e^{4cx}$ | $Y = e^{B}e^{cy}$ |
| $X = k_1e^{4cx}$ | $Y = k_2e^{cy}$ |

Now by putting values in equation (i)

$$u(x,y) = XY = k_1k_2e^{4cx+cy} \quad \text{...... (3)}$$

By using boundary condition $u(0,y) = 8e^{-3y}$ in equation 3, we have

$$u(0,y) = k\ e^{cy} = 8e^{-3y}$$

Here $k = 8\ \ and\ c = -3,$

Hence

$$u(x,y) = 8e^{-3(4x+y)} = 8e^{-12x-3y}$$

LECTURE NO. 15

**Heat Conduction Equation and its Physical Interpretation:**

Question:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \quad ...(1) \qquad \therefore 0 < x < 3\ , t > 0$$

Given that (boundary values)

$$u(0,t) = u(3,t) = 0 \qquad ...(i),$$

$$u(x,0) = 5sin4\pi x - 3sin8\pi x + 2sin10\pi x \qquad ...(ii) \therefore |u(x,t)| < M < \infty$$

Solution: Let

$$u(x,t) = X(x).T(t) = XT\ \ ...(z)$$

Taking derivative w.r.to $'t'$

$$\frac{\partial u}{\partial t} = XT'$$

Taking 1st derivative w.r.to $'x'$

$$\frac{\partial u}{\partial x} = X'T$$

Taking 2nd derivative w.r.to $'x'$

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

By putting values equation 1 becomes

$$XT' = 2X''T$$

$$\frac{X''}{X} = \frac{T'}{2T} = -\lambda^2\ (say)$$

So,

$$\frac{X''}{X} = -\lambda^2$$

$$X'' + \lambda^2 X = 0\ ...(a)$$

$$Let\ X = e^{mx}$$

$$X'' = m^2e^{mx} = m^2X$$

$$(a) => m^2X + \lambda^2 X = 0$$

$$m = \pm\lambda i\ \&\ X \neq 0$$

$$\therefore X(x) = \alpha_1 e^{\lambda ix} + \alpha_2 e^{-\lambda ix}$$

$$X(x) = \alpha_1[cos\lambda x + isin\lambda x] + \alpha_2[cos\lambda x - isin\lambda x]$$

$$X(x) = (\alpha_1 + \alpha_2)cos\lambda x + i(\alpha_1 - \alpha_2)sin\lambda x$$

$$X(x) = A_1cos\lambda x + B_1sin\lambda x$$

$$\frac{T'}{2T} = -\lambda^2$$

$$Taking\ derivative$$

$$\frac{dT}{dt} = -2\lambda^2 T$$

$$\frac{dT}{T} = -2\lambda^2\ dt$$

$$\int\frac{dT}{T} = -2\lambda^2\int dt$$

$$ln\ T = -2\lambda^2 t + c$$

$$T = e^{-2\lambda^2 t + c}$$

$$T = e^c.e^{-2\lambda^2 t}$$

$$T = c_1e^{-2\lambda^2 t}$$

$$Here\ e^c = c_1$$

Here we were solving both functions side by side. Now

Putting values in equation $(z)$, we get

$$u(x,t) = XT = e^{-2\lambda^2 t}(A_1cos\lambda x + B_1sin\lambda x) \quad ...(2)$$

$Calculating\ (i)\ (boundary\ value)\ by\ putting\ values\ in\ equaition\ 2,$

$$u(0,t) = 0 = e^{-2\lambda^2 t}(Acos\lambda(0) + Bsin\lambda(0)) = e^{-2\lambda^2 t}.A \quad => \quad A = 0 \quad \therefore e^{-2\lambda^2 t} \neq 0$$

Equation 2 becomes

$$u(x,t) = XT = B.e^{-2\lambda^2 t} sin\lambda x \qquad \therefore A = 0$$

$Calculating\ second\ part\ of\ (i)\ by\ putting\ values\ in\ above\ relation$

$$u(3,t) = 0 = e^{-2\lambda^2 t}.Bsin3\lambda$$

Two Cases

$$1;\ if\ B = 0 \ => u(x,t) = 0 \qquad as\ A = 0\ already$$

$$2;\ if\ B \neq 0\ but\ sin3\lambda = 0 => 3\lambda = m\pi \ => \ \lambda = \frac{m\pi}{3} \qquad \therefore m \in Z$$

Hence

$$u(x,t) = Be^{-2\frac{m^2\pi^2}{9}t} \sin\left(\frac{m\pi}{3}x\right) \quad \dots (3)$$

$Calculating\ (ii)(boundary\ value)$

$$u(x,0) = 5sin4\pi x - 3sin8\pi x + 2sin10\pi x \ \dots (b)$$

By applying superposition principle on equation 3,

$$u(x,t) = B_1 e^{-2\frac{m_1^2\pi^2}{9}t} \sin\left(\frac{m_1\pi}{3}x\right) + B_2 e^{-2\frac{m_2^2\pi^2}{9}t} \sin\left(\frac{m_2\pi}{3}x\right) + B_3 e^{-2\frac{m_3^2\pi^2}{9}t} \sin\left(\frac{m_3\pi}{3}x\right) \ \dots (4)$$

For

$$u(x,0) = B_1 \sin\left(\frac{m_1\pi}{3}x\right) + B_2 \sin\left(\frac{m_2\pi}{3}x\right) + B_3 \sin\left(\frac{m_3\pi}{3}x\right) \ \dots (c)$$

By comparing $b$ & $C$, we have

$$B_1 = 5\ , B_2 = -3\ , B_3 = 2\ , m_1 = 12\ , m_2 = 24 \ \& \ m_3 = 30$$

Finally by putting calculated values in equation 4, we get

$$u(x,t) = 5e^{-32\pi^2 t}\ sin4\pi x - 3e^{-128\pi^2 t}\ sin8\pi x + 2e^{-200\pi^2 t}\ sin10\pi x$$

LECTURE NO. 16

**Motivation Behind Fourier Series:**

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \quad \dots (1) \qquad \therefore 0 < x < 3\ , t > 0$$

Given that (boundary values)

$$u(0,t) = u(3,t) = 0 \qquad \dots (i),$$

$$u(x,0) = 5sin4\pi x - 3sin8\pi x + 2sin10\pi x \qquad \dots (ii) \therefore |u(x,t)| < M < \infty$$

Solution: If

$$u(x,0) = f(x) = ? \quad \dots (1)$$

As we know from previous lecture

$$u(x,t) = B_1 e^{-2\frac{m_1^2\pi^2}{9}t} \sin\left(\frac{m_1\pi}{3}x\right) + B_2 e^{-2\frac{m_2^2\pi^2}{9}t} \sin\left(\frac{m_2\pi}{3}x\right) + \dots = \sum_{m=1}^{\infty} B_m\, e^{-2\frac{m^2\pi^2}{9}t} \sin\left(\frac{m\pi x}{3}\right)$$

$Calculating\ (1)$

$$f(x) = u(x,0) = \sum_{m=1}^{\infty} B_m e^{-2\frac{m^2\pi^2(0)}{9}} \sin\left(\frac{m\pi x}{3}\right) = \sum_{m=1}^{\infty} B_m e^0 \sin\left(\frac{m\pi x}{3}\right)$$

As $e^0 = 1$

$$f(x) = u(x,0) = \sum_{m=1}^{\infty} B_m \sin\left(\frac{m\pi x}{3}\right)$$

LECTURE NO. 17

**Periodic & Piecewise Continuous Functions:**

Definition: A function $f(x)$ is said to have a period $'p'$ if $\forall x$, we have $f(x+p) = f(x)$, and $'p > 0'$. Further if $p > 0$ is least then it is said to be period of $f(x)$.

Example: $sinx = sin(x+2\pi) = sin(x+4\pi) = sin(x+6\pi) = \cdots$

But $'2\pi'$ is least $=> p = 2\pi$ is a period of $'sinx'$.

In general; for

$$sin\, nx = sin(nx + 2\pi)$$
$$sin\, n(x+0) = sin\, n(x + \frac{2\pi}{n})$$
$$Period = p = \frac{2\pi}{n}$$

**Piecewise Continuous Functions:**

$$f: [a,b] \to R$$

It is a function that has at most a finite number of finite discontinuities.

LECTURE NO. 18

**Fourier Series:**

Definition: let $f(x)$ be a periodic function of period $'2l'$ and defined on $(-l, l)$, then its Fourier series is defined as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \; cos\,(\frac{n\pi x}{l}) + b_n \; sin\,(\frac{n\pi x}{l})]$$

Where

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x)\, cos\,(\frac{n\pi x}{l})\, dx \quad , \quad b_n = \frac{1}{l} \int_{-l}^{l} f(x)\, sin\,(\frac{n\pi x}{l})\, dx \qquad \therefore n = 0,1,2,3 \ldots$$

These are called Fourier coefficient.

By changing the period, we can also define coefficients as

$$a_n = \frac{1}{l} \int_{c}^{c+2l} f(x)\, cos\,(\frac{n\pi x}{l})\, dx \quad , \quad b_n = \frac{1}{l} \int_{c}^{c+2l} f(x)\, sin\,(\frac{n\pi x}{l})\, dx \qquad \therefore n = 0,1,2,3 \ldots$$

$$\frac{a_0}{2} = \frac{1}{2l} \int_{-l}^{l} f(x) dx \quad \to Average\ value\ of\ f(x)\ in\ interaval\ (-l, l)$$

Example: Function

$$f(x) = x^2 \quad \& \quad interval\ (0,2)$$

$$\int_0^2 x^2 dx = \frac{1}{2} \left| \frac{x^3}{3} \right|_0^2 = \frac{1}{2} \cdot \frac{8}{3} = \frac{4}{3} \quad \to Average\ value\ in\ the\ interval\ (0,2)$$

LECTURE NO. 19

**Dirichlet Condition:**

Fourier series corresponding to $'f(x)'$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \; cos\,(\frac{n\pi x}{l}) + b_n \; sin\,(\frac{n\pi x}{l})]$$

Question is; Series converge or diverge? If converge then converges to $f(x)$ or not.

Suppose that:

$i)$ $f(x)$ is defined and single valued on $(-l, l)$
$ii)$ $f(x)$ is periodic on $(-l, l)$ with period $'2l'$
$iii)$ $f(x)$ & $f'(x)$ are piecewise continuous, then the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})] = \begin{cases} f(x) \text{ if } 'x' \text{ is point of continuity} \\ \frac{f(x+0)+f(x-0)}{2} \text{ if } 'x' \text{ is point of discontinuity} \end{cases}$$

LECTURE NO. 20

**Evaluation of Fourier Series:**

Theorem: If the series;

$$A + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})]$$

Converges uniformly to $f(x)$ in $(-l, l)$, then for $n = 0,1,2,3 \ldots$ show that

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos(\frac{n\pi x}{l})\, dx \quad \ldots (i)$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin(\frac{n\pi x}{l})\, dx \quad \ldots (ii)$$

$$A = \frac{a_0}{2} \quad \ldots (iii)$$

Solution: Given that

$$f(x) = A + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})] \ldots (1)$$

*<u>Calculating i)</u>*:

Multiplying (equation 1) both sides by $\cos(\frac{m\pi x}{l})$ and integrating from $-l$ to $l$, we have

$$\int_{-l}^{l} f(x) \cos(\frac{m\pi x}{l})\, dx = \int_{-l}^{l} A \cos(\frac{m\pi x}{l})\, dx + \sum_{n=1}^{\infty} \{a_n \int_{-l}^{l} \cos(\frac{n\pi x}{l}) \cos(\frac{m\pi x}{l})\, dx + b_n \int_{-l}^{l} \sin(\frac{n\pi x}{l}) \cos(\frac{m\pi x}{l})\, dx\} \ldots (2)$$

(a) (b) (c)

Now we'll calculate the values for above functions.

$$(a); \quad \int_{-l}^{l} A \cos(\frac{m\pi x}{l})\, dx = A\frac{l}{m\pi} \int_{-l}^{l} (\frac{m\pi}{l}) \cos(\frac{m\pi x}{l})\, dx = \frac{Al}{m\pi} \left| \sin(\frac{m\pi x}{l}) \right|_{-l}^{l} = 0$$

$$(b); \quad \int_{-l}^{l} \cos(\frac{n\pi x}{l}) \cos(\frac{m\pi x}{l})\, dx = \frac{1}{2} \int_{-l}^{l} \{\cos\frac{(m+n)\pi x}{l} + \cos\frac{(m-n)\pi x}{l}\}\, dx = 0 \quad ; \quad for\ m \neq n$$

∴ Here we use the trigonometric relation; $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

For $m = n$, we have

$$\int_{-l}^{l} \cos(\frac{n\pi x}{l}) \cos(\frac{m\pi x}{l})\, dx = \frac{1}{2} \int_{-l}^{l} \cos^2(\frac{m\pi x}{l})\, dx = \frac{1}{2} \int_{-l}^{l} \{1 + \cos 2(\frac{m\pi x}{l})\}\, dx = \frac{1}{2} |x|_{-l}^{l} = l$$

$$(c); \quad \int_{-l}^{l} \sin(\frac{n\pi x}{l}) \cos(\frac{m\pi x}{l})\, dx = \frac{1}{2} \int_{-l}^{l} \{\sin\frac{(m+n)\pi x}{l} + \sin\frac{(m-n)\pi x}{l}\}\, dx = 0$$

∴ Here we use the trigonometric relation; $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A+B)]$

LECTURE NO. 21

By putting values in equation 2, we get

$$\int_{-l}^{l} f(x)\cos\left(\frac{m\pi x}{l}\right)dx = 0 + a_m l + 0 \Rightarrow a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

This is our required solution for $(i)$.

<u>*Calculating (ii);*</u>

Multiplying (equation 1) both sides by $sin\left(\frac{m\pi x}{l}\right)$ and integrating from $-l$ to $l$, we have

$$\int_{-l}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)dx = \int_{-l}^{l} A\sin\left(\frac{m\pi x}{l}\right)dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-l}^{l}\sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx + b_n\int_{-l}^{l}\sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx\right\} \dots (3)$$

(a) (b) (c)

Now we'll calculate the values for above functions.

$$(a);\quad \int_{-l}^{l} A\sin\left(\frac{m\pi x}{l}\right)dx = 0 \qquad \therefore \sin\alpha \text{ is odd } \forall \alpha \in R$$

$$(b);\quad \int_{-l}^{l} \sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx = 0 \qquad \therefore \sin\alpha\ \cos\beta \text{ are odd functions } \forall \alpha,\beta \in R$$

$$(c);\quad \int_{-l}^{l} \sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}\left\{\cos\frac{(m-n)\pi x}{l} - \cos\frac{(m+n)\pi x}{l}\right\}dx = 0 \quad ; \quad m \neq n$$

For $m = n$, we have

$$\int_{-l}^{l} \sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}\sin^2\left(\frac{m\pi x}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}\left\{1-\cos 2\left(\frac{m\pi x}{l}\right)\right\}dx = \frac{1}{2}|x|_{-l}^{l} = l$$

By putting values in equation 3, we get

$$\int_{-l}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)dx = 0 + 0 + b_m l \Rightarrow b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

This is our required solution for $(ii)$.

<u>*Calculating (iii)*</u>:

Integrating equation 1 from $-l$ to $l$ on both sides,

$$\int_{-l}^{l} f(x) = \int_{-l}^{l} A\,dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)dx + b_n\int_{-l}^{l}\sin\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$\int_{-l}^{l} f(x)dx = 2Al \Rightarrow 2A = \frac{1}{l}\int_{-l}^{l} f(x)dx \quad \dots (a)$$

Put $n = 0$ in $a_n$;

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \Rightarrow a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx \quad \dots (b)$$

Comparing $a$ and $b$, we have $a_0 = 2A \Rightarrow A = \frac{a_0}{2}$

Hence equation 1 (Fourier Series) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

LECTURE NO. 22

**Problem:**

Expand $f(x) = x^2$ ; $0 < x < 2\pi$ in a Fourier Series.

Solution: Fourier Series

$$f(x) = A + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})]$$

According to given function, ($\therefore l = \pi$)

$$f(x) = A + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)] \ \dots (A)$$

As we know from previous lectures,

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x) \cos(\frac{n\pi x}{l})\, dx$$

According to given function,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos(\frac{n\pi x}{\pi})\, dx \ => \ \frac{1}{\pi}\int_0^{2\pi} x^2 \cos(nx) dx \quad \therefore l = \pi$$

By integrating and applying limits, we have

$$a_n = \frac{1}{\pi}[x^2(\frac{\sin(nx)}{n}) - (2x)(-\frac{\cos(nx)}{n^2}) + (2)(-\frac{\sin(nx)}{n^3})]_0^{2\pi}$$

$$a_n = \frac{1}{\pi}[4\pi^2(0) + \frac{4\pi(1)}{n^2} + 0] = \frac{1}{\pi}[\frac{4\pi}{n^2}] = \frac{4}{n^2} \ \dots (1)$$

Now

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin(\frac{n\pi x}{l})\, dx$$

According to given function,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin(nx) dx$$

By integrating and applying limits, we have

$$b_n = \frac{1}{\pi}[x^2(-\frac{\cos(nx)}{n}) - (2x)(-\frac{\sin(nx)}{n^2}) + 2(\frac{\cos(nx)}{n^3})]_0^{2\pi}$$

$$b_n = \frac{1}{\pi}[0 + \frac{4\pi^2(-1)}{n} + 0] = -\frac{4\pi}{n} \ \dots (2)$$

Now

$$A = \frac{a_0}{2} \ => \frac{1}{\pi}\int_0^{2\pi} f(x)dx \ => \frac{1}{\pi}\int_0^{2\pi} x^2 dx \ => \frac{1}{\pi}[\frac{x^3}{3}]_0^{2\pi} \ => \frac{8\pi^2}{3} \ \dots (3)$$

Putting values in Fourier series,

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} [\frac{4}{n^2} \cos(nx) - \frac{4\pi}{n} \sin(nx)]$$

LECTURE NO. 23

**Fourier Expansion of Even Functions:**

Theorem: Show that an even function does not have *sine* terms in its Fourier Series.

Proof: Since

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})] \ \dots (1)$$

If $f(x)$ is even function, then we know

$$f(x) = f(-x)$$

So, replacing $x \ by -x$, we have

$$f(-x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{-n\pi x}{l}) + b_n \sin(\frac{-n\pi x}{l})]$$

As $\cos(-x) = \cos$ & $\sin(-x) = -sinx$, hence

$$f(-x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{l}) - b_n \sin(\frac{n\pi x}{l})] \quad \dots (2)$$

By comparing equation 1 & 2,

$$f(x) = f(-x)$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})] = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{l}) - b_n \sin(\frac{n\pi x}{l})]$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{l}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{l}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{l}) - \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{l})$$

By cancelling the terms, we have

$$\sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{l}) = -\sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{l})$$

$$2\sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{l}) = 0 \quad => \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{l}) = 0$$

So equation (1) implies,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{l})$$

Hence proved that an even function does not have *sine* terms in its Fourier Series.

LECTURE NO. 24

**Half Range Expansion of Identity Function:**

Expand $f(x) = x \,; 0 < x < 2$ in half-range $sines$ & $cosine\ series$.

Solution:

$(i)$: For odd extension of $f(x) = x$ in $(-2,2) \ \therefore \ -2 < x < 2$

This implies $2l = 4 \quad so \quad l = 2$

For odd expansion; $a_n = 0$

$(\therefore a_0 = 0 \ also)$ And

$$b_n = \frac{2}{l}\int_0^l f(x) \sin(\frac{n\pi x}{l})\, dx$$

For given function,

$$b_n = \frac{2}{2}\int_0^2 x \sin(\frac{n\pi x}{2})\, dx \quad => \int_0^2 x \sin(\frac{n\pi x}{2})\, dx$$

By integrating and applying limits, we have

$$b_n = [x\{\frac{-2}{n\pi} \cos(\frac{n\pi x}{2})\} - (1)\{\frac{4}{n^2\pi^2} \sin(\frac{n\pi x}{2})\}]_0^2$$

$$b_n = -\frac{4}{n\pi} \cos(n\pi)$$

Now, putting values in Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})]$$

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{2}) \;=>\; \sum_{n=1}^{\infty} (-\frac{4}{n\pi} \cos(n\pi)) \sin(\frac{n\pi x}{2})$$

$$f(x) = \frac{4}{\pi}[\sin(\frac{\pi x}{2}) - \frac{1}{2}\sin(\frac{2\pi x}{2}) + \frac{1}{3}\sin(\frac{3\pi x}{2}) - \ldots]$$

Required result.

LECTURE NO. 25

***(ii)***: For even extension of $f(x) = |x|$ in $(-2,2)$

This implies $2l = 4 \quad so \quad l = 2$

For odd expansion; $b_n = 0$ and

$$a_n = \frac{2}{l}\int_0^l f(x) \cos(\frac{n\pi x}{l})\, dx$$

For given function,

$$a_n = \frac{2}{2}\int_0^2 x \cos(\frac{n\pi x}{2})\, dx \;=>\; \int_0^2 x \cos(\frac{n\pi x}{2})\, dx$$

By integrating and applying limits, we have

$$a_n = [x\{\frac{2}{n\pi} \sin(\frac{n\pi x}{2})\} - (1)\{\frac{-4}{n^2\pi^2} \cos(\frac{n\pi x}{2})\}]_0^2$$

$$a_n = 0 + \frac{4}{n^2\pi^2}(\cos n\pi - 1) \;=>\; \frac{4}{n^2\pi^2}[(-1)^n - 1]$$

For $n = 0$,

$$a_0 = \frac{2}{l}\int_0^l x dx \;=>\; \frac{2}{2}\int_0^2 x dx \;=>\; [\frac{x^2}{2}]_0^2 \;=>\; 2$$

$$\frac{a_0}{2} = 1$$

Now putting values in Fourier Series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l})]$$

$$f(x) = 1 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{2}) + 0 \;=>\; 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}[(-1)^n - 1] \cos(\frac{n\pi x}{2})$$

$$f(x) = 1 + \frac{4}{\pi^2}[-2\cos(\frac{\pi x}{2}) - \frac{2}{3^2}\cos(\frac{3\pi x}{2}) - \frac{2}{5^2}\cos(\frac{5\pi x}{2}) - \ldots]$$

$$f(x) = 1 - \frac{8}{\pi^2}[\cos(\frac{\pi x}{2}) + \frac{1}{3^2}\cos(\frac{3\pi x}{2}) + \frac{1}{5^2}\cos(\frac{5\pi x}{2}) + \ldots]$$

Required result.

LECTURE NO. 26

**Parseval's Identity:**

Theorem: If Fourier Series of $f(x)$ converges uniformly to $f(x) in\ (-l, l)$, then prove that

$$\frac{1}{l}\int_{-l}^{l} \{f(x)\}^2 dx = \frac{{a_0}^2}{2} + \sum_{n=1}^{\infty} ({a_n}^2 + {b_n}^2)$$

Proof:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\ cos\left(\frac{n\pi x}{l}\right)+b_n\ sin\left(\frac{n\pi x}{l}\right)]\ ...(1)$$

Multiplying both sides by $f(x)$ and integrating form $-l\ to\ l$,

$$\int_{-l}^{l}\{f(x)\}^2dx = \int_{-l}^{l}\frac{a_0}{2}f(x)dx+\sum_{n=1}^{\infty}[a_n\int_{-l}^{l}f(x)\,cos\left(\frac{n\pi x}{l}\right)dx+b_n\int_{-l}^{l}f(x)\,sin\left(\frac{n\pi x}{l}\right)dx]\ ...(A)$$

∴

$$a_n = \frac{1}{l}\int_{-l}^{l}f(x)\,cos\left(\frac{n\pi x}{l}\right)dx\ => \ a_n l = \int_{-l}^{l}f(x)\,cos\left(\frac{n\pi x}{l}\right)dx$$

Similarly for

$$b_n l = \int_{-l}^{l}f(x)\,sin\left(\frac{n\pi x}{l}\right)dx$$

$$a_0 l = \int_{-l}^{l}f(x)dx$$

So, equation (A) can be written as

$$\int_{-l}^{l}\{f(x)\}^2dx = \frac{a_0}{2}(a_0 l) + \sum_{n=1}^{\infty}[a_n(a_n l)+b_n(b_n l)]$$

$$\int_{-l}^{l}\{f(x)\}^2dx = \frac{{a_0}^2}{2}(l)+(l)\sum_{n=1}^{\infty}({a_n}^2+{b_n}^2)$$

$$\frac{1}{l}\int_{-l}^{l}\{f(x)\}^2dx = \frac{{a_0}^2}{2}+\sum_{n=1}^{\infty}({a_n}^2+{b_n}^2)$$

Required result proved.

LECTURE NO. 27

**Problem:**

Use Parseval's identity for $f(x) = |x|, -2 < x < 2$ to show that

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ ... = \frac{\pi^4}{96}$$

Where it is given

$$a_0 = 2\quad,\quad a_n = \frac{4}{n^2\pi^2}\{(-1)^n-1\}, n\neq 0\quad \&\quad b_n = 0$$

Also deduce that

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ ... = \frac{\pi^4}{90}$$

Solution: Parseval's identity

$$\frac{1}{l}\int_{-l}^{l}\{f(x)\}^2dx = \frac{{a_0}^2}{2}+\sum_{n=1}^{\infty}({a_n}^2+{b_n}^2)\ ...(1)$$

For $f(x) = |x|\,; (-2,2) => \ 2l = 4\quad so\quad l = 2$

Equation (1) becomes

$$\frac{1}{2}\int_{-2}^{2}x^2dx = \frac{2^2}{2}+\sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}\{(-1)^n-1\}^2+0$$

By solving, we have

$$\frac{8}{3} = 2 + \frac{64}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ \dots\right)$$
$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ \dots = \frac{\pi^4}{96}$$

Proved. Now let

$$S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$
$$S = \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ \dots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \ \dots\right)$$
$$S = \left(\frac{\pi^4}{96}\right) + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ \dots\right)$$
$$S = \left(\frac{\pi^4}{96}\right) + \frac{1}{2^4}(S)$$
$$S = \frac{\pi^4}{90}$$

Deduced.

LECTURE NO. 28

**Finding a Fourier Series by Integration:**

Theorem: Fourier series of $f(x)$ can be integrated by term form $a\ to\ x$ and the resulting series with converge uniformly to $\int_0^x f(x)$ provided that;

$(i)$: $f(x)$ is piecewise continued in $(-l, l)$ $(ii)$: $a, x \in (-l, l)$

Problem: Find the Fourier Series of $f(x) = x^2$ , $0 < x < 2$, by integrating

$$x = \frac{4}{\pi}\left[sin\frac{\pi x}{2} - \frac{1}{2}sin\frac{2\pi x}{2} + \frac{1}{3}sin\frac{3\pi x}{2} - \ \dots\right]$$

And further evaluate;

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = ?$$

Solution: Given that

$$x = \frac{4}{\pi}\left[sin\frac{\pi x}{2} - \frac{1}{2}sin\frac{2\pi x}{2} + \frac{1}{3}sin\frac{3\pi x}{2} - \ \dots\right]$$

Integrating from $0\ to\ x$ term by term

$$\int_0^x x\,dx = \frac{4}{\pi}\left[\int_0^x sin\frac{\pi x}{2}dx - \frac{1}{2}\int_0^x sin\frac{2\pi x}{2}dx + \frac{1}{3}\int_0^x sin\frac{3\pi x}{2}dx\right]$$
$$\left[\frac{x^2}{2}\right]_0^x = \frac{4}{\pi}\left[\frac{2}{\pi}\left|-cos\frac{\pi x}{2}\right|_0^x + \frac{2}{\pi}\frac{1}{2^2}\left|cos\frac{2\pi x}{2}\right|_0^x - \frac{2}{\pi}\frac{1}{3^2}\left|cos\frac{3\pi x}{2}\right|_0^x - \dots\right]$$
$$\frac{x^2}{2} = \frac{4}{\pi}\left[\frac{2}{\pi}\left(-cos\frac{\pi x}{2} + 1\right) + \frac{2}{\pi}\frac{1}{2^2}\left(cos\frac{2\pi x}{2} - 1\right) - \frac{2}{\pi}\frac{1}{3^2}\left(cos\frac{3\pi x}{2} - 1\right)\right]$$
$$x^2 = \frac{16}{\pi^2}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ \dots\right] - \frac{16}{\pi^2}\left[cos\frac{\pi x}{2} - \frac{1}{2^2}cos\frac{2\pi x}{2} + \frac{1}{3^2}cos\frac{3\pi x}{2} - \dots\right]$$

Comparing above relation with general form;

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\ cos\left(\frac{n\pi x}{l}\right) + b_n\ sin\left(\frac{n\pi x}{l}\right)\right]$$

We have

$$\frac{a_0}{2} => \frac{1}{l}\int_0^l f(x)dx = \frac{16}{\pi^2}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ \dots\right] \ \dots (1)$$

Here

$$\frac{1}{l}\int_0^l f(x)dx = \frac{1}{2}\int_0^2 x^2dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}$$

So equation (1) becomes,

$$\frac{4}{3} = \frac{16}{\pi^2}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right]$$

$$\frac{4}{3} = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

Required result.

LECTURE NO. 29

**Problem:**

Check the term by term differentiation of Fourier Series;

$$f(x) = \frac{4}{\pi}\left[\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \dots\right]$$

Solution: Taking derivative,

$$f'(x) = \frac{4}{\pi}\left[\frac{\pi}{2}\cos\frac{\pi x}{2} - \frac{1}{2}\frac{2\pi}{2}\cos\frac{2\pi x}{2} + \frac{1}{3}\frac{3\pi}{2}\cos\frac{3\pi x}{2} - \dots\right]$$

$$f'(x) = 2\left[\cos\frac{\pi x}{2} - \cos\frac{2\pi x}{2} + \cos\frac{3\pi x}{2} - \dots\right]$$

This implies

$$a_n = (-1)^{n-1}\left(\cos\frac{n\pi x}{2}\right) \nrightarrow 0$$

⇒ Series does not converge ⇒ It does not converges uniformly ⇒ Term by term differentiation is not possible.

LECTURE NO. 30

**Heat Flow Problem:**

A bar of length "$l$" whose entire surface is insulated including its ends at $x = 0$ and $x = l$. Its initial temperature is $f(x)$, then determine the subsequent temperature of the bar.

Solution: It is a heat flow boundary value problem (B.V.P.). As we know heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \dots (1)$$

Where

$$|u(x,t)| < M\ , \qquad u(x,0) = f(x)\ , \qquad u_x(0,t) = 0 = u_x(l,t)$$

Let $u(x,t) = X(x)T(t) = XT$ be its solution. Equation (1) implies,

$$XT' = kX''T$$

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda^2\ (say)$$

Here become two equations. We'll solve them side by side.

| | |
|---|---|
| $T' + \lambda^2 kT = 0 \quad \dots (i)$ | $X'' + \lambda^2 X = 0 \quad \dots (ii)$ |
| Let $T = e^{mt}$ | $X = e^{mx} \Rightarrow X'' = m^2e^{mx} = m^2X$ |
| $T' = me^{mt} \Rightarrow mT$ | $m^2X + \lambda^2X = 0 \Rightarrow m = \pm i\lambda\ (\because x \neq 0)$ |
| So $(i)$; $mT + \lambda^2kT = 0$ | $X = ae^{\lambda ix} + be^{-\lambda ix}$ |
| $m = -\lambda^2 k$ | $X = a(\cos\lambda x + i\sin\lambda x) + b(\cos\lambda x - i\sin\lambda x)$ |
| $\therefore\ T = ce^{-\lambda^2kt}$ | $X = (a+b)\cos\lambda x + i(a-b)\sin\lambda x = A\cos\lambda x + B\sin\lambda x$ |

By putting values,

$$u(x,t) = X(x)T(t) = XT = ce^{-\lambda^2 kt}[A\cos\lambda x + B\sin\lambda x] = e^{-\lambda^2 kt}[\alpha\cos\lambda x + \beta\sin\lambda x] \quad \dots (2)$$

Now

$$u_x = e^{-\lambda^2 kt}[-\alpha\lambda\cos\lambda x + \beta\lambda\sin\lambda x]$$

$$u_x(0,t) = \beta\,\lambda\, e^{-\lambda^2 kt} = 0 \Rightarrow \beta = 0$$

Equation (2) implies,

$$u(x,t) = \alpha\, e^{-\lambda^2 kt}\cos(\lambda x) \quad \dots (3)$$

$$u_x = -\lambda\,\alpha\, e^{-\lambda^2 kt}\sin(\lambda x)$$

$$u_x(l,t) = -\lambda\,\alpha\, e^{-\lambda^2 kt}\sin(\lambda l) = 0$$

$$\Rightarrow \sin(\lambda l) = 0 \Rightarrow \lambda l = m\pi \Rightarrow \lambda = \frac{m\pi}{l}$$

Equation (3) implies,

$$u(x,t) = \alpha\, e^{-\frac{m^2\pi^2}{l^2}kt}\cos\left(\frac{m\pi}{l}x\right)$$

By using super-position principle;

$$u(x,t) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\alpha_m\, e^{-\frac{m^2\pi^2}{l^2}kt}\cos\left(\frac{m\pi}{l}x\right) \quad \dots (A)$$

Now; as given

$$u(x,0) = f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\alpha_m\cos\left(\frac{m\pi}{l}x\right)$$

Where

$$\alpha_m = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{m\pi}{l}x\right) \quad , \quad \frac{a_0}{2} = \frac{1}{l}\int_0^l f(x)dx$$

So, equation (A) becomes (by putting values)

$$u(x,t) = \frac{1}{l}\int_0^l f(x)dx + \frac{2}{l}\sum_{m=1}^{\infty}\left\{\int_0^l f(x)\cos\left(\frac{m\pi}{l}x\right)e^{-\frac{m^2\pi^2}{l^2}kt}\cos\left(\frac{m\pi}{l}x\right)\right\}$$

Required result.

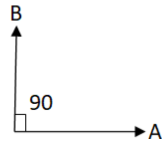
## LECTURE NO. 31

**Laplace Equation:**

Problem: Suppose that the three sides of a square plate kept at zero temperature and fourth one at $u_1$. Determine the steady state temperature at all parts on the plate.

For Solution; WATCH Lecture (Lengthy Calculations)

## LECTURE NO. 32




**Orthogonal Functions And Orthogonal Sets:**

$(i)$: Can we generalize the idea of vectors and orthogonality?

$\{f(x)$ *defined on* $(a,b)$; *value of* $'f'$ *at each point on* $(a,b)$ *represents its comoponents.*

$(ii)$: Say $A(x)$ & $B(x)$ defined on $(a,b)$ if $\int_a^b A(x).B(x) = 0$, then $A(x)$ & $B(x)$ are said to orthogonal.

$(iii)$: A vector say $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is said to normalized if its magnitude is unity i.e.

$$\hat{r} = \frac{\vec{r}}{|r|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$(iv)$: $A(x)$ normal or normalized in $(a,b)$ if

$$\int_a^b \{A(x)\}^2 dx = 1$$

**Orthogonal Sets:**

A set of function say $\{\Phi_k(x)\}, k = 1,2,3,\ldots$ defined on $(a, b)$ such that

$$(i): \quad \int_a^b \Phi_m(x).\Phi_n(x) = 0 \quad , \quad m \neq n$$

$$(ii): \quad \int_a^b \{\Phi_m(x)\}^2 = 1 \quad , \quad m = 1,2,3,\ldots$$

Then the set $\{\Phi_k(x)\}, k = 1,2,3,\ldots$ is orthogonal.

$(i)$& $(ii)$ implies,

$$\int_a^b \Phi_m(x).\Phi_n(x)dx = S_{mn} = \begin{cases} 0 & if \quad m \neq n \\ 1 & if \quad m = n \end{cases}$$

**Orthogonality w.r.to Weight Function:**

If

$$\int_a^b \Phi_m(x)\, \Phi_n(x)\, \omega(x)\, dx = S_{mn}$$

Where $\omega(x) \geq 0$, then $\{\Phi_m(x)\}_{k=1}^{\infty}$ is orthogonal w.r.to weight function.

LECTURE NO. 33

**Obtaining Normalizing Constants From orthogonal Sets:**

Show that the set

$$\{1, sin\frac{\pi x}{l}, cos\frac{\pi x}{l}, sin\frac{2\pi x}{l}, cos\frac{2\pi x}{l}, \ldots\}$$

Is an orthogonal set. Also find its corresponding normalizing constants, so that given set is orthogonal.

Solution: As we know condition for orthogonal set condition

$$\int_{-l}^{l} \Phi_m(x).\Phi_n(x)dx = \begin{cases} 0 & if \quad m \neq n \\ 1 & if \quad m = n \end{cases}$$

Possibilities for given set (using Fourier series results;

$$(i): \int_{-l}^{l} 1\, sin\frac{k\pi x}{l}dx = 0 = \int_{-l}^{l} 1\, cos\frac{k\pi x}{l}dx \quad \forall\, k = 1,2,3,\ldots$$

$$(ii): \int_{-l}^{l} sin\frac{k\pi x}{l}\, cos\frac{p\pi x}{l}\, dx = 0 \qquad k \neq p$$

$$(iii): \int_{-l}^{l} sin\frac{m\pi x}{l}\, sin\frac{n\pi x}{l}\, dx = \int_{-l}^{l} cos\frac{m\pi x}{l}\, cos\frac{n\pi x}{l}\, dx = \begin{cases} 0 & if \quad m \neq n \\ l & if \quad m = n \end{cases}$$

(iii) Implies

$$\int_{-l}^{l} sin^2\frac{m\pi x}{l}dx = l \qquad \& \qquad \int_{-l}^{l} cos^2\frac{m\pi x}{l}dx = l$$

$$\int_{-l}^{l} \left(\frac{1}{\sqrt{l}}\, sin\frac{m\pi x}{l}\right)^2 dx = 1 \qquad \& \qquad \int_{-l}^{l} \left(\frac{1}{\sqrt{l}}\, cos\frac{m\pi x}{l}\right)^2 dx = 1$$

Similarly for (i):

$$\int_{-l}^{l} (1)^2\, dx = 2l \quad => \quad \int_{-l}^{l} \left(\frac{1}{\sqrt{2l}}\right)^2 dx = 1$$

Hence, Orthonormalizing constants;

$$\frac{1}{\sqrt{2l}}, \frac{1}{\sqrt{l}}$$

Corresponding Orthonormal set;

$$\{\frac{1}{\sqrt{2l}}, \frac{1}{\sqrt{l}} sin\frac{\pi x}{l}, \frac{1}{\sqrt{l}} cos\frac{\pi x}{l}, \frac{1}{\sqrt{l}} sin\frac{2\pi x}{l}, \frac{1}{\sqrt{l}} cos\frac{2\pi x}{l}, \dots\}$$

LECTURE NO. 34

**Generalized Fourier Series:**

Given that $\{\Phi_m(x)\}_{m=1}^{\infty}$ be an orthogonal set of functions and if possible to expand a function "$f(x)$" in a set of orthonormal functions, i.e.

$$f(x) = \sum_{n=1}^{\infty} C_n\, \Phi_n(x) \quad ; \; a \le x \le b$$

Then such series is called Orthonormal Series or Generalized Fourier Series and "$C_n$" are called Generalized Fourier Coefficients.

If $f(x)$ & $f'(x)$ are piecewise continuous functions, then

$$\sum_{n=1}^{\infty} C_n\, \Phi_n(x) = \frac{1}{2}[f(x+0) + f(x-0)]$$

"If a function and its derivative are piecewise continuous, then the series converge to their average".

LECTURE NO. 35

**Theorem:**

Let $\{\Phi_n(x)\}$ be a set of mutually orthonormal functions in $(a, b)$. Show that if $\sum_{n=1}^{\infty} C_n\, \Phi_n(x)$ converges uniformly, then $C_n = \int_a^b f(x)\, \Phi_n(x)dx$.

Proof: As given that

The series $\sum_{n=1}^{\infty} C_n\, \Phi_n(x)$ converges uniformly to $f(x)$

This implies,

$$f(x) = \sum_{n=1}^{\infty} C_n\, \Phi_n(x) \quad \dots (1)$$

Multiplying both sides by $\Phi_m(x)$ and integrating form $'a \; to \; b'$

$$\int_a^b f(x)\, \Phi_m(x)dx = \sum_{n=1}^{\infty} C_n \int_a^b \Phi_m(x)\, \Phi_n(x)dx \quad \dots (2)$$

As

$$\int_a^b \Phi_m(x)\, \Phi_n(x)dx = \begin{cases} 0 & if \quad m \ne n \\ l & if \quad m = n \end{cases}$$

Hence equation (2) impies

$$C_m = \int_a^b f(x)\, \Phi_m(x)dx$$

Or it can be written as;

$$C_n = \int_a^b f(x)\, \Phi_n(x)dx$$

Which are called the generalized fourier constants.

LECTURE NO. 36

**MSE and RMSE:**

MSE stands for "Mean Square Error" and RMSE stands for "Root Mean Square Error".

Let $f(x)\ and\ f'(x)$ be continuous piecewise functions in $(a,b)$, $\{\Phi_m(x)\}_{m=1}^{\infty}$ be an orthogonal set in $(a,b)$.

Now suppose the sum $S_m(x)=\sum_{n=1}^{\infty}\alpha_n\,\Phi_n(x)$ be an approximation of $f(x)$, where "$\alpha_n$" are unknown constants. Then the mean square error of this approximation is given by,

$$MSE=\frac{\int_a^b\{f(x)+S_m(x)\}^2dx}{b-a}$$

And

$$RMSE=\sqrt{\frac{\int_a^b\{f(x)+S_m(x)\}^2dx}{b-a}}$$

These are used to compare the accuracy of different mathematical methods of a physical system.

We are aimed to find "$\alpha_n$" which with produce the least mean square error.

Use: (MSE; for large or greater errors/quantities & RMSE; for same units as that of given function)

LECTURE NO. 37

**Lease Square Approximation & Principle of Finality:**

Theorem: The root mean square error (RMSE) is least when the coefficients "$\alpha_n$" are equal to generalized Fourier Coefficient; i.e.

$$\alpha_n=c_n=\int_a^n f(x)\,\Phi_n(x)dx$$

Proof: As we known

$$RMSE=\sqrt{\frac{\int_a^b\{f(x)+S_m(x)\}^2dx}{b-a}}\Rightarrow\frac{1}{\sqrt{b-a}}\sqrt{\left(\int_a^b\{f(x)+S_m(x)\}^2dx\right.}$$

Now

$$f(x)-S_m(x)=f(x)-\sum_{n=1}^{\infty}\alpha_n\,\Phi_n(x)$$

Where $\{\Phi_n(x)\}_{n=1}^{\infty}$ is an orthonormal set and "$\alpha_n$" is unknown. Now,

$$\{f(x)-S_m(x)\}^2=\{f(x)\}^2-2\sum_{n=1}^{\infty}\alpha_n\,\Phi_n(x)f(x)+\left(\sum_{n=1}^{\infty}\alpha_n\,\Phi_n(x)\right)^2 \qquad \dots (1)$$

$$\left(\sum_{n=1}^{\infty}\alpha_n\,\Phi_n(x)\right)^2=\sum_{n=1}^{\infty}\alpha_n\,\Phi_n(x)\times\sum_{n=1}^{\infty}\alpha_n\,\Phi_n(x)=\sum_{n=1}^{\infty}\alpha_n\,\Phi_n(x)\times\sum_{m=1}^{\infty}\alpha_m\,\Phi_m(x)$$

$$\left(\sum_{n=1}^{\infty}\alpha_n\,\Phi_n(x)\right)^2=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n\,\alpha_m\,\Phi_n(x)\,\Phi_m(x)$$

By putting equation (1) becomes,

$$\{f(x)-S_m(x)\}^2=\{f(x)\}^2-2\sum_{n=1}^{\infty}\alpha_n\,\Phi_n(x)f(x)+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n\,\alpha_m\,\Phi_n(x)\,\Phi_m(x)$$

Integrating form "$a\ to\ b$"

$$\int_a^b\{f(x)-S_m(x)\}^2\,dx=\int_a^b\{f(x)\}^2\,dx-2\sum_{n=1}^{\infty}\alpha_n\int_a^b\Phi_n(x)f(x)\,dx+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_n\,\alpha_m\int_a^b\Phi_m\Phi_n\,dx \qquad \dots (2)$$

$$\therefore \int_a^b \Phi_n(x) f(x)\, dx = c_n = Generalized\ Fourier\ Coefficients$$

So, equation (2) becomes,

$$\int_a^b \{f(x) - S_m(x)\}^2\, dx = \int_a^b \{f(x)\}^2\, dx - 2\sum_{n=1}^{\infty} \alpha_n\, c_n + \sum_{n=1}^{\infty} (\alpha_n)^2$$

$$\int_a^b \{f(x) - S_m(x)\}^2\, dx = \int_a^b \{f(x)\}^2\, dx + \sum_{n=1}^{\infty} (\alpha^2 - 2\,\alpha_n c_n) \quad \dots (3)$$

$$\{\alpha^2 - 2\,\alpha_n c_n = \alpha^2 - 2\,\alpha_n c_n + (c_n)^2 - (c_n)^2 = (\,\alpha_n - c_n)^2 - (c_n)^2$$

So, equation (3) becomes,

$$\int_a^b \{f(x) - S_m(x)\}^2\, dx = \int_a^b \{f(x)\}^2\, dx + \sum_{n=1}^{\infty} \{(\alpha_n - c_n)^2 - (c_n)^2\}$$

$$\int_a^b \{f(x) - S_m(x)\}^2\, dx = \int_a^b \{f(x)\}^2\, dx + \sum_{n=1}^{\infty} (\alpha_n - c_n)^2 - \sum_{n=1}^{\infty} (c_n)^2$$

Error will be minimum when

$$\{\sum_{n=1}^{\infty} (\alpha_n - c_n)^2 = 0 \quad \Rightarrow \quad (\alpha_n - c_n)^2 = 0 \quad \Rightarrow \quad \alpha_n - c_n \quad \Rightarrow \quad \alpha_n = c_n$$

LECTURE NO. 38

**Bissel's Inequality:**

Theorem: For generalized Fourier coefficients "$c_n$" corresponding to $f(x)$, show that

$$\sum_{n=1}^{\infty} (c_n)^2 \le \int_a^b \{f(x)\}^2\, dx$$

Proof: Since $RMSE \ge 0$, so

$$\sqrt{\frac{\int_a^b \{f(x) + S_m(x)\}^2}{b-a}}\, dx \ge 0 \quad \Rightarrow \quad \left(\int_a^b \{f(x) + S_m(x)\}^2 dx\right)} \ge 0$$

$$\Rightarrow \quad \int_a^b \{f(x) + S_m(x)\}^2 dx \ge 0$$

$$\Rightarrow \quad \int_a^b \{f(x)\}^2\, dx - \sum_{n=1}^{\infty} (c_n)^2 \ge 0$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} (c_n)^2 \le \int_a^b \{f(x)\}^2\, dx$$

This is our required result. This inequality is known as Bissel's inequality.

LECTURE NO. 39

**Limiting Value of Generalized Fourier Coefficients:**

Theorem: Show that

$$\lim_{n\to\infty} \int_a^b f(x)\Phi_n(x)\, dx = 0$$

Proof: As we know, Generlized Fourier Coefficients

$$c_n = \int_a^b f(x)\Phi_n\,(x)\;dx$$

And from Bissel's inequality

$$\sum_{n=1}^{\infty}(c_n)^2 \le \int_a^b f(x)\Phi_n\,(x)\;dx$$

Since

$$\int_a^b f(x)\Phi_n\,(x)\;dx < k \qquad \forall\, k \in R$$

However large it may be.

$$\sum_{n=1}^{\infty}(c_n)^2 \; will\; converge \;\;\Rightarrow\;\; c_n \to 0 \;\; as\, n \to \infty$$

This implies that

$$\lim_{n\to\infty} c_n = 0 \;\;\Rightarrow\;\; \lim_{n\to\infty} \int_a^b f(x)\Phi_n\,(x)\;dx = 0$$

This is our required reslut.

LECTURE NO. 40

**Sturm-Liouville System:**

A boundary value problem of the form:

$$\frac{d}{dx}[p(x)\,\frac{dy}{dx}] + [q(x) + \lambda\, r(x)]y = 0 \qquad a \le x \le b \quad \dots(1)$$

Subjected to

$$\alpha_1\; y(a) + \alpha_2\; y'(a) = 0 \;\;,\;\; \beta_1\; y(b) + \beta_2\; y'(b) = 0 \quad \dots(2)$$

Where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are given constants and $p(x), q(x)\; and\; r(x)$ are given differentiable functions. And λ is unspecified parameter independent of $x$.

Here system has a squence of eigon value $\lambda_n$ and corresponding eigen functions satisfying boundary value problem.

**Example:** $y'' + \lambda y = 0$ ; $B.V.P$: $y(0) = y(1) = 0$

This implies

$$\frac{d}{dx}[1\,.\,\frac{dy}{dx}] + [(0 + \lambda.1)y] = 0 \qquad 0 \le x \le 1$$

Boundary values can be written as,

$$y(0) = 1\,y(0) + 0\,y'(0) = 0 = \alpha_1\,y(a) + \alpha_2\,y'(a)$$
$$y(1) = 1\,y(1) + 0\,y'(1) = 0 = \beta_1\,y(b) + \beta_2\,y'(b)$$

Comparing this with Sturm-Liouville System, we have

$$\alpha_1 = 1 \;,\;\; \alpha_2 = 0 \;,\;\; \beta_1 = 1 \;,\;\; \beta_2 = 0 \;,\;\; p(x) = 1 \;,\;\; q(x) = 0 \;,\;\; r(x) = 1$$

LECTURE NO. 41

**Heat Equation as Motivation for S-L System:**

Consider a B.V.P,

$$g(x)\;\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}[k(x)\,\frac{\partial u}{\partial x}] + ?(x)\,u \quad \dots(1)$$

Where

$$0 < x < l \;,\;\; t > 0 \;,\;\; u(0,t) = 0 \;,\;\; u(l,t) = 0 \;,\;\; u(x,0) = f(x) \;\;\&\;\; |u(x,t)| < M$$

Let

$$u(x,t) = X(x).T(t) = XT$$

So, equation (1) can be written as

$$g(x)\, XT' = \frac{\partial}{\partial x}[k(x)XT'] + \square(x)\, XT$$

$$g(x)\, XT' = T\frac{d}{dx}[k(x)\frac{dX}{dx}] + \square(x)\, XT$$

Dividing by $g(x)XT$ , we get

$$\frac{T'}{T} = \frac{1}{g(x)X}\frac{d}{dx}[k(x)\frac{dX}{dx}] + \frac{\square(x)}{g(x)} = -\lambda\ (say)$$

$$T' + \lambda T = 0 \quad \dots (2)$$

Boundary Values can be written as,

$$u(0,t) = 0 \ \Rightarrow\ X(0).T(t) = 0 \ \Rightarrow\ X(0) = 0$$

$$u(l,t) = 0 \ \Rightarrow\ X(l).T(t) = 0 \ \Rightarrow\ X(l) = 0$$

On comparing;

$$y = x\ ,\ p(x) = k(x)\ ,\ q(x) = \square(x)\ ,\ r(x) = g(x)$$

Solution: From equation (2)

$$T' + \lambda T = 0 \ \Rightarrow\ T = C\, e^{-\lambda t}$$

By superposition principle

$$u(x,t) = \sum_{n=1}^{\infty} C_n\, e^{-\lambda nt}\ X_n(x) \quad \dots (3)$$

As given

$$u(x,0) = f(x)$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} C_n\ X_n(x)$$

If $C_n$ is generalized coefficient, then

$$C_n = \int_0^l f(x)\, X_n(x)\, dx$$

Putting $C_n$ value in equation (3) we have

$$u(x,t) = \sum_{n=1}^{\infty}\int_0^l f(x)\, X_n(x)\, dx\ e^{-\lambda nt}\ X_n(x)$$

This is our required result.

LECTURE NO. 42

**Eigen Values and Eigen Functions of S-L System:**

Given a S-L System;

$$\frac{d}{dx}[p(x)\frac{dy}{dx}] + [q(x) + \lambda\, r(x)]y = 0$$

Subjected to;

$$\alpha_1\, y(a) + \alpha_2\, y'(a) = 0\ ,\ \beta_1\, y(b) + \beta_2\, y'(b) = 0 \quad a \le x \le b$$

Here non-trivial solution of S-L System exists in generalized for a particular value of λ. These values are called the Eigen values and the corresponding non-trivial solutions are called Eigen functions of S-L system.

Example:

$$y'' + \lambda y = 0\,; \quad y(0) = y(1) = 0\ ,\ 0 \le x \le 1$$

Let

$$y = e^{mx} \ \Rightarrow\ y'' = m^2 y \ \Rightarrow\ m = \pm i\sqrt{\lambda}\,; \quad y \ne 0$$

By superposition principle;

$$y = A\, cos\sqrt{\lambda x} + B\, sin\sqrt{\lambda x} \quad \dots (1)$$

For $y(0) = A = 0$ . So equation (1) becomes

$$y = B\, sin\sqrt{\lambda x}$$

And for

$$y(1) = B\, sin\sqrt{\lambda} = 0 \;\Rightarrow\; B \neq 0 \;\; \&\, so \;\; sin\sqrt{\lambda} = 0$$

This implies,

$$\sqrt{\lambda} = m\pi \quad \Rightarrow \quad \lambda_m = m^2\pi^2 \; ; \quad m \in Z$$

These are called Eigen values of given S-L System. And

$$y_m = B_m \; sin\, m\pi x$$

These are called the Corresponding Eigen Functions.

LECTURE NO. 43

**Orthogonality of Eigen Functions:**

$$y_m = B_m \; sin\, m\pi x \, ; \quad 0 \leq x \leq 1$$

$$\int_0^1 (B_m \; sin\; m\pi x)(B_n \; sin\; n\pi x)\, dx$$

$$= B_m\, B_n \int_0^1 (sin\; m\pi x)(sin\; n\pi x)\, dx$$

Using relation$\{2\; sinA\; sinB = \cos(A-B) - \cos(A+B)\}$, we have

$$= \frac{B_m B_n}{2} \int_0^1 \{\cos(m-n)\,\pi x - \cos(m+n)\,\pi x\} dx$$

By integrating and applying limits, we get

$$= \frac{B_m \; B_n}{2} \left[\frac{\sin(m-n)\,\pi x}{(m-n)\pi} - \frac{\sin(m+n)\,\pi x}{(m+n)\pi}\right]_0^1 = 0$$

Hence, $\{B_m \; sin\; m\pi x\}_{m=1}^{\infty}$ is an orthogonal set.

LECTURE NO. 44

**Normalization of Eigen Functions:**

Given set $\{\Phi_m(x)\}_{m=1}^{\infty}$ is orthonormal if

$$\int_a^b \Phi_m \Phi_n \; dx = S_{mn} \; = \begin{cases} 0\; ; & m \neq n \\ 1\; ; & m = n \end{cases}$$

Now for $\{B_m \; sin\; m\pi x\}_{m=1}^{\infty}$ in (0,1), we apply the condition of orthonormality, if

$$\int_0^1 (B_m \; sin\; m\pi x)^2 \; dx = 1 \;\; ; \quad iff\; m = n$$

$$(B_m)^2 \int_0^1 sin^2 \; m\pi x \; dx = 1 \quad \Rightarrow \quad (B_m)^2 \int_0^1 \left(\frac{1 - cos2m\pi x}{2}\right) dx = 1$$

$$(B_m)^2 \left[\frac{1}{2} x - \frac{sin2m\pi x}{42m\pi}\right]_0^1 = 1 \quad \Rightarrow \quad (B_m)^2 \left[\frac{1}{2}(1) - 0 - 0 + 0\right] = 1$$

$$(B_m)^2 = 2 \quad \Rightarrow \quad B_m = \sqrt{2} \;\; ; \quad \forall\, m \in N$$

Hence, $\{\sqrt{2}\; sin\; m\pi x\}_{m=1}^{\infty}$ is an orthonormal set.

LECTURE NO. 45

**Expansion in Terms of Orthonormal Functions:**

Let $f(x) = 1\; (say)$ and we want to express $f(x)$ as an expansion of orthonormal functions say;

$$\{\Phi_n(x)\}_{n=1}^{\infty} = \{\sqrt{2}\, sin\, n\pi x\}_{n=1}^{\infty}$$

i.e.

$$f(x) = \sum_{n=1}^{\infty} C_n\, \Phi_n(x) \quad \dots (1)$$

Where

$$C_n = \int_0^l f(x)\Phi_n(x)\, dx \quad \Rightarrow \quad \int_0^l 1.\sqrt{2}\, sin\, n\pi x\, dx \quad \Rightarrow \quad \sqrt{2}\int_0^l sin\, n\pi x\, dx$$

$$C_n = -\frac{\sqrt{2}}{n\pi}[cos\, n\pi x]_0^1 \quad \Rightarrow \quad -\frac{\sqrt{2}}{n\pi}[cos\, n\pi - 1] \quad \Rightarrow \quad \frac{\sqrt{2}}{n\pi}[1-(-1)^2]$$

So, equation (1) can be written as

$$f(x) = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{n\pi}[1-(-1)^2]\ sin\, n\pi x$$

This is our required result.

LECTURE NO. 46

**Characterizing the Eigen Values of S-L System:**

Theorem: show that the Eigen values of a S-L System are real.

Proof: Given a S-L System in $(a, b)$;

$$\frac{d}{dx}[p(x)\frac{dy}{dx}] + [q(x) + \lambda\, r(x)]y = 0$$

Or it can be written as,

$$y\frac{d}{dx}(p\, y') + [q(x) + \lambda\, r(x)]y = 0 \quad \dots (1)$$

Subjected to boundary condition;

$$\alpha_1\, y(a) + \alpha_2\, y'(a) = 0 \quad \dots (a) \quad , \quad \beta_1\, y(b) + \beta_2\, y'(b) = 0 \quad \dots (b)$$

Where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are given real numbers and $p(x), q(x)$ $and$ $r(x)$ are real valued functions.

Taking conjugate of equation (1),

$$\bar{y}\frac{d}{dx}(p\bar{y}) + [q(x) + \bar{\lambda}\, r(x)]\bar{y} = 0 \quad \dots (2)$$

Subjected to;

$$\alpha_1\bar{y}(a) + \alpha_2\bar{y}(a) = 0 \quad \dots (c) \quad , \quad \beta_1\bar{y}(b) + \beta_2\bar{y}(b) = 0 \quad \dots (d)$$

By $y(2) - y'(1)$;

$$y\frac{d}{dx}[p\bar{y}] - \bar{y}\ \frac{d}{dx}[p\, y'] + [q(x) + \bar{\lambda}\, r(x) - q(x) - \lambda\, r(x)]y\bar{y} = 0$$

$$y[p\bar{y} + p'\bar{y}] - \bar{y}[p\bar{y} + p'y'] + (\bar{\lambda} - \lambda)r(x)y\bar{y} = 0$$

$$p[y\bar{y} - \bar{y}y] + p'[y\bar{y} - \bar{y}y'] = (\lambda - \bar{\lambda})r(x)y\bar{y}$$

$$p\frac{d}{dx}[y\bar{y} - \bar{y}y'] + p'[y\bar{y} - \bar{y}y'] = (\lambda - \bar{\lambda})r(x)y\bar{y}$$

$$\frac{d}{dx}[p(x)\{y\bar{y} - \bar{y}y'\}] = (\lambda - \bar{\lambda})r(x)y\bar{y}$$

Integrating form "$a\ to\ b$",

$$(\lambda - \bar{\lambda})\int_a^b r(x)y\bar{y}dx = p(x)[\{y\bar{y} - \bar{y}y'\}]_{x=a}^{x=b}$$

$$= p(x)\{y(b)\bar{y}(b) - \bar{y}(b)y'(b) - y(a)\bar{y}(a) + \bar{y}(a)y'(a)\} \quad \dots (3)$$

Since,

$$\alpha_1\, y(a) + \alpha_2\, y'(a) = 0 \quad , \quad \beta_1\, y(b) + \beta_2\, y'(b) = 0$$

This implies,

$$\frac{y(a)}{y'(a)} = -\frac{\alpha_2}{\alpha_1} \quad , \quad \frac{\bar{y}(a)}{\bar{y}'(a)} = -\frac{\alpha_2}{\alpha_1}$$

$$\frac{y(a)}{y'(a)} = \frac{\bar{y}(a)}{\bar{y}'(a)} \quad \Rightarrow \quad y(a)\bar{y}'(a) - \bar{y}(a)y'(a) = 0$$

Similarly,

$$y(b)\bar{y}'(b) - \bar{y}(b)y'(b) = 0$$

So, equation (3) becomes, ($\because y\bar{y} = |y|^2$)

$$(\lambda - \bar{\lambda}) \int_a^b r(x)|y|^2 \, dx = 0$$

As,

$$r(x) \geq 0 \;\; and \;\; |y|^2 \geq 0 \;\; \Rightarrow \;\; \int_a^b r(x)|y|^2 \, dx \geq 0$$

So,

$$(\lambda - \bar{\lambda}) = 0 \;\; \Rightarrow \;\; \lambda = \bar{\lambda} \;\; \Rightarrow \;\; \lambda \; is \; real \; valued.$$

Hence proved.

LECTURE NO. 47

**The Gamma Function (Definition and Recurrence Relation):**

The Gamma function is defined and given by;

$$\gamma(n+1) = \int_0^\infty x^n \, e^{-x} \, dx \quad convergent \; for \quad n > 0.$$

$$= [x^n.(e^{-x})]_0^\infty - \int_0^\infty (-e^{-x}).nx^{n-1} \, dx = 0 \quad \Rightarrow \quad 0 + n\int_0^\infty x^{n-1}.e^{-x} \, dx$$

$$\gamma(n+1) = n\,\gamma(n) \;\; ; \quad \forall\, n > 0 \quad \dots (1)$$

If $n \in N$

$$For \;\; n = 1\,; \quad \gamma(2) = 1.\gamma(1) = 1.1 = 1!$$

$$For \;\; n = 2\,; \quad \gamma(3) = 2.\gamma(2) = 2.1! = 2!$$

$$For \;\; n = 3\,; \quad \gamma(4) = 3.\gamma(3) = 3.2! = 3!$$

By induction; $\gamma(n+1) = n!$

Example: $\gamma(6) = 5! = 5.4.3.2.1 = 120$

LECTURE NO. 48

**Analytic Continuation of Gamma Function:**

The Gamma function is defined and given by;

$$\gamma(n+1) = \int_0^\infty x^n \, e^{-x} \, dx\,; \quad n > 0.$$

It is technique which is used to extend the domain of a given function beyond its natural domain of definition.

$$\gamma(\alpha+1) = \alpha\,\gamma(\alpha) \;\; \Rightarrow \;\; \gamma(\alpha) = \frac{\gamma(\alpha+1)}{\alpha} \quad \dots (a)$$

Here

$$\gamma(\alpha+1) = \frac{\gamma(\alpha+2)}{(\alpha+1)}$$

Hence above (a) relation becomes,

$$\gamma(\alpha) = \frac{1}{\alpha(\alpha+1)}\,\gamma(\alpha+2)$$

Similarly, we can write it as

$$\gamma(\alpha) = \frac{1}{\alpha(\alpha+1)(\alpha+2)}\,\gamma(\alpha+3)$$

In general form it can be written as,

$$\gamma(\alpha) = \frac{1}{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+k)}\,\gamma(\alpha+k+1)$$

$\alpha > 0 \;\Rightarrow\; \gamma(\alpha)$ is not defined for $\alpha = 0, -1, -2, \dots$

Example:

$$\gamma(-0.5) = \frac{\gamma(-0.5+1)}{-0.5} = \frac{\gamma(0.5)}{-0.5} = \frac{\sqrt{\pi}}{-0.5} = -2\sqrt{\pi}$$

LECTURE NO. 49

**An Important Gamma Value:**

Theorem: Prove that $\gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof: Since,

$$\gamma(n+1) = \int_0^\infty x^{n-1} e^{-x}\,dx \;\Rightarrow\; \int_0^\infty x^{-1/2} e^{-x}\,dx$$

Let

$$x = u^2 \;\Rightarrow\; dx = 2u\,du$$
$$\{x^{-\frac{1}{2}} = u^{-1} \;\Rightarrow\; u^{-1} = \frac{1}{u}$$

By putting, we get

$$\gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{1}{u} e^{-u^2} 2u\,du \;\Rightarrow\; 2\int_0^\infty e^{-u^2}\,du$$

Taking square on both sides, $(\therefore u \to v)$

$$\left\{\gamma\left(\frac{1}{2}\right)\right\}^2 = 4\left(\int_0^\infty e^{-u^2}\,du\right)^2 \;\Rightarrow\; \left(2\int_0^\infty e^{-u^2}\,du\right)\left(2\int_0^\infty e^{-v^2}\,dv\right)$$

$$\left\{\gamma\left(\frac{1}{2}\right)\right\}^2 = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}du\,dv$$

Let

$$u = r\cos\theta, \qquad v = r\sin\theta \;\Rightarrow\; u^2+v^2 = r^2$$
$$\{As\ u, v \to \infty, \qquad so\ \ \theta = \tan^{-1}\left(\frac{v}{u}\right) \;\Rightarrow\; \theta = \frac{\pi}{2}$$
$$And \qquad du\,dv = r\,dr\,d\theta$$

By putting, we get

$$\left\{\gamma\left(\frac{1}{2}\right)\right\}^2 = 4\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r\to\infty} e^{-r^2} r\,dr\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\left\{\int_0^\infty e^{-r^2} r\,dr\right\}d\theta \;\Rightarrow\; 4\int_0^{\frac{\pi}{2}}\left[\frac{e^{-r^2}}{-2}\right]_0^\infty d\theta$$

$$= \frac{4}{-2}\int_0^{\frac{\pi}{2}}\left\{\frac{1}{\infty} - 1\right\}d\theta \;\Rightarrow\; 2\int_0^{\frac{\pi}{2}} d\theta \;\Rightarrow\; 2[\theta]_0^{\frac{\pi}{2}} \;\Rightarrow\; \pi$$

$$so, \qquad \left\{\gamma\left(\frac{1}{2}\right)\right\}^2 = \pi \;\Rightarrow\; \gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

This is required result.

LECTURE NO. 50

**Evaluating Some Improper Integrals by Gamma Function:**

Question: Evaluate

$$(i)\ \int_0^\infty x^m\, e^{-ax^n}\, dx \qquad (ii)\ \int_0^\infty e^{-\alpha\lambda^2} \cos\beta\lambda\, d\lambda$$

Solution: $(i)$ let

$$I = \int_0^\infty x^m\, e^{-ax^n}\, dx$$

Let

$$ax^n = u \ \Rightarrow\ x^n = \frac{u}{a}$$

$$du = nax^{n-1}dx \ \Rightarrow\ x = \left(\frac{u}{a}\right)^{\frac{1}{n}}$$

$$\because As\ x \to 0 \Rightarrow u = 0, \qquad x \to \infty \Rightarrow u = \infty$$

By putting, we get

$$I = \int_0^\infty \left\{\left(\frac{u}{a}\right)^{\frac{1}{n}}\right\}^m e^{-u} . d\left\{\frac{u}{a}\right\}^{\frac{1}{n}}$$

$$I = \frac{1}{a^{\frac{m}{n}}} \int_0^\infty u^{m/n}\, e^{-u}\ \left\{\frac{1}{n}\left(\frac{u}{a}\right)^{\frac{1-n}{n}} \cdot \frac{1}{a}\, du\right\}$$

By solving and simplifying, we have

$$I = \frac{1}{a^{\frac{m}{n}+\frac{1-n}{n}+1}} \int_0^\infty \frac{1}{n}\, u^{\frac{m+1}{n}-1}\, e^{-u}\, du \ \Rightarrow\ \frac{1}{na^{\frac{m+1}{n}}}\, \gamma\left(\frac{m+1}{n}\right)$$

This is our required result.

$(ii)$: $Do\ Your\ Self$.

LECTURE NO. 51

**Inverse Linear Motion:**

A particular is attracted toward a fixed point with a force inversely proportional to its instantaneous distance from fixed point. If the particle is released form rest, then find the time for it to reach fixed point.

Proof: Let at $t = 0$, the particle is a rest i.e. $v = 0$ and be at position $x = a$ and attracted towards $x = 0$. Then by given statement and by Newton's law;

$$m\frac{d^2x}{dt^2} = -\frac{k}{x} \quad \dots (1)$$

Here $m$ is the mass of particle and $k$ is constant of proportionality.

$$m\frac{d^2x}{dt^2} = m\frac{d}{dt}\left(\frac{dv}{dt}\right) \Rightarrow\ m\frac{dv}{dt} \Rightarrow\ m\frac{dv}{dx}\frac{dx}{dt} \Rightarrow mv\ \frac{dv}{dx} = -\frac{k}{2}$$

Separating the variables;

$$mv\, dv = -k\ \frac{dx}{x}$$

Integrating;

$$m\int v\, dv = -k\int \frac{dx}{x} \ \Rightarrow\ m\frac{v^2}{2} = -k\ lnx + c \quad \dots (2)$$

$\therefore v = 0\ at\ x = a \ \Rightarrow\ 0 = -k\ lna + c \ \Rightarrow\ c = k\ lna$

By putting, equation (2) becomes,

$$m\frac{v^2}{2} = k\ lna - k\ lnx = k\ ln\left(\frac{a}{x}\right) \Rightarrow v^2 = \frac{2k}{m}\ ln\left(\frac{a}{x}\right)$$

$$v = \sqrt{\frac{2k}{m}}\sqrt{ln\left(\frac{a}{x}\right)} \Rightarrow \frac{dx}{dt} = -\sqrt{\frac{2k}{m}}\sqrt{ln\left(\frac{a}{x}\right)} \ \ldots (3)$$

Here $-ve$ sign is taken because '$x$' decreases as '$t$' increases. So,

$$\sqrt{\frac{m}{2k}}\int_{x=a}^{x=0}\frac{dx}{\sqrt{ln\left(\frac{a}{x}\right)}} = -\int_{t=0}^{t=T} dt \Rightarrow T = \sqrt{\frac{m}{2k}}\int_{x=a}^{x=0}\frac{dx}{\sqrt{ln\left(\frac{a}{x}\right)}} \ \ldots (4)$$

Let

$$\begin{cases} ln\left(\frac{a}{x}\right) = u \Rightarrow e^u = \frac{a}{x} \Rightarrow x = ae^{-u} \Rightarrow dx = -ae^{-u}\,du \\ When\ x = 0 \Rightarrow u \to \infty, \quad x \to a \Rightarrow ln\left(\frac{a}{a}\right) = ln(1) = 0 = u \end{cases}$$

By putting, we get

$$T = \sqrt{\frac{m}{2k}}\int_0^\infty u^{-\frac{1}{2}}(-ae^{-u})du \Rightarrow \sqrt{\frac{m}{2k}}\int_0^\infty u^{-\frac{1}{2}}e^{-u}\,du$$

$$T = \sqrt{\frac{m}{2k}}.\gamma\left(\frac{1}{2}\right) = \sqrt{\frac{m}{2k}}\sqrt{\pi} \Rightarrow \sqrt{\frac{m\pi}{2k}}$$

This is our required result.

LECTURE NO. 52

**Beta Function (Definition and Some Properties):**

It is defined and given by;

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \ ; \quad m,n > 0$$

Let

$$x = 1 - y \Rightarrow y = 1 - x$$
$$\{x = 0 \Rightarrow y = 1, \quad x = 1 \Rightarrow 0$$
$$\therefore dx = -dy$$

By putting, we get

$$B(m,n) = \int_0^1 (1-y)^{m-1}y^{n-1}(-dy) \Rightarrow \int_0 y^{n-1}(1-y)^{m-1}\,dy$$

$$B(m,n) = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

**Question:** Prove that

$$B(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta.\,cos^{2n-1}\theta\ d\theta$$

Proof: Since,

$$B(m,n) = 2\int_0^{\pi/2} x^{m-1}(1-x)^{n-1}\,dx$$

Let

$$x = sin^2\theta \Rightarrow dx = 2sin\theta\ cos\theta$$
$$\{x = 0 \Rightarrow sin^2\theta = 0, x = 1 \Rightarrow sin^2\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

By putting, we get

$$B(m,n) = \int_0^{\pi/2} (sin^2\theta)^{m-1} (1 - sin^2\theta)^{n-1}\ 2sin\theta cos\theta\ d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} (sin^2\theta)^{m-1}(cos^2\theta)^{n-1}\ sin\theta cos\theta\ d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta . cos^{2n-1}\theta\ d\theta$$

Hence proved our require result.

LECTURE NO. 53

**Relation between Beta and Gamma Function:**

Prove that

$$B(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$$

Proof: Since

$$\gamma(m) = \int_0^{\infty} x^{m-1}\ e^{-u}\ du\ ; \quad n > 0.$$

Let $\{x^2 = u \Rightarrow du = 2x\ dx$

$$\gamma(m) = \int_0^{\infty}(x^2)^{m-1}\ e^{-x^2}\ 2xdx \Rightarrow 2\int_0^{\infty} x^{2m-1}\ e^{-x^2}\ dx$$

Similarly,

$$\gamma(n) = 2\int_0^{\infty} y^{2n-1}\ e^{-y^2}\ dy$$

$$\gamma(m)\gamma(n) = 4\int_0^{\infty}\int_0^{\infty} x^{2m-1}\ y^{2n-1}\ e^{-(x^2+y^2)}dx\ dy$$

Let

$$\left\{\begin{array}{c} x = rcos\theta\ , y = rsin\theta\ ,\ x^2 + y^2 = r^2 \\ \theta = tan^{-1}\left(\frac{y}{x}\right) \\ dx\ dy = r\ dr\ d\theta \end{array}\right.$$

By putting, we get

$$\gamma(m)\gamma(n) = 4\int_0^{\pi/2}\int_0^{\infty}(r^{2(m+n)-1}\ e^{-r^2}\ cos^{2m-1}\theta\ sin^{2n-1}\theta)d\theta\ dr$$

$$\gamma(m)\gamma(n) = 2\int_0^{\pi/2} (cos^{2m-1}\theta\ sin^{2n-1}\theta\ d\theta)\ (2\int_0^{\infty} r^{2(m+n)-1}\ e^{-r^2}\ dr)$$

$$\gamma(m)\gamma(n) = B(m,n)\ \gamma(m+n)$$

$$B(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$$

For better understanding, WATCH lecture.

LECTURE NO. 54

**Legendre's Duplication Formula for Gamma Function:**

By evaluating the integrals,

$$I = \int_0^{\pi/2} sin^{2p} x\, dx \quad and \quad J = \int_0^{\pi/2} sin^{2p}(2x)\, dx$$

Derive the Legendre's duplication formula for Gamma function.

Proof: As given

$$I = \int_0^{\pi/2} sin^{2p} x\,(cosx)^0\, dx \quad \dots (1)$$

Comparing with,

$$\int_0^{\pi/2} sin^{2m-1}\theta . cos^{2n-1} x\, dx = \frac{\gamma(m)\gamma(n)}{2\gamma(m+n)} \quad \dots (2)$$

We have,

$$2p = 2m - 1 \Rightarrow m = p + \frac{1}{2} \quad \& \quad 2n - 1 = 0 \Rightarrow n = \frac{1}{2}$$

So, equation (2) can be written as,

$$\frac{\gamma\left(p+\frac{1}{2}\right)\gamma\left(\frac{1}{2}\right)}{2\gamma(p+1)} = \frac{\sqrt{\pi}\,\gamma\left(p+\frac{1}{2}\right)}{2p\,\gamma(p)} = I \quad \dots (A)$$

LECTURE NO. 55

Now for

$$J = \int_0^{\frac{\pi}{2}} sin^{2p}(2x)\, dx$$

Let $\{2x = y \Rightarrow dx = \frac{1}{2} dy \,;\, y = \pi\}$ by putting, we get

$$= \frac{1}{2}\int_0^{\pi} (sin2y)^{2p}\, dy \Rightarrow \frac{2}{2}\int_0^{\pi/2} sin^{2p} x\, dx \Rightarrow \int_0^{\pi/2} sin^{2p} x\, dx = I$$

Again

$$J = \int_0^{\frac{\pi}{2}} sin^{2p}(2x)\, dx \Rightarrow \int_0^{\frac{\pi}{2}} (sin2x)^{2p}\, dx \Rightarrow \int_0^{\frac{\pi}{2}} (2sinx\, cosx)^{2p}\, dx$$

$$J = 2^{2p}\int_0^{\frac{\pi}{2}} sin^{2p} x . cos^{2p} x\, dx \Rightarrow \frac{2^{2p}\,\gamma\left(p+\frac{1}{2}\right)\gamma\left(p+\frac{1}{2}\right)}{2\gamma(2p+1)} \quad \dots (B)$$

$\therefore I = J$

$$\frac{\sqrt{\pi}\,\gamma\left(p+\frac{1}{2}\right)}{2p\,\gamma(p)} = \frac{2^{2p}\,\gamma\left(p+\frac{1}{2}\right)\gamma\left(p+\frac{1}{2}\right)}{2\gamma(2p+1)}$$

By solving we get,

$$\gamma(2p) = 2^{2p-1}\,(\pi)^{-1/2}\,\gamma\left(p+\frac{1}{2}\right)\,\gamma(p)$$

This is our required result.

LECTURE NO. 56

**The Walli's Product Formula (By Beta Function):**

Prove that

$$\int_0^{\frac{\pi}{2}} sin^p\,\theta\, d\theta = \int_0^{\frac{\pi}{2}} cos^p\,\theta\, d\theta = \begin{cases} \dfrac{1.3.5\ldots(p-1)}{2.4.6\ldots p}.\dfrac{\pi}{2}; & if\ p\ is\ even \\ \dfrac{2.4.6\ldots(p-1)}{1.3.5\ldots p}; & if\ p\ is\ odd \end{cases}$$

Solution: Since

$$B(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)} = 2\int_0^{\pi/2} sin^{2m-1}\theta.\,cos^{2n-1}\theta\ d\theta$$

$$\frac{\gamma(m)\gamma(n)}{2\gamma(m+n)} = \int_0^{\pi/2} sin^{2m-1}\theta.\,cos^{2n-1}\theta\ d\theta \quad \ldots(1)$$

Let $\{2m-1=p\ \ and\ \ 2n-1=0 \ \Rightarrow\ m=(P+1)\frac{1}{2},\ n=\frac{1}{2}\}$ by putting, we get

$$\int_0^{\frac{\pi}{2}} sin^p\,\theta\,d\theta = \frac{\gamma\left(\frac{1}{2}(p+1)\right)\gamma\left(\frac{1}{2}\right)}{2\gamma\left(\frac{1}{2}(p+2)\right)} \quad \ldots(2)$$

$(i)$: If $p\ is\ even \Rightarrow p=2l,\ l\in N$, then equation (2) becomes

$$\int_0^{\frac{\pi}{2}} sin^p\,\theta\,d\theta = \frac{\gamma\left(l+\frac{1}{2}\right)\gamma\left(\frac{1}{2}\right)}{2\gamma(l+1)}$$

$$= \frac{\left(l-\frac{1}{2}\right)\left(l-\frac{3}{2}\right)\ldots\frac{1}{2}\gamma\left(\frac{1}{2}\right)\gamma\left(\frac{1}{2}\right)}{2.\,l(l-1)\ldots1}$$

$$= \frac{(2l-1)(2l-3)\ldots1\ \left\{\gamma\left(\frac{1}{2}\right)\right\}^2}{2^l\,.\,2l(l-1)\ldots3.2.1}$$

$$= \frac{(2l-1)(2l-3)\ldots5.3.1\ (\sqrt{\pi})^2}{2l(2l-2)\ldots6.4.2}$$

$$= \frac{1.3.5\ldots(2l-3)(2l-3)}{2.4.6\ldots(2l-2)2l}\frac{\pi}{2}$$

$(ii)$: If $p\ is\ odd \Rightarrow p=2k+1,\ \ k\in N$

$$\int_0^{\frac{\pi}{2}} sin^p\,\theta\,d\theta = \frac{\gamma\left[\frac{1}{2}(2k+1+1)\right]\gamma\left(\frac{1}{2}\right)}{2\,\gamma\left[\frac{1}{2}(2k+3)\right]} = \frac{\gamma(k+1)\,\gamma\left(\frac{1}{2}\right)}{2\,\gamma\left(k+\frac{2}{3}\right)}$$

$$= \frac{k(k-1)(k-2)\ldots3.2.1\,\sqrt{\pi}}{2\left[\left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)\ldots\frac{1}{2}\,\gamma\left(\frac{1}{2}\right)\right]} = \frac{k(k-1)\ldots3.2.1\ (2^{k+1})}{2(2k+1)(2k-1)\ldots5.3.1}$$

$$= 2k(2k-2)\ldots6.4.2 \ \Rightarrow\ 2.4.6\ldots(2k-2)2k$$

This is our required result.

LECTURE NO. 57

**Fourier Integral (Definition and Related Theorem):**

Since Fourier series of a function $f(x)$ in $(-l,l)$ is defined and given by;

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\ cos\left(\frac{n\pi x}{l}\right) + b_n\ sin\left(\frac{n\pi x}{l}\right)\right]$$

Where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\,cos\left(\frac{n\pi x}{\pi}\right)dx \quad \&\quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\,sin\left(\frac{n\pi x}{l}\right)dx\,; \quad n=1,2,3,\ldots$$

What if $l \to \infty$, then in this case, the Fourier series becomes the Fourier Integral.

**Fourier Integral Theorem:**

Let $f(x)$ be a function define on $(-\infty, \infty)$ and satisfying;

$(i)$: $f(x)$ *and* $f'(x)$ are piecewise continuous in every finite interval.

$(ii)$: $f(x)$ is absolutely integerable in $(-\infty, \infty)$ then

$$\int_0^2 \{A(\alpha)cos\alpha x + B(\alpha)sin\alpha x\}\,dx = \begin{cases} f(x) if\ x\ is\ a\ point\ of\ continuity \\ \dfrac{f(x+0)+f(x-0)}{2}\ if\ disconti. \end{cases}$$

The L.H.S side is called Fourier Integral Expansion of $f(x)$.

LECTURE NO. 58

**Some Equivalent Forms of Fourier Integral:**

$$f(x) = \int_{\alpha=0}^{\infty} \{A(\alpha)cos\alpha x + B(\alpha)sin\alpha x\}dx \quad \dots(p)$$

$$f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{+\infty} f(u)\, cos\alpha(x-u)\,du\,d\alpha \quad \dots(q)$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\alpha(x-u)}\,du\,d\alpha \quad \dots(r)$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\,d\alpha \int_{-\infty}^{\infty} f(u)\,e^{-i\alpha x}\,du \quad \dots(s)$$

All these forms imply each other.

**Question:** Prove that $(p \rightleftarrows q)$

$$\int_0^{\infty}\{A(\alpha)cos\alpha x + B(\alpha)sin\alpha x\}dx \rightleftarrows \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{+\infty} f(u)\,cos\alpha(x-u)\,du\,d\alpha$$

Where

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)cos\alpha x\,dx \quad \& \quad B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)sin\alpha x\,dx$$

Proof: Since *__Equation (q)__*

$$f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{+\infty} f(u)\,cos\alpha(x-u)\,du\,d\alpha$$

$$= \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{+\infty} f(u)\{cos\alpha x\ cos\alpha u + sin\alpha x\ sin\alpha u\}du\,dx$$

$$= \int_{\alpha=0}^{\infty}[\{\frac{1}{\pi}\int_{u=-\infty}^{+\infty} f(u)cos\alpha u\}\,cos\alpha x + \{\frac{1}{\pi}\int_{u=-\infty}^{+\infty} f(u)\,sin\alpha u\}\,sin\alpha x]$$

$$= \int_{\alpha=0}^{\infty}\{A(\alpha)cos\alpha x + B(\alpha)sin\alpha x\}\,dx$$

Where

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)cos\alpha u\,du \quad \& \quad B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)sin\alpha u\,du$$

$(Proved\ q \Rightarrow p)$

Now $\underline{Equation\ (p)}$

$$f(x) = \int_{\alpha=0}^{\infty}\{A(\alpha)cos\alpha x + B(\alpha)sin\alpha x\}dx \quad ...(1)$$

Where

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)cos\alpha u\, du \quad \& \quad B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)sin\alpha u\, du$$

Putting $A(\alpha)$ $and$ $B(\alpha)$ values in equation (1),

$$f(x) = \int_{\alpha=0}^{\infty} [\{\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)cos\alpha u\, du\}\, cos\alpha x + \{\frac{1}{\pi}\int_{-\infty}^{\infty} f(u)sin\alpha u\, du\}\, sin\alpha x]\, d\alpha$$

$$= \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{-\infty}^{\infty} f(u)\, [cos\alpha x\ cos\alpha u + sin\alpha x\ sin\alpha u]\, du\, dx$$

$$f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{+\infty} f(u)\, cos\alpha(x-u)\, du\, d\alpha$$

$(Proved\ p \Rightarrow q)$

Hence Proved. $(q\ implies\ p\ \&\ p\ implies\ q)$.

LECTURE NO. 59

**Fourier Transform and its Inverse (Definition):**

For a function $f(x)$ defined on $(-\infty, \infty)$, Fourier Integral is defined and given by;

$$(x) = \int_{0}^{\infty}\{A(\alpha)cos\alpha x + B(\alpha)sin\alpha x\}dx$$

Where

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)cos\alpha x\, dx \quad \& \quad B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)sin\alpha x\, dx$$

And one of its equivalent form is;

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\, d\alpha \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha x}\, du$$

Taking

$$F(\alpha) = \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha x}\, du \quad ...Kernal$$

Then above relation becomes,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha$$

Here $F(\alpha)$ is called Fourier Transform of $f(x)$ and $f(x)$ is called Inverse Fourier Transform of $F(\alpha)$.
It can also be expressed as;

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x}\, f(x)\, d\alpha \quad \& \quad f^{-1}\{F(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\alpha x}\, F(x)\, d\alpha = f(x)$$

Note that in both cases/forms/formulations "$product\ of\ constants\ should\ be\ \frac{1}{2\pi}$."

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**"Good Luck For The Mid-Term Exam"**

LECTURE NO. 60

**Fourier Transform of Unit Step Function:**

Find the Fourier series of

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

Solution: Fourier Transform of $f(x)$

$$F(\alpha) = \int_{-\infty}^{\infty} f(u)e^{-i\alpha u}\ du$$

For given problem,

$$F(\alpha) = \int_{-a}^{a} 1.\, e^{-i\alpha u}\ du \quad \dots (1)$$

$$F(\alpha) = \left|\frac{e^{-i\alpha u}}{-i\alpha}\right|_{-a}^{a} \Rightarrow \frac{e^{-i\alpha a} - e^{-i\alpha(-a)}}{-i\alpha} \Rightarrow \frac{e^{i\alpha\, a} - e^{-i\alpha\, a}}{i\alpha}$$

Divide and multiply by 2,

$$\frac{2}{\alpha}\left[\frac{e^{i\alpha\, a} - e^{-i\alpha\, a}}{2i}\right]$$

Using relation "$\sin(x) = \frac{e^{i(x)} - e^{-i(x)}}{2i}$" we have

$$F(\alpha) = \frac{2}{\alpha}\ \sin\alpha a$$

It is the required function when $\alpha \neq 0$.

If we put $\alpha = 0$ in equation (1), we get

$$F(\alpha) = \int_{-a}^{a} 1\ du \Rightarrow 2a$$

LECTURE NO. 61

**Fourier Transform of Exponential Functions:**

Fourier function of

$$e^{-px^2}, p > 0$$

Solution: As we know from previous lecture,

$$F(\alpha) = \int_{-\infty}^{\infty} f(u)e^{-i\alpha u}\ du$$

For given problem,

$$F(\alpha) = \int_{-\infty}^{\infty} e^{-pu^2}.\, e^{-i\alpha u}\ du$$

$$F(\alpha) = \int_{-\infty}^{\infty} e^{-p\{u^2 - \frac{i\alpha}{p}u\}}\, du \Rightarrow \int_{-\infty}^{\infty} e^{-p\{u^2 - \frac{i\alpha}{p}u + (\frac{i\alpha}{2p})^2 - (\frac{i\alpha}{2p})^2\}}\, du$$

$$F(\alpha) = \int_{-\infty}^{\infty} e^{-p\{(u - \frac{i\alpha}{2p})^2 + \frac{\alpha^2}{4p^2}\}}\, du \Rightarrow e^{\frac{\alpha^2}{4p^2}} \int_{-\infty}^{\infty} e^{-p(u - \frac{i\alpha}{2p})^2}\, du$$

Let

$$\sqrt{p}\,(u - \frac{i\alpha}{2p}) = y$$

$$dy = \sqrt{p}\, du \Rightarrow du = \frac{1}{\sqrt{p}}\, dy$$

By putting in above relation, we have

$$F(\alpha) = \frac{e^{\frac{\alpha^2}{4p^2}}}{\sqrt{p}} \int_{-\infty}^{\infty} e^{-y^2}\, dy \Rightarrow \frac{e^{\frac{\alpha^2}{4p^2}}}{\sqrt{p}} \sqrt{\pi} \Rightarrow \sqrt{\frac{\pi}{p}}\, e^{\frac{\alpha^2}{4p^2}}$$

This is our required result.

LECTURE NO. 62

**Conjugate of a Fourier Series:**

Theorem: For a real valued function $f(x)$ on $(-\infty, \infty)$, show that $\overline{F(\alpha)} = F(-\alpha)$

Proof: As we know

$$F(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-i\alpha u}\, du$$

Taking conjugate, we have

$$\overline{F(\alpha)} = \overline{\int_{-\infty}^{\infty} f(u) e^{-i\alpha u}\, du} \Rightarrow \int_{-\infty}^{\infty} \overline{f(u)}\, \left(\overline{e^{-i\alpha u}}\right) du \Rightarrow \int_{-\infty}^{\infty} f(u) e^{i\alpha u}\, du$$

$$\overline{F(\alpha)} = \int_{-\infty}^{\infty} f(u) e^{-(-i\alpha u)} du \Rightarrow \int_{-\infty}^{\infty} f(u) e^{-i(-\alpha)u}\, du \Rightarrow F(-\alpha)$$

Hence proved that

$$\overline{F(\alpha)} = F(-\alpha)$$

LECTURE NO. 63

**Fourier Transforms of Even and Odd Functions:**

Let $f(x)$ be an odd function

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(u) e^{-i\alpha u}\, du = F(\alpha)$$

Here put $u = -t \Rightarrow du = -dt$

As $u \to \infty\ ,\ t \to \infty\ \ and\ \ u \to -\infty\ \Rightarrow\ t \to \infty$

$$F(\alpha) = -\int_{-\infty}^{\infty} f(-t) e^{-i\alpha(-t)}\, dt \Rightarrow -\int_{-\infty}^{\infty} f(t) e^{-i\alpha(-t)}\, dt = -\int_{-\infty}^{\infty} f(t) e^{-i(-\alpha)t}\, dt$$

Hence proved that

$$F(\alpha) = -F(-\alpha) = -\overline{F(\alpha)}$$

LECTURE NO. 64

**Attenuation Property of Fourier Transforms:**

Theorem:

$$F\{f(x)e^{-px}\} = F(\alpha - pi) \quad where \quad F(\alpha) = F\{f(x)\}$$

Proof: Aswe know

$$F\{f(x)\} = F(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-i\alpha u}\, du$$

For given function,

$$F\{f(x)e^{-px}\} = \int_{-\infty}^{\infty} f(u) e^{-pu} e^{-i\alpha u}\, du$$

$$= \int_{-\infty}^{\infty} f(u) e^{i^2 pu} e^{-i\alpha u}\, du \Rightarrow \int_{-\infty}^{\infty} f(u) e^{i^2 pu - i\alpha u}\, du$$

$$F\{f(x)e^{-px}\} = \int_{-\infty}^{\infty} f(u)e^{-i(\alpha-ip)u}\ du = F(\alpha - ip)$$

Hence proved that

$$F\{f(x)e^{-px}\} = F(\alpha - pi)$$

LECTURE NO. 65

**Shifting Property of Fourier Transform:**

Theorem:

$$If \quad F\{f(x)\} = F(\alpha) \quad then \quad F\{f(x-p)\} = e^{-i\alpha p}\ F(\alpha)$$

Proof: As we know

$$F(\alpha) = F\{f(x)\} = \int_{-\infty}^{\infty} f(u)e^{-i\alpha u}\ du$$

For given function,

$$F\{f(x-p)\} = \int_{-\infty}^{\infty} f(u-p)e^{-i\alpha u}\ du$$

Put $\quad u - p = t \Rightarrow u = p + t\ \ and\ \ dt = du$

$$F\{f(x-p)\} = \int_{-\infty}^{\infty} f(t)e^{-i\alpha(p+t)}\,dt \Rightarrow \int_{-\infty}^{\infty} f(t)e^{-i\alpha p}\,e^{-i\alpha t}\ du$$

$$= e^{-i\alpha p}\int_{-\infty}^{\infty} f(t)e^{-i\alpha t}\ du \Rightarrow e^{-i\alpha p}\ F\{f(x)\} = e^{-i\alpha p}\ F(\alpha)$$

This is our required result.

LECTURE NO. 66

**Fourier Transform of Derivatives:**

Theorem: For a function $f(x)$on $(-\infty,\infty)$, show that if $f(x)$ is n-times differentiable and $f^{n-1}(x) \to 0$ as $x \to \pm\infty$, then

$$F\{f^{(n)}(x)\} = (i\alpha)^n\ F(u)$$

Proof:

$$F\{\frac{d^n y}{du^n}\} = \int_{-\infty}^{\infty} \frac{d^n y}{du^n} e^{-i\alpha u}\ du$$

By integrating,

$$= e^{-i\alpha u}\left|\frac{d^{n-1}y}{du^{n-1}}\right|_{-\infty}^{\infty} + i\alpha \int_{-\infty}^{\infty} \frac{d^{n-1}y}{du^{n-1}} e^{-i\alpha u}\,du$$

$$= 0 + (i\alpha)^1\left[\{e^{-i\alpha u}\frac{d^{n-2}y}{du^{n-2}}\}_{-\infty}^{\infty} + i\alpha \int_{-\infty}^{\infty} \frac{d^{n-2}y}{du^{n-2}} e^{-i\alpha u}\,du\right]$$

$$= (i\alpha)^1\left[0 + (i\alpha)F\{\frac{d^{n-2}y}{du^{n-2}}\}\right] \Rightarrow (i\alpha)^2\,F\{\frac{d^{n-2}y}{du^{n-2}}\}$$

Now by induction,

$$F\{\frac{d^n y}{du^n}\} = (i\alpha)^n F\{f(x)\} = (i\alpha)^n\ F(u)$$

Hence proved that

$$F\{f^{(n)}(x)\} = (i\alpha)^n\ F(u)$$

## LECTURE NO. 67

**Parseval's Theorems:**

Theorem: If $f(x)$ and $g(x)$ are real valued on $(-\infty,\infty)$, then show

$$I)\quad \int_{-\infty}^{\infty} F(\alpha)\, G(-\alpha) d\alpha = \int_{-\infty}^{\infty} f(u) g(u) du$$

$$II)\quad \int_{-\infty}^{\infty} |f(u)|^2\, du = \int_{-\infty}^{\infty} |F(\alpha)|^2\, d\alpha$$

Proof: **I)**

$$\int_{-\infty}^{\infty} F(\alpha)\, G(-\alpha) d\alpha = \int_{-\infty}^{\infty} F(\alpha)\left\{\int_{-\infty}^{\infty} g(u) e^{-i(-\alpha)u} du\right\}$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} F(\alpha) e^{i\alpha u}\right\} g(u)\, du = \int_{-\infty}^{\infty} f(u)\, g(u)\, du$$

**II):** Putting $f = g$

$$\int_{-\infty}^{\infty} F(\alpha) G(-\alpha) d\alpha = \int_{-\infty}^{\infty} F(\alpha)\overline{G}(\alpha) d\alpha = \int_{-\infty}^{\infty} F(\alpha) G(\alpha) d\alpha = \int_{-\infty}^{\infty} f(u)\, g(u)\, du$$

$$\Rightarrow \int_{-\infty}^{\infty} |G(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} |g(u)|^2\, du \;Or\; \int_{-\infty}^{\infty} |f(u)|^2\, du = \int_{-\infty}^{\infty} |F(\alpha)|^2\, d\alpha$$

Required result proved.

## LECTURE NO. 68

**Convolution (Definition and Related Theorem):**

$$F(f) = F(\alpha)\ ,\ \ G(g) = G(\alpha)$$
$$F(f\ .\ g) \neq F(\alpha) G(\alpha)$$
$$F(f * g) = F(\alpha)\, G(\alpha)$$

Convolution: of function $f(x)$ *and* $g(x)$ is defined and given,

$$f * g = \int_{-\infty}^{\infty} f(u) g(x-u) du$$

Now we have to prove some theorems.

1) Commutative $(f * g = g * f)$: By definition

$$f * g = \int_{-\infty}^{\infty} f(u) g(x-u) du$$

Put $\quad x - u = t\,, u = x - t \Rightarrow du = -dt \quad and\ as\ \begin{cases} u \to \infty \;\Rightarrow\; t \to \pm\infty \\ u \to -\infty \;\Rightarrow\; t \to \infty \end{cases}$

So, we get

$$f * g = \int_{+\infty}^{-\infty} f(x-t) g(t)(-dt) \Rightarrow \int_{-\infty}^{\infty} g(t) f(x-t)(dt) = g * f$$

Hence proved

$$f * g = g * f$$

2) $F\{f * g\} = F(\alpha)\, G(\alpha)\ \ Or\ \ F^{-1}[F(\alpha)\, G(\alpha)] = f * g$

By definition,

$$F^{-1}[F(\alpha)\, G(\alpha)] = \frac{1}{2\pi}\int^{\infty} F(\alpha)\, G(\alpha)\, e^{i\alpha x}\, dx \Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)$$

$-\infty$

$$e^{i\alpha x}\, dx\, .\, G(\alpha)$$

As we know (from previous lectures)

$$G(\alpha) = \int_{-\infty}^{\infty} g(t)e^{-i\alpha t} \; dt$$

By putting, we have

$$F^{-1}[F(\alpha)\; G(\alpha)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha)\; e^{i\alpha x}\, dx \int_{-\infty}^{\infty} g(t)e^{-i\alpha t} \; dt$$

$$F^{-1}[F(\alpha)\; G(\alpha)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t)\; dt \int_{-\infty}^{\infty} F(\alpha)\; e^{i\alpha x}\, e^{-i\alpha t} \; dt$$

$$F^{-1}[F(\alpha)\; G(\alpha)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t)\; dt \int_{-\infty}^{\infty} F(\alpha)\; e^{i\alpha (x-t)} \; dt$$

By using inverse Fourier transformation,

$$F^{-1}[F(\alpha)\; G(\alpha)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t)\; f(x-t)dt \Rightarrow g * f$$

As $"g * f = f * g"$ so

$$F^{-1}[F(\alpha)\; G(\alpha)] = f * g$$

Applying "$F$" on both sides, we get

$$F\{f * g\} = F(\alpha)\; G(\alpha)$$

**Examples:**

$$i): \quad f(x) = e^{-x^2} \quad ; \quad g(x) = \begin{cases} 1 & |x| \le p \\ 0 & |x| \ge p \end{cases}$$

$$ii) \quad f(x) = e^{-\alpha x^2} \quad ; \quad g(x) = e^{-\beta x^2} \quad ; \quad \alpha, \beta > 0$$

Do Your Self.

LECTURE NO. 69

**Solving Integral Equation by Convolution:**

Solve the integral equation,

$$y(x) = g(x) + \int_{-\infty}^{\infty} y(u)r(x-u)du$$

Solution: For the given integral equation, taking fourier transformation on both sides,

$$F(y(x)) = F\{g(x)\} + F\left\{ \int_{-\infty}^{\infty} y(u)r(x-u)du\right\}$$

Convolution of last function,

$$\int_{-\infty}^{\infty} y(u)r(x-u)du = y(x) * r(x)$$

So, above relation can be writtena as

$$F(y(x)) = F\{g(x)\} + F\{y(x) * r(x)\} \;\; \dots (1)$$

Let $\begin{cases} F(y(x)) = Y(\alpha) \\ F\{g(x)\} = G(\alpha) \\ F\{r(x)\} = R(\alpha) \end{cases}$ By putting equation (1) becomes

$$Y(\alpha) = G(\alpha) + F\{y(x) * r(x)\} \Rightarrow G(\alpha) + Y(\alpha)R(\alpha)$$

By solving we have

$$Y(\alpha) = \frac{G(\alpha)}{1 - R(\alpha)}$$

Taking inverse fourier transformation,

$$F^{-1}\{Y(\alpha)\} = F^{-1}\{\frac{G(\alpha)}{1-R(\alpha)}\} \Rightarrow \int_{-\infty}^{\infty} (\frac{G(\alpha)}{1-R(\alpha)})\, e^{i\alpha x}\, dx$$

$Since\ \ F \to 1\ \ then\ \ F^{-1} \to \dfrac{1}{2\pi}\quad and\ here\ \ F^{-1}\{Y(\alpha)\} = y(x)$

So the above equation becomes,

$$y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \{\frac{G(\alpha)}{1-R(\alpha)}\}\, e^{i\alpha x}\, dx$$

This is our required result.

LECTURE NO. 70

**Fourier Sine and Cosing Tranforms:**

***I***): If $f(x)$ is an odd function, then Fourier $sine$ transform is defined and given by

$$F_s(\alpha) = \int_0^{\infty} f(u)\, sin\alpha u\, du$$

And its inverse Fourier $sine$ transfrom is given by

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\alpha)\ \ sin\alpha u\ d\alpha$$

***II***): If $f(x)$ is an even function, then fourier $cosine$ transform of $f(x)$ is defined and given by

$$F_c(\alpha) = \int_0^{\infty} f(u)\, cos\alpha u\, du$$

And its inverse Fourier $cosine$ transfrom is given by

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(\alpha)\ \ cos\alpha u\ d\alpha$$

Example:

$$If;\quad f(x) = e^{-mx}\quad ,\ m > 0\quad then\quad F_c(\alpha) = ?$$

As we know

$$F_c(\alpha) = \int_0^{\infty} f(u)\, cos\alpha u\, du$$

For given function, it can be written as

$$F_c(\alpha) = \int_0^{\infty} e^{-mu}\ cos\alpha u\ du \qquad \dots (1)$$

Using integral formula, which is given as

$$[\int e^{px}\ cosqx\ dx = \frac{e^{px}}{p^2+q^2}\{p\ cosqx + q\ sinqx\}]$$

Here $\quad p \to -m\ ,\ \ q \to \alpha\ \ and\ \ x \to u$

So, by applying integral formula, equation (1) becomes

$$F_c(\alpha) = [\frac{e^{-mu}}{m^2+\alpha^2}\{-m\ cos\alpha u + \alpha\ sin\alpha u\}]_0^{\infty}$$

By applying limits, we get

$$F_c(\alpha) = \frac{m}{m^2+\alpha^2} = F_c\{e^{-mx}\}$$

Now its inverse, $cosine$ transform;

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(\alpha)\ \ cos\alpha u\, d\alpha$$

For given function, it can be written as (Putting $F_c(\alpha)$ calculated value)

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{m}{m^2+\alpha^2}\, cos\alpha u\, d\alpha$$

As $f(x) = e^{-mx}$ ,

$$e^{-mx} = \frac{2m}{\pi}\int_0^\infty \frac{cos\alpha u}{m^2+\alpha^2}\, d\alpha$$

$$\frac{\pi e^{-mx}}{2m} = \int_0^\infty \frac{cos\alpha u}{m^2+\alpha^2}\, d\alpha$$

This is our required result.

LECTURE NO. 71

**Integral Equations Solution by Fourier Sine Transform:**

Solve the integral equation

$$\int_0^\infty f(x)\, sin\alpha x\, dx = \{\begin{matrix} 1-\alpha & 0 \le \alpha \le 1 \\ 0 & \alpha > 1 \end{matrix}$$

Solution: The given function can be writen as

$$F_s\{f(\alpha)\} = F_s(\alpha) = \int_0^\infty f(x)\, sin\alpha x\, dx = \{\begin{matrix} 1-\alpha & 0 \le \alpha \le 1 \\ 0 & \alpha > 1 \end{matrix}$$

Its $F^{-1}$ for given function,

$$F_s^{-1}\{f(x)\} = f(x) = \frac{2}{\pi}\int_0^\infty F_s(\alpha)\, sin\alpha x\, d\alpha$$

$$f(x) = \frac{2}{\pi}\int_0^1 (1-\alpha)\, sin\alpha x\, d\alpha$$

Integrating and applying limits, we have

$$= \frac{2}{\pi}[\{(1-\alpha)\,(-\frac{cos\alpha x}{x})\}_0^1 + \int_0^1 \frac{cos\alpha x}{x}(-1)\, d\alpha]$$

$$= \frac{2}{\pi}[0-(1)\,(-\frac{1}{x}) - \frac{1}{x}|\frac{sin\alpha x}{x}|_0^1\,] \Rightarrow \frac{2}{\pi}[\frac{1}{x} - \frac{sinx}{x^2}]$$

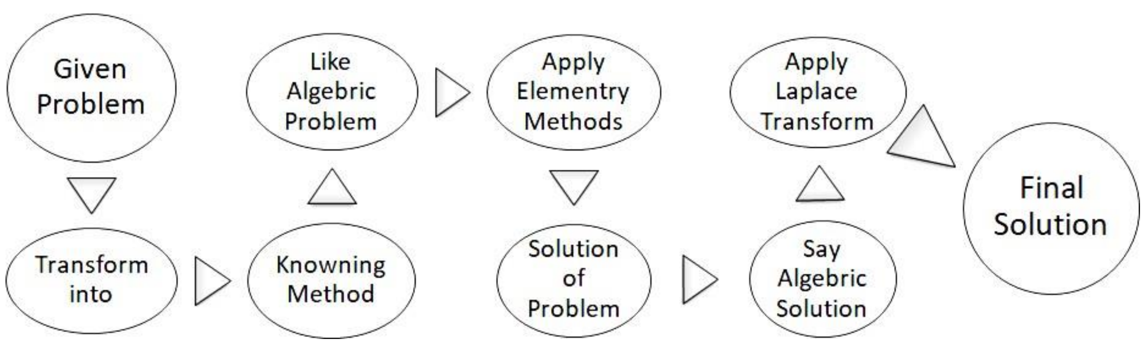
$$f(x) = \frac{2}{\pi}(\frac{x-sinx}{x^2})$$

This is our required result.

LECTURE NO. 72

**Laplace Transforms:**

Objectives/Aim/Steps:



Usefulness:

Deals with discontinuos functions (like electrical signals, eechanical forces etc) and sine, cosines forms (like Fourier analysis).

It is Direct method to solve differential equations.

Applications:

Physics, Engineering Problems → Transform into → Differential Equations (Like ODEs, PDEs, BVPs , etc.)

LECTURE NO. 73

**Integral Transform:**

For given function $f(t)$ defined on $[0,\infty)$ $or\ t \geq 0$, if improper function

$$\int_0^\infty K(s,t)\, f(t)\, dt$$

Is convergent, then it is called an integral transformation of $f(t)$ where $s \in C$. And if choosen

$$K(s,t) = e^{-st} \quad then \quad F(s) = L(f(t)) = \int_0^\infty e^{-st}\, f(t)\, dt \quad , \quad s \in C.$$

Is said to be Laplace transform of $f(t)$ provided that it is convergent. Here $K(s,t) = e^{-st}$ is known as Kernal.

Q: Why $K(s,t) = e^{-st}$ ? Why we choose/use Kernal?

Ans: 1) Kernal smooth out the given functions (Like sharp egdges, corners etc.)

**2)** Easy to integrate or differetinate.

**3)** By kernal, differntial equation transform into algebric equation. So that it can be easily solved.

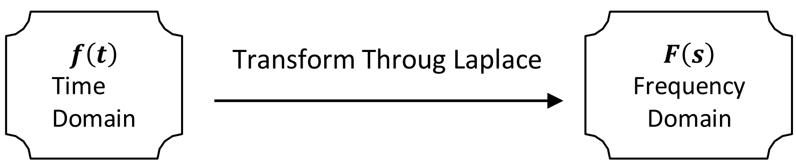
LECTURE NO. 74

**Geometical Interpretition of Laplace Transform:**

Since we know

$$L(f(t)) = F(s) = \int_0^\infty e^{-st}\, f(t)\, dt \quad , t \geq 0\ , \ s \in C, R.$$

Trnasformation of $f(t)\ into\ f(s)$:




$F(s)$ gives us information about the different frequency components that makes the $f(t)$.

Laplace relations:

$$L(f(t)) = F(s)$$
$$f(t) = L^{-1}(F(s))$$
$$L^{-1}[L(f(t))] = f(t)$$
$$L[L^{-1}(f(t))] = F(s)$$

LECTURE NO. 75

**Laplace Transform of "1":**

Since we know

$$L(f(t)) = F(s) = \int_0^\infty e^{-st}\, f(t)\, dt$$

For given function,

$$L(1) = \int_0^\infty 1.e^{-st}\, dt \quad \Rightarrow \lim_{T\to\infty} \int_0^T e^{-st}\, dt$$

Integrating and applying limits, we have

$$L(1) = \left|\lim_{T\to\infty} \frac{e^{-st}}{-s}\right|_0^T \Rightarrow \left|-\frac{1}{s} \lim_{T\to\infty} e^{-st}\right|_0^T \Rightarrow -\frac{1}{s}\left|\lim_{T\to\infty} \frac{1}{e^{st}}\right|_0^T \quad \dots (1)$$

$$L(1) = -\frac{1}{s}\left[\lim_{T\to\infty} \frac{1}{e^{sT}} - \frac{1}{e^{s(0)}}\right] \Rightarrow -\frac{1}{s}\left[\frac{1}{e^\infty} - \frac{1}{1}\right]$$

$$L(1) = \frac{1}{s}$$

This is our required result for $s > 0$. What if $s < 0$?

$$If; \qquad s < 0 \Rightarrow -s > 0 \Rightarrow -s = m\ (say) \in R^+$$

If so, then equation (1) becomes

$$\lim_{T\to\infty} \frac{1}{e^{st}} = \lim_{T\to\infty} e^{-st} = \lim_{T\to\infty} e^{mt} \Rightarrow e^\infty = \infty$$

So, whenever $s < 0$, we cannot calculate the transform of "1". Hence, we can take only $s > 0$.

## LECTURE NO. 76

**Laplace Transform of Exponential Function:**

Let $f(t) = e^{at}$. Here "a" is a constant, then evaluate $L(e^{at})$

Solution: Laplace transform for given function,

$$L(e^{at}) = \int_0^\infty e^{-st}\, e^{at}\, dt \Rightarrow \int_0^\infty e^{-(s-a)t}\, dt$$

Integrating and applying limits, we have

$$L(e^{at}) = \left|\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty \Rightarrow \frac{-1}{(s-a)}\left[e^{-(s-a)t}\right]_0^\infty$$

$$L(e^{at}) = \frac{-1}{(s-a)}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty = \frac{-1}{(s-a)}\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right]$$

$$L(e^{at}) = \frac{1}{(s-a)}$$

This is our required result. $(For\ s - a > 0 \Rightarrow s > a)$

## LECTURE NO. 77

**Linearity of Laplace Transform:**

Theorem: If $L(f(t))$ and $L(g(t))$ exist, then for any constants "$\alpha$" and "$\beta$", transform of

$$\alpha\, L(f(t)) + \beta\, L(g(t))$$

Also exist and further

$$L[\alpha\,(f(t)) + \beta\,(g(t))] = \alpha\, L(f(t)) + \beta\, L(g(t))$$

Also holds.

Proof: Applying Laplace transform on L.H.S.

$$= L[\alpha\,(f(t)) + \beta\,(g(t))]$$

$$= \int_0^\infty e^{-st}\,[\alpha\,(f(t)) + \beta\,(g(t))]dt \Rightarrow \int_0^\infty [\alpha\{e^{-st} f(t)\} + +\beta\,\{e^{-st} g(t)\}]dt$$

$\therefore$ Integration is a linear transformation on $R$.

$$= \int_0^\infty \alpha\, e^{-st} f(t)dt + \int_0^\infty \beta\, e^{-st} g(t)dt \Rightarrow \alpha \int_0^\infty e^{-st} f(t)dt + \beta \int_0^\infty e^{-st} g(t)dt$$

$$= \alpha\, L(f(t)) + \beta\, L(g(t)) = R.H.S$$

Required result proved.

$$\therefore \quad L(f(t)) \quad and \quad L(g(t)) \quad exist.$$

$$\Rightarrow \quad \alpha\, L(f(t)) + \beta\, L(g(t)) < \infty \qquad also\ exist.\ (Improper\ Integral)$$

LECTURE NO. 78

**Corollary:**

$$L[\alpha\, \{f(t)\} + \beta\, \{g(t)\}] = \alpha\, L\{f(t)\} + \beta\, L\{g(t)\}$$

If $\beta = 0$;

$$L[\alpha\, \{f(t)\}] = \alpha\, L\{f(t)\}$$

Here we use "Scalar Composition" property.

This property also used in calculus. For example:

$$Linear\ Transformation:\ T(\alpha v) = \alpha(Tv)$$

$$Derivative:\quad \frac{d}{dx}(\alpha\, sinx) = \alpha \frac{d}{dx} sins$$

$$Integration:\quad \int c\, f(x)dx = c \int f(c)dx$$

LECTURE NO. 79

**Laplace Transform of Hyperbolic Function:**

As we know

$$cos(x) = \frac{e^x + e^{-x}}{2}$$

Calculate: $L(cos\ at)$, where "a" is a constant

Solution: As given

$$L(cos\ at) = L\,[\frac{1}{2}\,(e^{at} + e^{-at})] \Rightarrow \frac{1}{2}\,[L(e^{at} + e^{-at})]$$

$$L(cos\ at) = \frac{1}{2}\,[L(e^{at}) + L(e^{-at})] \Rightarrow \frac{1}{2}[\frac{1}{s-a} + \frac{1}{s+a}]$$

$\therefore\ L(e^{at}) = \frac{1}{s-a} \quad and \quad L(e^{-at}) = \frac{1}{s+a}$, so

$$L(cos\ at) = \frac{1}{2}[\frac{s+a+s-a}{s^2-a^2}] \Rightarrow \frac{s}{s^2-a^2}$$

This is our required result.

LECTURE NO. 80

**Laplace Transform of Cosine and Sine:**

Show that

$$L(cos\ at) = \frac{s}{s^2+a^2} \qquad and \qquad L(sin\ at) = \frac{a}{s^2+a^2}$$

Proof: Suppose that

$$L(cos\ at) = F_c \qquad and \qquad L(sin\ at) = F_s$$

$\boldsymbol{L(cos\ at)}$**:** Applying Laplace Transform

$$F_c = \int_0^\infty e^{-st}\,(cos\ at)\,dt$$

Integrating and applying limits, we have

$$F_c = |cos\ at\,(\frac{e^{-st}}{-s})|_0^\infty - \int_0^\infty (\frac{e^{-st}}{-s})\,(-asin\ at)dt$$

$$F_c = -\frac{1}{s}[\frac{cos\ at}{e^{st}}]_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\,(sin\ at)dt$$

$$F_c = -\frac{1}{s}\left[\frac{\cos\infty}{e^{\infty}} - \frac{\cos(0)}{e^0}\right] - \frac{a}{s}\left(L(\sin at)\right)$$
$$F_c = -\frac{1}{s}[0-1] - \frac{a}{s}F_s$$
$$F_c = \frac{1}{s} - \frac{a}{s}F_s \quad \ldots(1)$$

$\boldsymbol{L(\sin at)}$:

$$F_s = \int_0^{\infty} e^{-st}(\sin at)\,dt \Rightarrow \left|\sin at\left(\frac{e^{-st}}{-s}\right)\right|_0^{\infty} - \int_0^{\infty}\left(\frac{e^{-st}}{-s}\right)(a\cos at)dt$$
$$F_s = -\frac{1}{s}\left[\frac{\sin at}{e^{st}}\right]_0^{\infty} + \frac{a}{s}\int_0^{\infty} e^{-st}(\cos at)dt$$
$$F_s = -\frac{1}{s}\left[\frac{\sin\infty}{e^{\infty}} - \frac{\sin(0)}{e^0}\right] + \frac{a}{s}\left(L(\cos at)\right)$$
$$F_s = -\frac{1}{s}[0-0] + \frac{a}{s}F_c$$
$$F_s = \frac{a}{s}F_c \quad \ldots(2)$$

By putting $F_s$ value in equation (1), we get

$$F_c = \frac{1}{s} - \frac{a}{s}\frac{a}{s}F_c \Rightarrow \frac{1}{s} - \frac{a^2}{s^2}F_c$$
$$F_c + \frac{a^2}{s^2}F_c = \frac{1}{s} \Rightarrow F_c\left(\frac{s^2+a^2}{s^2}\right) = \frac{1}{s}$$
$$F_c = \frac{1}{s}\left(\frac{s^2}{s^2+a^2}\right) \Rightarrow \frac{s}{s^2+a^2}$$

As $L(\cos at) = F_c$, so

$$L(\cos at) = \frac{s}{s^2+a^2}$$

This is our required result.

Now, from equation (2)

$$F_c = \frac{s}{a}F_s$$

By putting in equation (1) we get

$$\frac{s}{a}F_s = \frac{1}{s} - \frac{a}{s}F_s \Rightarrow \left(\frac{s}{a} + \frac{a}{s}\right)\frac{s}{a}F_s = \frac{1}{s}$$
$$\left(\frac{s^2+a^2}{as}\right)F_s = \frac{1}{s} \Rightarrow F_s = \left(\frac{a}{s^2+a^2}\right)$$

As $L(\sin at) = F_s$, so

$$L(\sin at) = \frac{a}{s^2+a^2}$$

This is our required result.

LECTURE NO. 81

**Laplace Transform of $\boldsymbol{t^n}$ ($\boldsymbol{n \in N}$):**

Theorem: Prove that

$$L(t^n) = \frac{n!}{s^{n+1}} \quad ; \quad n \in N \qquad \ldots(1)$$

Proof: We'll prove it by mathematical induction.

For $n = 0$;

$$L(t^0) = \frac{0!}{s^{0+1}} \Rightarrow L(1) = \frac{1}{s}$$

Suppose equation (1) is true for fixed "$n$" and we will prove it for "$n+1$".
For $n+1$;

$$L(t^{n+1}) = \frac{(n+1)!}{s^{n+2}}$$

Applying Laplace Transform,

$$L(t^{n+1}) = \int_0^\infty e^{-st}\, t^{n+1}\, dt$$

Integrating and applying limits, we have

$$L(t^{n+1}) = \left| t^{n+1} \left(\frac{e^{-st}}{-s}\right)\right|_0^\infty - \int_0^\infty \left(\frac{e^{-st}}{-s}\right)(n+1)t^n\, dt$$

$$L(t^{n+1}) = -\frac{1}{s}\left[\frac{t^{n+1}}{e^{st}}\right]_0^\infty + \frac{n+1}{s}\int_0^\infty e^{-st}\, t^n\, dt$$

$$L(t^{n+1}) = -\frac{1}{s}\left[\lim_{t\to\infty}\frac{t^{n+1}}{e^{st}} - 0\right] + \frac{n+1}{s} L(t^n)$$

As given $L(t^n) = \frac{n!}{s^{n+1}}$, so

$$L(t^{n+1}) = -\frac{1}{s}[0-0] + \frac{n+1}{s}\left(\frac{n!}{s^{n+1}}\right)$$

$$L(t^{n+1}) = \frac{(n+1)!}{s^{n+2}}$$

i.e. it is true for "$n+1$". Hence, proved our required result.

LECTURE NO. 82

**Laplace Transform of $t^a$ $(a > -1)$:**

Aaplying Laplace Transorm,

$$L(t^\alpha) = \int_0^\infty e^{-st}\, t^\alpha\, dt$$

Put $\begin{cases} v = st \Rightarrow dv = sdt \Rightarrow dt = \frac{1}{s}dv \ \ and\ \ t = \frac{s}{v} \\ As\ t \to 0 \Rightarrow v \to 0\ \ and\ as\ t \to \infty \Rightarrow v \to \infty \end{cases}$

$$L(t^\alpha) = \int_0^\infty e^{-v}\left(\frac{v}{s}\right)^\alpha \frac{1}{s}\, dv \Rightarrow \int_0^\infty e^{-v}\left(\frac{v^\alpha}{s^{\alpha+1}}\right) dv$$

$$L(t^\alpha) = \frac{1}{s^{\alpha+1}}\int_0^\infty e^{-v}\, v^\alpha\, dv$$

Here

$$\int_0^\infty e^{-v}\, v^\alpha\, dv = Gamma\ Function = \gamma(\alpha+1)$$

So,

$$L(t^\alpha) = \frac{\gamma(\alpha+1)}{s^{\alpha+1}}$$

This is our required result.

LECTURE NO. 83

**S-Shifting Theorem:**

If $f(t)$ has the transform $F(s)$ where $s > k$, then "$e^{\alpha t} f(t)$" has the transform $F(s-\alpha)$ where $(s-\alpha) > k$ i.e.

$$L\{e^{\alpha t} f(t)\} = F(s-\alpha)$$

In term of inverse Laplace transform,

$$L^{-1}\{F(s-\alpha)\} = e^{\alpha t} f(t)$$

Proof: As we know Laplace Transform,

$$L(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

For given function,

$$F(s-\alpha) = \int_0^\infty e^{-(s-\alpha)t} f(t)\, dt \Rightarrow \int_0^\infty e^{-st}\, e^{\alpha t} f(t)\, dt \Rightarrow \int_0^\infty e^{-st} \left(e^{\alpha t} f(t)\right) dt$$

$$F(s-\alpha) = L(e^{\alpha t} f(t))$$

It can also be written as,

$$L^{-1}\{F(s-\alpha)\} = e^{\alpha t} f(t)$$

This is our required result.

## LECTURE NO. 84

**Application of S-Shifting Theorem:**

Evaluate

$$i):\ L(e^{\alpha t}\cos\omega t) \qquad ii):\ L(e^{\alpha t}\sin\omega t) \qquad iii)\ L^{-1}\left(\frac{2s-27}{s^2+2s+401}\right)$$

Solution: (Here we are going to use previous results)

***i*): $L(e^{\alpha t}\cos mt)$**

$$L(\cos\omega t) = \frac{s}{s^2+\omega^2} \Rightarrow \frac{s-o}{(s-o)^2+\omega^2}$$

$$L(e^{\alpha t}\cos\omega t) = \frac{s-\alpha}{(s-\alpha)^2+\omega^2}$$

***ii*): $L(e^{\alpha t}\sin mt)$**

$$L(e^{\alpha t}\sin\omega t) = \frac{\omega}{(s-\alpha)^2+\omega^2}$$

***iii*) $L^{-1}\left(\frac{2s-27}{s^2+2s+401}\right)$**

$$L^{-1}\left(\frac{2s-27}{s^2+2s+401}\right) = L^{-1}\left(\frac{2s+2-2-27}{s^2+2s+1-1+401}\right) \Rightarrow L^{-1}\left(\frac{2(s+1)-29}{(s+1)^2+400}\right)$$

$$= L^{-1}\left(\frac{2(s+1)}{(s+1)^2+(20)^2}\right) - L^{-1}\left(\frac{29}{(s+1)^2+(20)^2}\right)$$

$$= L^{-1}\left(\frac{2(s+1)}{(s+1)^2+(20)^2}\right) - L^{-1}\left(\frac{29}{(s+1)^2+(20)^2}\times\frac{20}{20}\right)$$

$$= 2L^{-1}\left(\frac{(s+1)}{(s+1)^2+(20)^2}\right) - 29L^{-1}\left(\frac{20}{20\{(s+1)^2+(20)^2\}}\right)$$

$$= 2L^{-1}\left(\frac{s-(-1)}{((s-(-1))^2+(20)^2}\right) - \frac{29}{20}L^{-1}\left(\frac{20}{((s-(-1))^2+(20)^2}\right)$$

Using method of calculations i) and ii)

$$= 2e^{-t}\cos 20t - \frac{29}{20}e^{-t}\sin 20t$$

$$= e^{-t}\left[2\cos 20t - \frac{29}{20}\sin 20t\right]$$

This is our required result.

## LECTURE NO. 85

**Piecewise Continuous Function:**

Infromlly, a function is a piecwise continuous on an interval if it has finite jumps (discontinuities) on that interval.

A function $f(t)$ is a piecewise continuous on a finite interval $[a,b]$ if satisfy the following conditions.

1) Function $f(t)$ defined on an/that interval. (i.e. $[a,b]$)
2) Interval can be sub-divided into finitely many sub-intervals in each of which function $f(t)$ is continuous.
3) Function $f(t)$ has finite limits as "$t$" approaches either end points of sub-intervals from interior of $[a,b]$. $\{f(a_0+0)<\infty\,,\, f(a_0-0)<\infty); Finite\}$

Example:

$$f(t)=\begin{cases} sinx & if \quad 0\le x\le \pi/2 \\ e^x & if \quad \pi/2\le x\le \pi \\ -2 & if \quad \pi\le x\le 2\pi \end{cases} \qquad Overall\ interval\ [0,2\pi]$$

LECTURE NO. 86

**Existence Theorem for Laplace Transform:**

Let $f(t)$ be a piecewise continuous function on every finite interval in $[0,\infty)$ and satisfies $|f(t)e^{-kt}|\le m$, $\forall\ t\ge 0$ and for some "$k\ and\ m$" then Laplace Transform of $f(t)$ exists for all $s>k$.

Answer: As given

$$|f(t)e^{-kt}|\le m \quad \dots(1) \qquad \forall\ t\ge 0\ \&\ for\ some\ "k\ and\ m"$$

Here equation (1) is known as "Growth Restriction". Equation (1) implies,

$$|f(t)|\le m\,e^{kt}$$

$$f(t)e^{-kt}\to 0 \quad as \quad t\to\infty \quad \Rightarrow \quad \lim_{t\to\infty}\frac{f(t)}{e^{kt}}\to 0$$

Proof: As given that $f(t)$ be a piecewise continuous function. $f(t)e^{-kt}$ is integerable over any finite interval on $[0,\infty)$. Laplace Transform,

$$|L(f(t))| = |\int_0^\infty e^{-st}\ f(t)\ dt| \le \int_0^\infty |f(t)|\ e^{-st}\ dt$$

For given function,

$$|L(f(t))|\le \int_0^\infty m\,e^{kt}\ e^{-st}\ dt \ \Rightarrow\ m\int_0^\infty e^{-(s-k)t}\,dt \quad \dots(2)$$

Integrating and applying limits, (without "m") , we have

$$\int_0^\infty e^{-(s-k)t}\ dt = -\frac{1}{s-k}|e^{-(s-k)t}|_0^\infty \ \Rightarrow\ -\frac{1}{s-k}|\frac{1}{e^{(s-k)t}}|_0^\infty$$

$$= -\frac{1}{s-k}[\frac{1}{e^\infty}-\frac{1}{e^0}] \Rightarrow \frac{1}{s-k} \quad ; \ for\ s>k$$

So, equation (2) becomes,

$$|L(f(t))|\le \frac{m}{s-k} \ \Rightarrow\ L(f)\ exist\ if\ growth\ restriction\ is\ satisfied.$$

Hence prove our required result.

LECTURE NO. 87

**Counter Example of Existence Theorem:**

Example:

$$f(t)=\frac{1}{\sqrt{t}} \qquad As\ t\to 0\ \ then\ \ f(t)\to\infty$$

Here $f(t)$ is not a piecewise continuous function on $[0,\infty)$. Also

$$|f(t)|\le m\,e^{kt} \qquad If\ m=1, k=0$$

Applying Laplace transform on given function,

$$L(f(t)) = \int_0^\infty e^{-st} \frac{1}{\sqrt{t}} \, dt$$

Put $\begin{cases} Let \quad \sqrt{st} = x \Rightarrow st = x^2 \\ x = \sqrt{s} \frac{1}{2\sqrt{t}} dt \Rightarrow \frac{dt}{\sqrt{t}} = \frac{2}{\sqrt{s}} dx \end{cases}$

$$L(f(t)) = \int_0^\infty e^{-x^2} \frac{2}{\sqrt{s}} dx \Rightarrow \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} dx \quad \dots (1)$$

Here

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \; (Gaussian\ Integral)$$

Hence relation (1) becomes,

$$L(f(t)) = \frac{2}{\sqrt{s}} \times \frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{s}} \; ; \; s > 0$$

Hence $\frac{1}{\sqrt{t}}$ is not a piecewise continuous function but its Laplace Transform exists. This is our required result.

LECTURE NO. 88

**Laplace Transform of Derivatives:**

Prove that

$$i) \quad L(f') = sL(f) - f(0) \qquad ii) \; L(f'') = s^2L(f) - sf(0) - f'(0)$$

Proof: since we know

$$If; \quad f \; is \; differentiable \Rightarrow f \; is \; continuous.$$

**I):**

Laplace transform fro given function

$$L(f') = L(f'(t)) = \int_0^\infty e^{-st} f'(t) \, dt$$

Here $f(t)$ is continuous. Integrating and applying limits, we have

$$L(f') = |e^{-st} f(t)|_0^\infty - \int_0^\infty f(t) \, (-s) e^{-st} \, dt$$

$$L(f') = |\lim_{t\to\infty} \frac{f(t)}{e^{st}} - \frac{f(0)}{e^0}| + s \int_0^\infty e^{-st} f'(t) \, dt$$

$$L(f') = 0 - f(0) + sL(f)$$

$$L(f') = sL(f) - f(0) \quad \dots (1)$$

**II):**

Taking derivative of equation (1),

$$L(f'') = sL(f') - f'(0)$$

Putting value from equation (1),

$$L(f'') = s[sL(f) - f(0)] - f'(0)$$

$$L(f'') = s^2L(f) - sf(0) - f'(0)$$

Hence prove.

LECTURE NO. 89

**Laplace Transform of nth Derivatives:**

$$L(f^n) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) \dots - s^0 f^{n-1}(0)$$

Watch Lecture(s).

LECTURE NO. 90

**Applications of Laplace of Derivatives:**

Find $L(cos\omega t)$ y using derivative expression of Laplace.

Solution: since we know from previous lectures

$$L(f') = sL(f) - f(0) \quad \dots(1)$$
$$L(f'') = s^2L(f) - sf(0) - f'(0) \quad \dots(2)$$

Here

$$f(t) = cos\omega t \;\Rightarrow\; f'(t) = -\omega sin\omega t \;\; \& \;\; f''(t) = -\omega^2 cos\omega t$$
$$f(0) = 1 \Rightarrow f'(0) = 0$$

By putting values in equation 2,

$$L(f'') = s^2L(f) - sf(0) - f'^{(0)}$$
$$L(-\omega^2 cos\omega t) = s^2L(f) - s$$
$$-\omega^2 L(cos\omega t) = s^2L(f) - s$$
$$-\omega^2 L(f) = s^2L(f) - s$$
$$s = L(f)\{s^2 + \omega^2\}$$
$$L(f) = \frac{s}{s^2 + \omega^2}$$

LECTURE NO. 91

**Evaluate:**

$$L(t\ sin\omega t)$$

Solution: Here

$$f(t) = t\ sin\omega t \;\Rightarrow\; f(0) = 0$$
$$f'(t) = t(\omega\ cos\omega t) + 1\ sin\omega t \;\Rightarrow\; f'(0) = 0$$
$$f''(t) = \omega[t(-\omega\ sin\omega t) + 1\ cos\omega t] + \omega\ cos\omega t$$
$$f''(t) = -\omega^2(t\ sin\omega t) + 2\omega\ cos\omega t$$

By putting values in equation 2,

$$L(f'') = s^2L(f) - sf(0) - f'(0)$$
$$L\{-\omega^2(t\ sin\omega t) + 2\omega\ cos\omega t\} = s^2L(f)$$
$$-\omega^2 L(t\ sin\omega t) + 2\omega L(cos\omega t) = s^2L(f)$$
$$-\omega^2 L(f) + 2\omega L\left(\frac{s}{s^2 + \omega^2}\right) = s^2L(f)$$
$$\frac{2\omega s}{s^2 + \omega^2} = L(f)\{s^2 + \omega^2\}$$
$$L(f) = \frac{2\omega s}{(s^2 + \omega^2)^2}$$

This is our required result.

LECTURE NO. 92

**Laplace Transform of Integral:**

Theorem: let $f(t)$ is a piecewise continuous function for $t \geq 0$, & $d[f(t)] = F(s)$. Further $|f(t)| \leq me^{kt}$ for some$m > 0\ and\ k > 0$, then

$$L\{\int_0^t f(r)\ dr = \frac{1}{s}\ F(s) \quad and \quad L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^\infty f(r)\ dr\}$$

Proof: Let

$$g(t) = \int_0^t f(r)\ dr$$

$$\therefore \quad f(t)\ is\ piecewise\ continuous \Rightarrow g(t)\ is\ continuous$$

$$|g(t)| = |\int_0^t f(r)\,dr| \le \int_0^t |f(r)|\,dr \le \int_0^t me^{kt}\,dr$$

$$|g(t)| = m\left|\frac{e^{kt}}{k}\right|_0^t \Rightarrow \frac{m}{k}(e^{kt}-1) \le \frac{m}{k}e^{kt}$$

$\Rightarrow g(t)$ satisfies growth restriction. Now

$$g'(t) = \frac{d}{dt}\left(\int_0^t f(r)\,dr\right) = f(t)$$

Except at point of discontinuities. This implies $g(t)\ is\ continuous.$

$$L(g'(t)) = L(f(t)) \quad and \quad sL(g(t)) - g(0) = F(s)$$

$$sL\left\{\int_0^t f(r)\,dr\right\} = F(s) \Rightarrow L\left\{\int_0^t f(r)\,dr\right\} = \frac{F(s)}{s}$$

## LECTURE NO. 93

**Evaluate:**

$$i)\ L^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} \quad and \quad ii)\ L^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\}$$

**I):**

$$L^{-1}\left\{\frac{1}{(s^2+\omega^2)}\right\} = L^{-1}\left\{\frac{1}{\omega}\cdot\frac{\omega}{s^2+\omega^2}\right\} \Rightarrow \frac{1}{\omega}L^{-1}\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$L^{-1}(F(s)) = \frac{1}{\omega}\ sin\omega t \quad ; \quad \therefore\ L^{-1}\left(\frac{\omega}{s^2+\omega^2}\right) = sin\omega t$$

Now, (Going to calculate required function)

$$L^{-1}\left(\frac{1}{s}F(s)\right) = \int_0^t f(r)\,dr$$

$$L^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \int_0^t \frac{1}{\omega}\ sin\omega r\,dr$$

Integrating and applying limits, we have

$$L^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = -\frac{1}{\omega^2}|cos\omega r|_0^t$$

$$L^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}(1-cos\omega t)$$

This is our required result.

**II):**

$$L^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = L^{-1}\left\{\frac{1}{s}\,\frac{1}{s(s^2+\omega^2)}\right\} \Rightarrow L^{-1}\left\{\frac{1}{s}\,F(s)\right\}$$

$$L^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \int_0^t f(r)\,dr \Rightarrow \int_0^t \frac{1}{\omega^2}(1-cos\omega t)\,dr$$

$$L^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\left[r - \frac{sin\omega r}{\omega}\right]$$

$$L^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\left[t - \frac{sin\omega t}{\omega}\right]$$

This is our required result.